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On the Uniqueness of Quantum Equilibrium in Bohmian Mechanics

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Abstract In Bohmian mechanics the distribution $|\psi|^2$ is regarded as the equilibrium distribution. We consider its uniqueness, finding that it is the unique equivariant distribution that is also a local functional of the wave function ψ.

1 Introduction

Bohmian mechanics (often called the de Broglie–Bohm theory) yields the same predictions as standard quantum theory provided the configuration of a system with wave function ψ is random, with distribution given by $|\psi|^2$. This distribution, the *quantum equilibrium distribution* [1, 2], satisfies the following natural property: If the distribution of the configuration at some time t_0 is given by $|\psi_{t_0}|^2$, then the distribution of the configuration at any other time t will be given by $|\psi_t|^2$—i.e., with respect to the wave function it will have the same functional form at the other time—provided, of course, that the wave function evolves according to Schrödinger's equation between the two times and the configuration evolves according to the law of motion for Bohmian mechanics. This property was already emphasized by de Broglie in 1927 [3] and was later formalized and called *equivariance* by Dürr et al. [2], who used it to establish the typicality of empirical statistics given by the quantum equilibrium distribution.

The notion of equivariance is a natural generalization of that of the stationarity of a distribution in statistical mechanics and dynamical systems theory [2]. Just as stationarity is regarded as a basic requirement for a description of equilibrium in statistical mechanics, one can regard equivariance as a basic requirement for what might be called equilibrium in

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Bohmian mechanics. Of course, this equilibrium need not be a complete equilibrium, since the wave function in general changes with time and need not be in equilibrium—even if the configuration is. Rather, equivariance concerns an equilibrium relative to the wave function: a quantum equilibrium.

An interesting question which then arises is whether the quantum equilibrium distribution $|\psi|^2$ is the unique equivariant distribution. In this paper we show that $|\psi|^2$ is the only local functional of the wave function that is equivariant.

The uniqueness proof is of particular value for the approach of Dürr et al. [2, 4–6] to explaining equilibrium in Bohmian mechanics, an approach first advocated by Bell [7]. Dürr et al. base their justification of the $|\psi|^2$ distribution on a "typicality" argument. They argue that a "typical" Bohmian universe yields $|\psi|^2$ probabilities as empirical distributions. What this means is that the set of initial configurations of the universe that yield the $|\psi|^2$ distribution is very large: it has measure near one for the measure P_e^{Ψ} having density $|\Psi|^2$, with Ψ the wave function of the universe. One reason P_e^{Ψ} is invoked is that it is equivariant.

After recalling Bohmian mechanics in Sect. 2, we define in Sect. 3 the notion of equivariance, providing some illustrative examples. Some of these touch upon the connection between the uniqueness of equivariant distributions for Bohmian mechanics and the notion of the ergodicity of a dynamical system, a connection that is developed in Sects. 6 and 7. While some familiarity with elementary ergodic theory would be helpful for some of the discussion in Sect. 3, the uniqueness results for the quantum equilibrium distribution presented in Sects. 4 and 5 require no such familiarity.

2 Bohmian Mechanics

In Bohmian mechanics the state of a quantum system is given by the positions of its particles as well as its wave function; the motion of the particles is determined by the wave function. For a system of N spinless particles the wave function $\psi_t(q) = \psi_t(q_1, \ldots, q_{3N})$ is a complex-valued function on the configuration space $\mathbb{R}^{3N}$, and satisfies the non-relativistic Schrödinger equation

$$i\hbar\partial_t\psi_t(q) = H\psi_t(q) = \left(-\sum_{k=1}^{M}\frac{\hbar^2}{2m_k}\partial_{q_k}^2 + V(q)\right)\psi_t(q), \tag{1}$$

with $M = 3N$, $\partial_{q_k} = \partial/\partial q_k$ and where $m_1 = m_2 = m_3$ is the mass of the first particle and similarly for the other particles. The particles move in physical space $\mathbb{R}^3$. We denote the actual positions of the particles by $\mathbf{Q}_i \in \mathbb{R}^3$. Thus the actual configuration Q of the system of particles, collectively representing their N actual positions, is given by the vector $Q = (Q_1, \ldots, Q_M) = (\mathbf{Q}_1, \ldots, \mathbf{Q}_N) \in \mathbb{R}^M = (\mathbb{R}^3)^N$. (The Cartesian coordinates of the first particle are given by $\mathbf{Q}_1 = (Q_1, Q_2, Q_3)$ and similarly for the other particles.) The possible trajectories Q_t for the system of particles are given by solutions to the *guidance equation*

$$\frac{dQ_t}{dt} = v^{\psi_t}(Q_t), \tag{2}$$

where the velocity field $v^{\psi} = (v_1^{\psi}, \ldots, v_M^{\psi})$ on $\mathbb{R}^M$ is given by

$$v_k^{\psi}(q) = \frac{\hbar}{m_k}\,\mathrm{Im}\,\frac{\partial_{q_k}\psi(q)}{\psi(q)}. \tag{3}$$

We denote the flow associated to the velocity field by $q_t : \mathbb{R}^M \to \mathbb{R}^M$.[1] Thus $Q_t = q_t(q)$ is the solution to the guidance equation for which $Q_0 = q$, so that $q_0(q) = q$. In this notation we have suppressed the dependence on the wave function. We keep the initial time $t = 0$ fixed throughout the paper, and let ψ usually denote the initial wave function, so that $\psi_0 = \psi$.

3 Equivariance

Suppose we have a (measure-valued) functional $P : \psi \mapsto P^\psi$ from (nontrivial, i.e. not everywhere 0) wave functions to probability distributions on configuration space $\mathbb{R}^M$. There exist then two natural time evolutions for P^ψ. On the one hand, with $\psi_t(q) = e^{-iHt/\hbar}\psi(q)$ a solution to the Schrödinger equation with initial wave function $\psi_0(q) = \psi(q)$, we have the probability distribution P^{ψ_t} for all $t \in \mathbb{R}$. On the other hand, under the Bohm flow (2, 3) the distribution P^ψ is carried to the distribution $P_t^\psi = P^\psi \circ q_t^{-1}$ at time t. This means that if the initial configuration Q_0 is random, with distribution P^ψ, then the distribution of the configuration $Q_t = q_t(Q_0)$ at time t is P_t^ψ.

The functional P is called *equivariant* [2] if

$$P_t^\psi = P^{\psi_t} \quad \text{for all } t \in \mathbb{R}. \tag{4}$$

In other words P is equivariant if P^ψ retains its form as a functional of the wave function ψ when the time evolution of the distribution is governed by the flow q_t associated to the velocity field v^{ψ_t}. When the equivariant functional P is given by a density, i.e., when it is of the form $P^\psi(dq) = p^\psi(q)dq$, we will also call the density-valued functional $p : \psi \mapsto p^\psi(q)$ equivariant. This will of course be so precisely when $p_t^\psi(q) = p^{\psi_t}(q)$ for all t, with $p_t^\psi(q) = p^\psi(q_t^{-1}(q))|\frac{\partial q_t}{\partial q}(q_t^{-1}(q))|^{-1}$ the density for P_t^ψ. (We will also say that the distribution P^ψ and the density p^ψ are equivariant when the functionals are.)

We can also characterize equivariance as follows. Suppose P^ψ is given by the density p^ψ. Then the density $p(q,t) = p_t^\psi(q)$ satisfies the continuity equation

$$\partial_t p(q,t) + \sum_{k=1}^{M} \partial_{q_k}(v_k^{\psi_t}(q)p(q,t)) = 0. \tag{5}$$

Thus the functional P is equivariant precisely if $\tilde{p}(q,t) = p^{\psi_t}(q)$ also satisfies the continuity equation (5) for all ψ. This follows from the uniqueness of solutions of partial differential equations and the fact that the functions $p^{\psi_t}(q)$ and $p_t^\psi(q)$ are equal at $t = 0$.

Let us now give some examples. The first example is the distribution $|\psi|^2$. In the following we don't assume the wave functions to be normalized. If the distributions are given by $|\psi|^2$, then it is natural to normalize the wave functions so that they have L^2-norm one. But for other distributions, other normalizations might be more appropriate.

Example 1 The *quantum equilibrium* functional is $P_e(dq) = p_e(q)dq$ where $p_e : \psi \mapsto p_e^\psi = N_e^\psi|\psi|^2$, with $N_e^\psi = 1/\int_{\mathbb{R}^M}|\psi|^2 dq$. Obviously p_e, respectively P_e, maps wave functions to probability densities, respectively probability distributions. This functional is equivariant since $p_e^{\psi_t}$ satisfies the continuity equation (5) for all wave functions ψ.

[1] The Bohmian dynamics, defined by (1–3), is well defined on the subset of $L^2(\mathbb{R}^M) \times \mathbb{R}^M$ consisting of pairs (ψ, q) with ψ sufficiently smooth and q such that $\psi(q) \neq 0$, see [8]. We shall usually ignore such details.

In general, whether or not a distribution P^ψ is equivariant would be expected to depend on the potential V. Note, however, that the quantum equilibrium distribution P_e is equivariant for all V.

Example 2 Suppose ϕ is a real-valued eigenstate of the Hamiltonian H, for example the ground state. For this stationary state the associated velocity field v^{ϕ_t} (3) vanishes, so that the Bohm motion is trivial in this case. Thus any functional P will trivially obey (4) for $\psi = \phi$.

The previous example illustrates the fact that equivariance is a property of a mapping $\psi \mapsto P^\psi$; it concerns a family $\{P^\psi\}$ and not merely the satisfaction of (4) for a single wave function ψ. Equivariance means that $P^{\psi_t} = P_t^\psi$ for *all* wave functions ψ in the Hilbert space.

We may also consider the equivariance of a functional P defined on an invariant subset of Hilbert space: Let $\mathcal{I}$ be an invariant set of wave functions, i.e., such that $\psi \in \mathcal{I}$ if and only if $\psi_t = e^{-iHt/\hbar}\psi \in \mathcal{I}$. We say that the functional $\psi \mapsto P^\psi$, defined for $\psi \in \mathcal{I}$, is *equivariant on* $\mathcal{I}$ if (4) is obeyed by all $\psi \in \mathcal{I}$.

We have so far not explicitly imposed any conditions on the distribution-valued functional P^ψ beyond equivariance. A condition that would be natural is that the functional be projective, i.e., that if ψ' is a (non-vanishing) scalar multiple of ψ then $P^{\psi'} = P^\psi$, but we shall not do so. We shall, however, insist on the following: When we speak of an equivariant functional P^ψ, it is to be understood that the mapping $\psi \mapsto P^\psi$ is measurable. When P^ψ is given by the density p^ψ, the measurability of P^ψ amounts to that of $p^\psi(q)$ as a function of ψ and q. Measurability is the weakest sort of regularity condition invoked in analysis, probability theory, and ergodic theory, much weaker than differentiability or continuity. We do not wish to specify here precisely what is meant by the measurability of P^ψ (or of $p^\psi(q)$), since the main result of this paper involves a much stronger condition, that P^ψ be suitably local. As a rule of thumb, however, we can say the following: Any mapping $\psi \mapsto P^\psi$ given by an explicit formula will be measurable.

In order to appreciate the importance of measurability, one should note that when a dynamical system is analyzed, it is often necessary to consider random initial conditions. For the Bohmian system the initial condition is given by the quantum state ψ as well as the initial configuration, and hence one should allow for the possibility that the initial wave function ψ is random, with distribution $\mu(d\psi)$. When this is combined with a functional $P^\psi(dq)$, one is naturally led to consider the joint distribution $\mu(d\psi)P^\psi(dq)$ of ψ and q, see Sect. 6. But this will be meaningful—i.e., define a genuine probability distribution—only when P^ψ is measurable.

Furthermore, there is a sense in which the equivariance condition (4) says that P^ψ is a constant of the motion for the Schrödinger evolution of wave functions: With each $\psi \in \mathcal{H} = L^2(\mathbb{R}^M)$ associate a "fiber" Γ_ψ, namely the set of probability distributions on configuration space $\mathbb{R}^M$. The Bohm flow acting on distributions provides a natural identification of Γ_{ψ_t} with Γ_ψ (and in fact defines a connection on the fiber bundle $\mathcal{H} \times \Gamma = \{(\psi, \mu) | \psi \in \mathcal{H}, \mu \in \Gamma_\psi\}$). The equivariance condition (4) then says that the function P^ψ is a constant of the Schrödinger motion under this identification.

Now if a dynamical system is ergodic, there can be no nontrivial functions (i.e., functions that are not almost everywhere equal to a constant) that are constants of the motion. However, it is understood that only measurable functions are to be considered; in fact, there are more or less always many nontrivial constants of the motion that are not measurable. Any function of the orbits of the motion will define a constant of the motion. Most such constants of the motion will be nontrivial, and these will also fail to be measurable when the dynamics is ergodic.

Similarly, one might expect that there will more or less always be a great many functionals satisfying (4) if measurability is not demanded, and this is indeed the case, as we indicate in the next example. (See Sect. 7 for more on equivariance and ergodicity.)

Example 3 For any fixed ψ let $\mathcal{O}_\psi = \{e^{-iHt/\hbar}\psi\} \equiv \{\psi_t\}$ denote the orbit of ψ under the Schrödinger evolution—the smallest invariant set containing ψ. If $\mathcal{O}_\psi$ is not a periodic orbit (one such that $\psi_t = \psi$ for some $t \neq 0$), we may let P^ψ, for this ψ, be *any* probability distribution on configuration space, and extend it to $\mathcal{O}_\psi$ via (4). The resulting function P is then obviously equivariant on $\mathcal{O}_\psi$. If $\mathcal{O}_\psi$ is periodic, let $P = P_e$ on $\mathcal{O}_\psi$. In this way we may obtain a great many different functionals P—one for each assignment of probability distributions to representatives of each non-periodic orbit—defined on the union of all orbits, and hence for all ψ in Hilbert space. All of them obey (4) for all ψ. Most of these, however, will not be measurable, and hence should not count as equivariant functionals.

In the previous example, suppose we were to choose P^ψ in an explicit way, for example as in Example 2, on the representatives. It might seem then, on the one hand, that we have provided in effect an explicit formula for the functional P constructed in this way, so that it would then be measurable. On the other hand, if the Bohmian dynamics is suitably ergodic, see Sect. 7, as is likely often to be the case, P (if it is given by a density) must then agree with P_e on many non-periodic orbits, which it clearly does not. What gives? The answer is that the specification just mentioned is much less explicit than it might at first appear to be, since in general there is no canonical way to choose a representative for each orbit, and the functional so constructed need not be measurable.

A flow on the line or an autonomous flow on the plane can't have strong ergodic properties. One might thus expect the Bohm motion on the line to also fail to have strong ergodic properties. That this is so was shown in [9]. Accordingly, since dynamical systems that are not ergodic have many stationary distributions, one should expect there to be a great many distributions that are equivariant for this case.

Example 4 Consider a Bohmian particle moving on the line. Since trajectories can't cross, it is easy to see that the function $F(\psi, q) = P_e^\psi((-\infty, q))$ is a constant of the motion, $F(\psi_t, q_t(q)) = F(\psi, q)$. For fixed ψ, F is a map $\mathbb{R} \to [0, 1]$, and for every probability distribution μ on $(0, 1)$ there is an equivariant functional

$$P_\mu^\psi(B) = \mu(F(\psi, B)), \tag{6}$$

the image of μ under F_ψ^{-1}, the inverse of the map $q \mapsto F(\psi, q)$. (When μ is the Lebesgue measure, $\mu(dq) = dq$, we have that $P_\mu = P_e$.) Perhaps the simplest way to understand this is in terms of the change of variables $(\psi, q) \mapsto (\psi, \tilde{q})$ with $\tilde{q} = F(\psi, q)$. In these new coordinates the Bohmian dynamics becomes trivial: ψ evolves as usual according to Schrödinger's equation and $\tilde{q}$ does not change under the dynamics. Thus any distribution μ for $\tilde{q}$ defines an equivariant functional.[2]

[2] Moreover, every equivariant functional for a particle on the line corresponds a.e. to a (possibly different) choice μ for each ergodic component of the Schrödinger dynamics.

A difference between the functional in Example 1 and those in (Example 3 and) Example 4 is that the former is a local functional, whereas the latter (except for the quantum equilibrium functional) are not. We call a functional p^ψ *local* if $p^\psi(q)$ can be written, up to normalization, as a (sufficiently differentiable) function of q, $\psi(q)$, and finitely many derivatives of ψ, evaluated at q. That is, for a local functional p^ψ we can write

$$p^\psi(q) = N^\psi g^\psi(q) \tag{7}$$

where N^ψ does not depend on q and where

$$g^\psi(q) = g(q, \psi(q), \ldots, \partial_{q_1}^{n_1} \ldots \partial_{q_M}^{n_M} \psi(q), \ldots) \tag{8}$$

depends on at most finitely many partial derivatives of ψ (and is sufficiently differentiable). We shall say that a functional of the form (8) is *strictly local*. (A local density $p^\psi(q)$, because of the normalization factor N^ψ, need not be strictly local.) We note that a local functional that, as demanded above, is differentiable will of course be measurable. In fact, for measurability, continuity—indeed mere measurability of g—would suffice.

In the following section we will see that equivariance, together with the requirement that the functional be local, leads uniquely to quantum equilibrium p_e.

4 Uniqueness of Equivariant Densities

Let $p : \psi \mapsto p^\psi$ be a functional from wave functions to probability densities. We show that p is uniquely given by p_e, with $p_e^\psi = N_e^\psi |\psi|^2$ as in Example 1, under the assumptions that p^ψ is equivariant and local. The locality implies that p^ψ can be written in the form $p^\psi(q) = N_g^\psi g^\psi(q)$, where $N_g^\psi = 1/\int_{\mathbb{R}^M} g^\psi(q)dq$ and where $g^\psi(q)$ is a strictly local functional, see (8). We split the proof into two parts, successively showing that:

(P1) $g^{\psi_t}(q)$ satisfies the equation

$$\partial_t g^{\psi_t}(q) + \sum_{k=1}^{M} \partial_{q_k}(v_k^{\psi_t}(q) g^{\psi_t}(q)) + h g^{\psi_t}(q) = 0, \tag{9}$$

with h a constant, i.e., independent of q and the wave function.

(P2) $p^\psi(q) = p_e^\psi(q) = N_e^\psi |\psi(q)|^2$.

We now give the proofs.

Proof of (P1) Equivariance implies that $p^{\psi_t}(q)$ satisfies the continuity equation (5). Since $p^{\psi_t}(q) = N_g^{\psi_t} g^{\psi_t}(q)$ the continuity equation for $p^{\psi_t}(q)$ can be written as

$$\frac{1}{g^{\psi_t}(q)} \left(\partial_t g^{\psi_t}(q) + \sum_{k=1}^{M} \partial_{q_k}(v_k^{\psi_t}(q) g^{\psi_t}(q)) \right) = -\partial_t \ln N_g^{\psi_t} \tag{10}$$

(wherever $g^{\psi_t}(q) > 0$).

Let us introduce the functional $h : \psi \mapsto h^\psi$, from wave functions to the real numbers, defined by

$$h^{\psi_t} = \partial_t \ln N_g^{\psi_t}. \tag{11}$$

Since $\partial_t \ln N_g^{\psi_t}$ is independent of q, h^{ψ_t} is well-defined as a real number. We will show that this functional is constant, i.e. independent of ψ.

First note that $\partial_t g^{\psi_t}(q)$ can be expressed as a function of q and of the variables $\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_t(q)$. This is because g^ψ is a strictly local functional and because the time derivatives of any of the variables $\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_t(q)$ can be replaced by spatial derivatives by making use of the Schrödinger equation. As a result we have from (10) that

$$h^\psi = h(q, \psi(q), \dots, \partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi(q), \dots), \tag{12}$$

so that h^ψ is a strictly local functional.

It follows that $h^\psi = h^{\psi'}$ for any two wave functions ψ and ψ' for which all derivatives agree at a configuration $q \in \mathbb{R}^M$. But this means that for any ψ and ψ', $h^\psi = h^{\psi'}$, since there is always a third wave function ψ'' such that all the derivatives of ψ and ψ'' agree at one configuration $q \in \mathbb{R}^M$ and such that all the derivatives of ψ' and ψ'' agree at another configuration $q' \in \mathbb{R}^M$.

Thus h^ψ is independent of ψ. We write $h^\psi = h$. The continuity equation (10) then reduces to (9). $\square$

Proof of (P2) Let us introduce the functional $f^\psi(q) = g^\psi(q)/|\psi(q)|^2$.[3] From the continuity equation for $|\psi_t(q)|^2$ and the equation (9) for $g^{\psi_t}(q)$ it follows that

$$\frac{df^{\psi_t}}{dt} + hf^{\psi_t} = 0, \tag{13}$$

with

$$\frac{d}{dt} = \partial_t + \sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k}. \tag{14}$$

Because f^ψ is a strictly local functional we have that

$$f^\psi(q) = f(q, \psi(q), \dots, \partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi(q), \dots). \tag{15}$$

Relation (13) can therefore be written as

$$\begin{aligned} 0 &= \frac{df^{\psi_t}}{dt} + hf^{\psi_t} \\ &= \sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k} f + \sum_{m_1,\dots,m_M}\left(\frac{d}{dt}(\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r})\frac{\partial f}{\partial(\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r})}\right. \\ &\quad \left. + \frac{d}{dt}(\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i})\frac{\partial f}{\partial(\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i})}\right) + hf, \end{aligned} \tag{16}$$

where $\psi_{t,r}$ and $\psi_{t,i}$ are respectively the real part and the imaginary part of ψ_t.

[3] $f^\psi(q)$ is defined on $\{q \in \mathbb{R}^M | \psi(q) \neq 0\}$. Since the Bohm flow (2, 3) is defined only on this set, we consider only densities on this set, i.e., for which $g^\psi > 0$ only on this set.

This expression can be rewritten by making use of the Schrödinger equation (1), since for every variable $\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r}$ and $\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i}$ we have that

$$\begin{aligned}\frac{d}{dt}(\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r}) &= \partial_t\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r} + \sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k}\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r}\\ &= -\sum_{k=1}^{M}\frac{\hbar}{2m_k}\partial_{q_k}^2\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i} + \frac{1}{\hbar}\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}(V\psi_{t,i})\\ &\quad + \sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k}\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r}\end{aligned}\tag{17}$$

and

$$\begin{aligned}\frac{d}{dt}(\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i}) &= \partial_t\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i} + \sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k}\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i}\\ &= \sum_{k=1}^{M}\frac{\hbar}{2m_k}\partial_{q_k}^2\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r} - \frac{1}{\hbar}\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}(V\psi_{t,r})\\ &\quad + \sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k}\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i}.\end{aligned}\tag{18}$$

In this way (16) expresses a functional relation between the variables q and all the real variables $\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,r}$ and $\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,i}$ which has to hold identically, i.e. for all possible values of these variables. Since all these variables can be treated as independent, we can show that the function f must be a constant as follows.

First select a variable $\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,r}$ or $\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,i}$ such that f depends on this variable and such that, if f depends on another variable $\partial_{q_1}^{\bar{n}_1}\dots\partial_{q_M}^{\bar{n}_M}\psi_{t,r}$ or $\partial_{q_1}^{\bar{n}_1}\dots\partial_{q_M}^{\bar{n}_M}\psi_{t,i}$, then $\bar{n}_1\le n_1$. Suppose the selected variable is, say, $\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,r}$. Then, from (16), (17) and (18) it follows that the only term in $df^{\psi_t}/dt + hf^{\psi_t}$ that contains the variable $\partial_{q_1}^{n_1+2}\dots\partial_{q_M}^{n_M}\psi_{t,i}$ is

$$-\frac{\hbar}{2m_1}\partial_{q_1}^{n_1+2}\dots\partial_{q_M}^{n_M}\psi_{t,i}\frac{\partial f}{\partial(\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,r})}.\tag{19}$$

Because $\partial_{q_1}^{n_1+2}\dots\partial_{q_M}^{n_M}\psi_{t,i}$ can be treated as an independent variable, the term above has to be zero. Hence

$$\frac{\partial f}{\partial(\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,r})} = 0.\tag{20}$$

But this contradicts the fact that f depends on the variable $\partial_{q_1}^{n_1}\dots\partial_{q_M}^{n_M}\psi_{t,r}$. It follows that f does not depend on any of the variables $\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,r}$ or $\partial_{q_1}^{m_1}\dots\partial_{q_M}^{m_M}\psi_{t,i}$. Hence we have that $f = f(q)$.

Equation (16) now reduces to

$$\sum_{k=1}^{M} v_k^{\psi_t}\partial_{q_k} f + hf = 0\tag{21}$$

and we can use a reasoning similar to the above to conclude that $\partial_{q_k} f = 0$, $k = 1, \dots, M$. Hence f is a constant independent of q and the wave function and any of its derivatives. Since $g^\psi(q) = f|\psi|^2$ with f now a constant and since p^ψ is assumed to be a probability density we have that

$$p^\psi(q) = p_e^\psi(q) = N_e^\psi |\psi(q)|^2. \quad \square \tag{22}$$

5 A Stronger Result?

There is a weaker version of the locality of the functional $p^\psi(q) = N^\psi g^\psi(q)$, which we shall call *weak locality*, that is worth considering. This requires that $g^\psi(q)$ be determined by ψ in a neighborhood of q, i.e., that if ψ and ψ' agree in some neighborhood of q, then $g^\psi(q) = g^{\psi'}(q)$. This is indeed a weaker notion of locality than used earlier, and allows in particular for $g^\psi(q)$ to depend on all derivatives of ψ at q.

It is reasonable to ask whether the uniqueness result would continue to be valid if the equivariant functional p^ψ were assumed only to be weakly local. We believe that the answer is yes. There is an argument for this that, while not entirely rigorous, is quite compelling. At the same time, the argument provides some perspective on our uniqueness result. It is this:

The Bohmian dynamics defines a flow on (a subset of) the space $\mathcal{X} = \mathcal{H} \times \mathbb{R}^M$, where $\mathcal{H} = L^2(\mathbb{R}^M)$ is the Hilbert space of the Bohmian system. We shall denote the action of this flow by T_t, so that for $\eta = (\psi, q) \in \mathcal{X}$, we have that $T_t\eta = (\psi_t, q_t(q))$. In terms of this flow, the equivariance of the density $p^\psi(q)$ can be conveniently expressed as follows: Let

$$G(\eta) = p^\psi(q)/p_e^\psi(q). \tag{23}$$

Then the equivariance of p^ψ amounts to the requirement that G be a constant of the motion for the flow T_t,

$$G(T_t\eta) = G(\eta). \tag{24}$$

(This is an easy consequence of the fact that p_e^ψ is equivariant.) And uniqueness amounts to the statement that G is constant on (the relevant subset of) $\mathcal{X}$. This would be so if the flow T_t were sufficiently ergodic (see Sect. 7): ergodicity means that there are no nontrivial constants of the motion—that the only constants of the motion are in fact functions that are almost everywhere constant, and hence trivially constants of the motion—as would be the case if the set of possible states η consisted of a single trajectory. This, of course, is impossible. Nonetheless, the ergodicity of a motion on a space means roughly that the motion is sufficiently complicated to produce trajectories that almost connect any two points in the space, so that functions that don't change along a trajectory must be more or less everywhere constant.

In fact, it is easy to see that for uniqueness it is sufficient that G be constant on the subsets $\mathcal{X}_\psi = \{(\psi, q) \in \mathcal{X} | q \in \mathbb{R}^M\}$ of $\mathcal{X}$ corresponding to fixed ψ, and for this it is of course sufficient that G be locally constant on $\mathcal{X}_\psi$, i.e., that every $q \in \mathbb{R}^M$ has a neighborhood O_q such that G is constant on $\{(\psi, q') \in \mathcal{X} | q' \in O_q\}$. It is also easy to see that for uniqueness it is sufficient that $F(\eta) = g^\psi(q)/|\psi(q)|^2$ be constant on $\mathcal{X}$—or (locally) constant on $\mathcal{X}_\psi$.

While F is not obviously invariant under the flow T_t, it is clearly *quasi-invariant*, which is almost as good: In terms of F, (24) becomes

$$F(T_t\eta) = e^{ht} F(\eta), \tag{25}$$

for all $t \in \mathbb{R}$, where h is the constant defined by (11). (That h^ψ is constant follows from weak locality much as it does from locality. Moreover, it seems likely on general grounds that $h = 0$, in which case F would be strictly invariant.)

Now the (weak) locality of p^ψ implies that F is invariant under a much larger set of transformations than the one-dimensional set $\{T_t\}$, defining an action of the group $\mathbb{R}$ on $\mathcal{X}$. It implies invariance under the action T_ϕ of the infinite-dimensional (additive) group $\mathcal{N} = \{\phi \in \mathcal{H} | \phi(q') = 0$ in a neighborhood of $q = 0\}$, where $T_\phi \eta = T_\phi(\psi, q) = (\psi + \phi_q, q)$, with $\phi_q(q') = \phi(q' - q)$. Thus with weak locality we have, in addition to (25), that for all $\phi \in \mathcal{N}$

$$F(T_\phi \eta) = F(\eta). \tag{26}$$

Now while the action of $\mathbb{R}$ on $\mathcal{X}$ given by the Bohmian flow T_t may fail to be suitable ergodic, it is hard to imagine this for the action T_ξ, $\xi \in \mathcal{G}$, of the group $\mathcal{G}$ generated by the actions of $\mathbb{R}$ and $\mathcal{N}$ on $\mathcal{X}$. Indeed, it seems very likely that $\mathcal{X}$ consists of a single orbit $\{T_\xi(\psi, q) | \xi \in \mathcal{G}\}$ of this action, and more likely still that $\mathcal{G}$ connects any two points in any sufficiently small neighborhood of any point in $\mathcal{X}_\psi$.

If h were 0 this would imply uniqueness. For general h we have that

$$F(T_\xi \eta) = e^{ht_\xi} F(\eta) \tag{27}$$

for all $\xi \in \mathcal{G}$. But what was suggested above for $\mathcal{G}$ should still be true of the subgroup $\mathcal{G}_0 = \{\xi \in \mathcal{G} | t_\xi = 0\}$, under the action of which F is invariant, and this would imply uniqueness in the general case.

Indeed, consider only the transformations in $\mathcal{G}_0$ of the form $T_{\phi_2, -t, \phi_1, t} = T_{\phi_2} T_{-t} T_{\phi_1} T_t$, with $\phi_i \in \mathcal{N}$ and $t \in \mathbb{R}$. Since the dimension of the set of such transformations should be regarded as roughly twice the dimension of $\mathcal{X}$, the set obtained by applying all such transformations to a given point $\eta \in \mathcal{X}$—the range of the mapping $(\phi_1, \phi_2, t) \mapsto T_{\phi_2, -t, \phi_1, t}\eta$—should be all of $\mathcal{X}$, at the very least, locally.

The previous argument also suggests that for the uniqueness of the equivariant distribution, the locality condition can be weakened further to that of having finite range $r > 0$: that $g^\psi(q)$ depend at most on the restriction of ψ to the ball B_r of radius r centered at q. (The weak locality condition is then that of having finite range r for all $r > 0$.)

6 Equivariance and Stationarity

We have already indicated that an equivariant functional can be regarded as generalizing the notion of a stationary probability distribution for a dynamical system—one that is invariant under the time-evolution. We wish here to tighten this connection a bit, and observe that the equivariance of the functional P^ψ is more or less equivalent to (it implies and is almost implied by) the following: For every measure $\mu(d\psi)$ on Hilbert space $\mathcal{H}$ that is stationary under the Schrödinger evolution, the measure $\mu(d\psi)P^\psi(dq)$ is a stationary measure on $\mathcal{X} = \mathcal{H} \times \mathbb{R}^M$ for the Bohmian dynamics. (The "almost" and "more or less" refer to the following: The stationarity of $\mu(d\psi)P^\psi(dq)$ implies that the condition (4) for equivariance is satisfied by all ψ with the possible exception of a set of ψ's with μ-measure 0. If there are exceptional ψ's, $P^\psi(dq)$ can be changed, on a set with μ-measure 0 so that it continues to define the same measure $\mu(d\psi)P^\psi(dq)$ on $\mathcal{X}$, so as to become strictly equivariant.)

A general probability measure on $\mathcal{X}$ can be regarded as of the form $\mu(d\psi)P^\psi(dq)$: $\mu(d\psi)$ is the first marginal, the distribution of the first component ψ of $\eta = (\psi, q) \in \mathcal{X}$, and $P^\psi(dq)$ is the conditional distribution of the configuration q given ψ, a probability measure on the fiber of the product space $\mathcal{X}$ that "lies above ψ". Consider now any measure on $\mathcal{X}$ of the form $\mu(d\psi)P^{\psi}(dq)$, with now μ any measure on $\mathcal{H}$ and $P^{\psi}(dq)$ a probability measure on $\mathbb{R}^M$. (Here μ need not be a probability measure, nor even normalizable.) For this measure to be stationary $\mu(d\psi)$ obviously must be. Suppose this is so. Then, for stationarity, we still must have that the measure $P^{\psi}(dq)$ on the ψ-fiber evolves to the correct measure on the ψ_t-fiber, namely $P^{\psi_t}(dq)$ (with the possible exception of a set of ψ's having μ-measure 0). But equivariance says more or less precisely that this is so: it says that for all ψ, $P^{\psi_t} = P^{\psi}_t$, the measure to which P^{ψ} evolves.

Thus a probability measure on $\mathcal{X}$ is stationary if and only if it is of the form $\mu(d\psi)P^{\psi}(dq)$ with μ stationary and P^{ψ} equivariant. In particular, the measure $\mu(d\psi) \times P^{\psi}_e(dq)$, where P_e is the quantum equilibrium distribution, is stationary whenever $\mu(d\psi)$ is. Suppose this is so. Consider a measure $\mu(d\psi)P^{\psi}(dq)$ having a density with respect to $\mu(d\psi)P^{\psi}_e(dq)$. This density is given by the function G (23) on $\mathcal{X}$. The measure $\mu(d\psi)P^{\psi}(dq)$ will be stationary precisely if its density G is a constant of the motion, consistent with our earlier assertion that this amounts to the equivariance of P^{ψ}.

7 Uniqueness and Ergodicity

The ergodicity of a dynamical system, defined by a dynamics and a given stationary probability distribution, is equivalent to the statement that any stationary probability distribution with a density with respect to the given one must in fact be the given one. Thus ergodicity amounts to the uniqueness, in an appropriate sense, of a stationary measure. So a uniqueness statement for an equivariant functional—a uniqueness statement for quantum equilibrium—can be regarded as expressing a sort of generalized ergodicity. We wish now to sharpen this connection by observing that certain uniqueness statements for quantum equilibrium are more or less equivalent to the ergodicity of certain dynamical systems. (One should bear in mind that the ergodicity of a dynamical system is usually extremely difficult to establish.)

The relevant dynamical systems for our purposes here are defined by the Bohmian dynamics on $\mathcal{X}$, with this space equipped with a stationary probability measure of the form $\mu(d\psi)P^{\psi}_e(dq)$, with $\mu(d\psi)$ stationary under the Schrödinger dynamics, as described in Sect. 6. In order for this dynamical system to be ergodic, it is of course necessary for $\mu(d\psi)$ to be an ergodic measure for the Schrödinger dynamics. Suppose that this is so. Then it is easy to see that the ergodicity of $\mu(d\psi)P^{\psi}_e(dq)$ under the Bohmian dynamics amounts to the uniqueness of quantum equilibrium "modulo $\mu(d\psi)$": $\mu(d\psi)P^{\psi}_e(dq)$ is ergodic if and only if every equivariant density p^{ψ} agrees with quantum equilibrium, $p^{\psi} = p^{\psi}_e$, for μ-a.e. ψ.[4]

There is, however, perhaps less in this equivalence than first meets the eye. The set of ψ's of μ measure 1 for which, as a consequence of the ergodicity of $\mu(d\psi)P^{\psi}_e(dq)$, we must have that $p^{\psi} = p^{\psi}_e$ when p^{ψ} is an equivariant density will be rather small. The set is large only relative to the "support" of μ, an invariant subset $\mathcal{I}_{\mu}$ of $\mathcal{H}$, with μ-measure 1, defined by specified values of the constants of the Schrödinger motion such as $\langle\psi|H^n|\psi\rangle, n = 0, 1, 2, \ldots$.

[4] A genuinely different equivariant distribution P^{ψ} with density p^{ψ}—one that does not agree with P^{ψ}_e for μ-a.e. ψ—would yield a stationary probability distribution on $\mathcal{X}$ that is given by a density with respect to the one arising from P^{ψ}_e but that differs from it, contradicting ergodicity. Conversely, by the discussion of Sect. 6 and the ergodicity of μ, a stationary probability distribution on $\mathcal{X}$ that is given by a density with respect to $\mu(d\psi)P^{\psi}_e(dq)$ must be of the form $\mu(d\psi)P^{\psi}(dq)$ with $P^{\psi}(dq)$ equivariant.

For every such "ergodic component" $\mathcal{I}_\mu$ of the Schrödinger dynamics, with $\mu(d\psi) \times P_e^\psi(dq)$ also ergodic, we have the uniqueness of quantum equilibrium for almost all ψ in $\mathcal{I}_\mu$. Taking the totality of such ergodic components of the Schrödinger dynamics, we obtain the uniqueness of quantum equilibrium for almost all of the union of these components. In particular, if $\mathcal{H}$ were completely decomposable into such ergodic components, we would have the uniqueness of quantum equilibrium for almost all ψ in $\mathcal{H}$.[5]

Here is an example of a typical ergodic component of the Schrödinger dynamics, to which the discussion of this section could be applied. Suppose $\phi_1, \dots, \phi_n$ are eigenstates of the Hamiltonian H, with corresponding eigenvalues $E_1, \dots, E_n$ that are rationally independent. For $c_j > 0$, $j = 1, \dots, n$, let $\mathcal{I}_{c_1,\dots,c_n} = \{\psi \in \mathcal{H} | \psi = \sum_{j=1}^n c_j e^{i\theta_j}\phi_j, 0 \le \theta_j < 2\pi, j = 1, \dots, n\}$. The Schrödinger dynamics on $\mathcal{I}_{c_1,\dots,c_n}$ is quasi-periodic, with stationary probability distribution, corresponding to a uniform distribution of the phases θ_j, that is ergodic.

8 Properties of Quantum Equilibrium

The quantum equilibrium functional $P^\psi = P_e^\psi$ satisfies many natural conditions, some of which play an important role in the analysis of a Bohmian universe:

(i) It is *universally* equivariant: it is equivariant for all Schrödinger Hamiltonians H, of the form expressed on the right hand side of (1), i.e., for all V and for all choices m_k of the masses of the particles.

(ii) It is projective: $P^{c\psi} = P^\psi$ for every constant $c \neq 0$.

(iii) It is covariant: $P_R^\psi = P^{R\psi}$ for all the usual symmetries of non-relativistic quantum mechanics, for example for space-translations, rotations, time-reversal, Galilean boosts, and particle permutations. Here P_R^ψ is the distribution to which P^ψ is carried by the action of R on configurations.

(iv) It is *factorizable*. Suppose a Bohmian system is a composite of two systems, with Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and configuration variable $q = (q^{(1)}, q^{(2)})$. Then $P^{\psi_1 \otimes \psi_2}(dq^{(1)} \times dq^{(2)}) = P^{\psi_1}(dq^{(1)}) P^{\psi_2}(dq^{(2)})$. (If $H = H_1 \otimes I_2 + I_1 \otimes H_2$, with I_i the identity on $\mathcal{H}_i$, then it follows immediately from the equivariance of P for the composite system that the P^{ψ_i} are equivariant for the respective components.)

(v) More generally, it is *hereditary*. Consider a composite system as in (iv), and suppose that the conditional wave function of, say, system 1 is ψ when the composite has wave function Ψ and system 2 has configuration $Q^{(2)}$, i.e., that $\psi(q^{(1)}) = \Psi(q^{(1)}, Q^{(2)})$. Then the conditional distribution of the configuration of system 1, given that the configuration of system 2 is $Q^{(2)}$, depends only on ψ and not on the choice of wave function Ψ and configuration $Q^{(2)}$ that yields ψ: for fixed ψ, $P^\Psi(dq^{(1)} \mid Q^{(2)})$ is independent of Ψ and $Q^{(2)}$.

[5] Such a decomposition, of all of $\mathcal{H}$, probably never exists. For many stationary states ψ the Bohm motion is trivial, so that, with μ the uniform distribution on the orbit $\mathcal{O}_\psi$ of ψ, which is ergodic for the Schrödinger dynamics, $\mu(d\psi)P_e^\psi(dq)$ is not ergodic, see Example 2. And for wave functions belonging to the spectral subspace of $\mathcal{H}$ corresponding to the continuous spectrum the situation is even worse. For example, for a free Hamiltonian H, with $V = 0$, there are no ergodic components to begin with. There are in fact, in this case, no probability measures on $\mathcal{H}$ that are stationary under the Schrödinger dynamics. (Consider the free Schrödinger dynamics. As time goes on the wave function should spread, never to become narrow again. But this conflicts with Poincaré recurrence, and thus implies that there is no finite invariant measure, and in particular no stationary probability measure.) And in this case as well, there are, presumably, equivariant densities p^ψ that disagree with p_e^ψ for all ψ.

It remains to be seen to what extent these properties, individually or in various combinations, uniquely characterize quantum equilibrium among equivariant distributions. (It presumably follows, along the lines of the discussion in Sect. 5, that the satisfaction of the equivariance condition (4) for all V's implies uniqueness—with the exception of the case of a single particle on a line.) Be that as it may, it is noteworthy that locality alone, with no additional conditions beyond equivariance, is sufficient to guarantee the uniqueness of quantum equilibrium.

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