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On the Spectrum of an Hamiltonian in Fock Space. Discrete Spectrum Asymptotics

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A model operator H associated with the energy operator of a system describing three particles in interaction, without conservation of the number of particles, is considered. The location of the essential spectrum of H is described. The existence of infinitely many eigenvalues (resp. the finiteness of eigenvalues) below the bottom $\tau_{\mathrm{ess}}(H)$ of the essential spectrum of H is proved for the case where the associated Friedrichs model has a threshold energy resonance (resp. a threshold eigenvalue). For the number $N(z)$ of eigenvalues of H lying below $z < \tau_{\mathrm{ess}}(H)$ the following asymptotics is found

$$\lim_{z \to \tau_{\mathrm{ess}}(H)-0} \frac{N(z)}{|\log|z - \tau_{\mathrm{ess}}(H)||} = \mathcal{U}_0 \, (0 < \mathcal{U}_0 < \infty).$$

KEY WORDS: model operator, conservation of number of particles, Efimov effect, infinitely many eigenvalues, Birman-Schwinger principle, essential spectrum, Hilbert-Schmidt operator, Friedrichs model, conditionally negative definite function.

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1. INTRODUCTION

The main goal of the present paper is to give a thorough mathematical treatment of the spectral properties for a model operator H with emphasis on the asymptotics for the number of infinitely many eigenvalues (Efimov's effect case). The model operator H is associated with a system describing three particles in interaction, without conservation of the number of particles.

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The Efimov effect is one of the more remarkable results in the spectral analysis for continuous three-particle Schrödinger operators: if none of the three two-particle Schrödinger operators (corresponding to the two-particle subsystems) has negative eigenvalues, but at least two of them have a zero energy resonance, then this three-particle Schrödinger operator has an infinite number of discrete eigenvalues, accumulating at zero.

Since its discovery by Efimov in 1970[13] many works have been devoted to this subject. See, for example Refs. 2, 8, 10, 15, 32, 37–39, 41.

The main result obtained by Sobolev[37] (see also Ref. 39). is an asymptotics of the form $\mathcal{U}_0|\log|\lambda||$ for the number of eigenvalues on the left of $\lambda, \lambda < 0$, where the coefficient $\mathcal{U}_0$ does not depend on the two-particle potentials v_α and is a positive function of the ratios $m_1/m_2, m_2/m_3$ of the masses of the three-particles.

Recently the existence of the Efimov effect for N-body quantum systems with $N \geq 4$ has been proved by X.P. Wang in Ref. 40.

In fact in Ref. 40 a lower bound on the number of eigenvalues of the total (reduced) Hamiltonian of the form

$$C_0|\log(E_0 - \lambda)|$$

is given, when λ tends to E_0, where C_0 is a positive constant and E_0 is the bottom of the essential spectrum.

The kinematics of the quantum systems describing three quasi-particles on lattices has peculiar features. For instance, due to the fact that the discrete analogue of the Laplacian (or its generalizations) is not rotationally invariant, the Hamiltonian of a system does not separate into two parts, one relating to the center-of-mass motion and the other one to the internal degrees of freedom. In particular, the Efimov effect exists only for the zero value of the three-particle quasi-momentum $K \in \mathbb{T}^3 = (-\pi, \pi]^3$ (see, e.g., Refs. 3, 5, 7, 20, 23, 24, 28) for relevant discussions and Refs. 11, 12, 19, 28, 29, 31, 34, 42, 44 for the general study of the low-lying excitation spectrum for quantum systems on lattices).

In statistical physics,[27,30] solid-state physics[31] and the theory of quantum fields[18] some important problems arise where the number of quasi-particles is bounded, but not fixed. In Ref. 36 geometric and commutator techniques have been developed in order to find the location of the spectrum and to prove absence of singular continuous spectrum for Hamiltonians without conservation of the particle number.

The study of systems describing n particles in interaction, without conservation of the number of particles is reduced to the investigation of the spectral properties of self-adjoint operators acting in the *cut subspace* $\mathcal{H}^{(n)}$ of the Fock space, consisting of $r \leq n$ particles.[18,30,31,36,43]

The perturbation problem of an operator (the Friedrichs model), with point and continuous spectrum (which acts in $\mathcal{H}^{(2)}$) has played a considerable role in the study of spectral problems connected with the quantum theory of fields.[18]

In the present paper we consider the perturbation problem with a particular attention to the two- and three-particle essential and point spectrum. Under some smoothness assumptions on the parameters of a family of Friedrichs models $h(p),\ p \in \mathbb{T}^3$, we obtain the following results:

(i) We describe the location of the essential spectrum of H via the spectrum of $h(p),\ p \in \mathbb{T}^3$.

(ii) We prove that the operator H has infinitely many eigenvalues below the bottom of the essential spectrum $\sigma_{\text{ess}}(H)$, if the operator $h(0)$ has a threshold energy resonance. Moreover, we establish the following asymptotic formula for the number $N(z)$ of eigenvalues of H lying below $z < m = \inf \sigma_{\text{ess}}(H)$

$$\lim_{z \to m-0} \frac{N(z)}{|\log|z-m||} = \mathcal{U}_0\,(0 < \mathcal{U}_0 < \infty).$$

(iii) We prove the finiteness of eigenvalues of H below the bottom of $\sigma_{\text{ess}}(H)$, if $h(0)$ has a threshold eigenvalue.

We remark that the presence of a zero energy resonance for the Schrödinger operators is due to the two-particle interaction operators V, in particular, the coupling constant (if V has in front of it a coupling constant) (see, e.g., Refs. 1, 22, 23, 41)

It is remarkable that for the Friedrichs model $h(0)$ the presence of a threshold energy resonance (consequently the existence of infinitely many eigenvalues of H) is due to the annihilation and creation operators acting in the *symmetric Fock space*.

We pointout that the assertion (iii) is surprising and similar assertions have not yet been proved for the three-particle Schrödinger operators on $\mathbb{Z}^3$.

We remark that the operator H has been considered before, but only the existence of infinitely many eigenvalues below the bottom of the essential spectrum of H has been announced in Ref. 25 and only the location of the essential spectrum of H has been described in terms of zeroes of the Friedholm determinant in Ref. 26, in the case where the functions u, v and w are analytic.

The organization of the present paper is as follows. Section 1 is an introduction to the whole work. In Sec. 2 the model operator is described as a bounded self-adjoint operator H in $\mathcal{H}^{(3)}$ and the main results of the present paper are formulated. Some spectral properties of $h(p)$, $p \in \mathbb{T}^3$, are studied in Sec. 3. In Sec. 4 the location and structure of the essential spectrum of H is studied. In Sec. 5 we prove the Birman-Schwinger principle for the operator H. In Sec. 6 the finiteness of the number of eigenvalues of the operator H is established. In Sec. 7 an asymptotic formula for the number of eigenvalues is proved. Some technical material is collected in Appendix A.

Throughout the present paper we adopt the following conventions: Denote by $\mathbb{T}^3$ the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides. The torus $\mathbb{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbb{R}^3$ modulo $(2\pi\mathbb{Z})^3$.

For each $\delta > 0$ the notation $U_\delta(0) = \{p \in \mathbb{T}^3 : |p| < \delta\}$ stands for a δ-neighborhood of the origin.

For any $n = 1, 2, \ldots$ let $\mathcal{B}(\theta, (\mathbb{T}^3)^n)$ with $1/2 < \theta \leq 1$, be the Banach spaces of Hölder continuous functions on $(\mathbb{T}^3)^n$ with exponent θ obtained by the closure of the space of smooth (periodic) functions f on $(\mathbb{T}^3)^n$ with respect to the norm

$$\|f\|_\theta = \sup_{\substack{t,\ell \in (\mathbb{T}^3)^n \\ \ell \neq 0}} [|f(t)| + |\ell|^{-\theta}|f(t+\ell) - f(t)|].$$

The set of functions $f : \mathbb{T}^3 \to \mathbb{R}$ having continuous partial derivatives up to order n inclusive will be denoted $C^{(n)}(\mathbb{T}^3)$. In particular $C^{(0)}(\mathbb{T}^3) = C(\mathbb{T}^3)$ by our convention that $f^{(0)}(x) = f(x)$.

2. THE MODEL OPERATOR AND STATEMENT OF RESULTS

Let us introduce some notations used in this work. Let $\mathbb{C} = \mathbb{C}^1$ be the field of complex numbers and let $L_2(\mathbb{T}^3)$ be the Hilbert space of square-integrable (complex) functions defined on $\mathbb{T}^3$ and $L_2^s((\mathbb{T}^3)^2)$ be the Hilbert space of square-integrable symmetric (complex) functions on $(\mathbb{T}^3)^2$.

Denote by $\mathcal{H}^{(3)}$ the direct sum of spaces $\mathcal{H}_0 = \mathbb{C}^1$, $\mathcal{H}_1 = L_2(\mathbb{T}^3)$ and $\mathcal{H}_2 = L_2^s((\mathbb{T}^3)^2)$, that is, $\mathcal{H}^{(3)} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$.

Let H be the operator in $\mathcal{H}^{(3)}$ with the entries $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i, i, j = 0, 1, 2,$:

$$(H_{00}f_0)_0 = u_0 f_0, \quad (H_{01}f_1)_0 = \int_{\mathbb{T}^3} v(q')f_1(q')dq', \quad H_{02} = 0,$$

$$H_{10} = H_{01}^*, \quad (H_{11}f_1)_1(p) = u(p)f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbb{T}^3} v(q')f_2(p,q')dq',$$

$$H_{20} = 0, \quad H_{21} = H_{12}^*, \quad (H_{22}f_2)_2(p,q) = w(p,q)f_2(p,q),$$

where $H_{ij}^* : \mathcal{H}_i \to \mathcal{H}_j, (j = i+1, i = 0, 1)$ denotes the adjoint operator to H_{ij} and $f_i \in \mathcal{H}_i, i = 0, 1, 2.$

Here u_0 is a real number, u is a real-valued essentially bounded function on $\mathbb{T}^3$, v is a real-valued function belonging to $L_2(\mathbb{T}^3)$ and w is a real-valued essentially bounded symmetric function on $(\mathbb{T}^3)^2$.

Under these assumptions the operator H is bounded and self-adjoint in $\mathcal{H}^{(3)}$.

We remark that the operators H_{10} and H_{21} resp. H_{01} and H_{12} defined in the Fock space are called creation resp. annihilation operators.

Throughout this paper we assume the following additional assumptions.

Assumption 2.1. *(i) The symmetric function w on $(\mathbb{T}^3)^2$ is even with respect to (p,q), and has a unique non-degenerate minimum at the point $(0,0) \in (\mathbb{T}^3)^2$ and all third order partial derivatives of w belong to $\mathcal{B}(\theta, (\mathbb{T}^3)^2)$.*
(ii) There exist positive definite matrix W and real numbers $l_1, l_2 (l_1 > 0, l_2 \neq 0)$ such that

$$\left(\frac{\partial^2 w(0,0)}{\partial p_i \partial p_j}\right)_{i,j=1}^3 = l_1 W, \quad \left(\frac{\partial^2 w(0,0)}{\partial p_i \partial q_j}\right)_{i,j=1}^3 = l_2 W.$$

Remark 2.2. *It is easy to check that Assumption 2.1 implies the inequality $l_1 > \mid l_2 \mid$.*

Assumption 2.3. *The function $u \in C^{(2)}(\mathbb{T}^3)$ is even on $\mathbb{T}^3$ and u has a unique non-degenerate minimum at the point $0 \in \mathbb{T}^3$. The function $v \in C^{(2)}(\mathbb{T}^3)$ is either even or odd on $\mathbb{T}^3$.*

Remark 2.4. *If the function v is equivalent to zero then the operator H will be direct sum of the operators $H_{ii}, i = 0, 1, 2,$ and hence in this case the spectrum of H is only the union of the spectra of H_{00}, H_{11} and H_{22}. Therefore throughout the present paper we assume that $v \neq 0$.*

Remark 2.5. *The function w resp. u is even and has a unique non-degenerate minimum on $\mathbb{T}^3$ and hence without loss of generality we assume that the function w resp. u has a unique minimum at the point $(0,0) \in (\mathbb{T}^3)^2$ resp. $0 \in \mathbb{T}^3$.*

Set

$$m = \min_{p,q \in \mathbb{T}^3} w(p,q), \qquad M = \max_{p,q \in \mathbb{T}^3} w(p,q)$$

and

$$\Lambda(p,z) = \int\limits_{\mathbb{T}^3} \frac{v^2(t)dt}{w(p,t) - z}, \; p \in \mathbb{T}^3, \; z \in \mathbb{C} \setminus [m(p), M(p)],$$

where the numbers $m(p)$ and $M(p)$ are defined by

$$m(p) = \min_{q \in \mathbb{T}^3} w(p,q) \quad \text{and} \quad M(p) = \max_{q \in \mathbb{T}^3} w(p,q).$$

For any $p \in \mathbb{T}^3$ the function $\Lambda(p, \cdot)$ is increasing in $(-\infty, m(p))$ and hence there exists a finite or infinite positive limit

$$\lim_{z \to m(p)-0} \Lambda(p, z) = \Lambda(p, m(p)).$$

For any $p \in U_\delta(0)$, $\delta > 0$-sufficiently small, the function $w_p(\cdot) = w(p, \cdot)$ has a unique non-degenerate minimum on $\mathbb{T}^3$ and by Lebesgue's dominated convergence theorem the following equality holds

$$\Lambda(p, m(p)) = \int_{\mathbb{T}^3} \frac{v^2(t)dt}{w_p(t) - m(p)}, \quad p \in U_\delta(0).$$

Assumption 2.6. *(i) The function $\Lambda(\cdot, m(\cdot))$ has a unique maximum at $p = 0 \in \mathbb{T}^3$. (ii) There exist positive numbers δ and c such that for all nonzero $p \in U_\delta(0)$ the following inequality holds*

$$\Lambda(0, m) - \Lambda(p, m) > cp^2.$$

We recall (see, e.g., Refs. 4, 35) that a complex-valued bounded function $\varepsilon : \mathbb{T}^d \to \mathbb{C}$, $d \geq 1$, is called conditionally negative definite if $\varepsilon(p) = \overline{\varepsilon(-p)}$ and

$$\sum_{i,j=1}^{n} \varepsilon(p_i - p_j) z_i \bar{z}_j \leq 0$$

for any $n \in \mathbb{N}$, for all $p_1, p_2, \ldots, p_n \in \mathbb{T}^d$ and all $\mathbf{z} = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n$ satisfying $\sum_{i=1}^{n} z_i = 0$.

Remark 2.7. *Let ε be a real-analytic conditionally negative definite function on $\mathbb{T}^3$ with a unique non-degenerate minimum at the origin and such that all third order partial derivatives of ε belong to $\mathcal{B}(\theta, \mathbb{T}^3)$. The function $v \in C^{(2)}(\mathbb{T}^3)$ (entering H_{01}, H_{12}) is either even or odd on $\mathbb{T}^3$ and the functions u (entering H_{11}) and w (entering H_{22}) satisfy*

$$u(p) = \varepsilon(p) + c, \; w(p, q) = \varepsilon(p) + \varepsilon(p + q) + \varepsilon(q) \tag{2.1}$$

for some real c. Then v, u and w satisfy Assumptions 2.1, 2.3 and 2.6 (see Lemma A.1).

To formulate the main results of the paper we introduce a family of Friedrichs model $h(p)$, $p \in \mathbb{T}^3$, which acts in $\mathcal{H}^{(2)} \equiv \mathcal{H}_0 \oplus \mathcal{H}_1$ with the entries

$$(h_{00}(p) f_0)_0 = u(p) f_0, \; h_{01} = \frac{1}{\sqrt{2}} H_{01}, \tag{2.2}$$

$$h_{10} = h_{01}^*, \; (h_{11}(p) f_1)_1(q) = w_p(q) f_1(q),$$

where $w_p(q) = w(p, q)$.

Let the operator $h_0(p),\ p \in \mathbb{T}^3$, acts in $\mathcal{H}^{(2)}$ as

$$h_0(p)\begin{pmatrix} f_0 \\ f_1(q) \end{pmatrix} = \begin{pmatrix} 0 \\ w_p(q)f_1(q) \end{pmatrix}.$$

The perturbation $h(p) - h_0(p)$ of the operator $h_0(p)$ is a self-adjoint operator of rank 2. Therefore in accordance with the invariance of the essential spectrum under finite rank perturbations the essential spectrum $\sigma_{\rm ess}(h(p))$ of $h(p)$ fills the following interval on the real axis:

$$\sigma_{\rm ess}(h(p)) = [m(p), M(p)].$$

Remark 2.8. *We remark that for some $p \in \mathbb{T}^3$ the essential spectrum of $h(p)$ may degenerate to the set consisting of the unique point $[m(p), m(p)]$ and hence we can not state that the essential spectrum of $h(p)$ is absolutely continuous for any $p \in \mathbb{T}^3$. For example, this is the case if the function w is of the form 2.1, where $p = (\pi, \pi, \pi) \in \mathbb{T}^3$ and*

$$\varepsilon(q) = 3 - \cos q_1 - \cos q_2 - \cos q_3, \quad q = (q_1, q_2, q_3) \in \mathbb{T}^3$$

The following theorem describes the essential spectrum of the operator H.

Theorem 2.9. *For the essential spectrum $\sigma_{\rm ess}(H)$ of the operator H the equality*

$$\sigma_{\rm ess}(H) = \cup_{p\in\mathbb{T}^3}\sigma_d(h(p)) \cup [m, M]$$

holds, where $\sigma_d(h(p))$ is the discrete spectrum of the operator $h(p)$, $p \in \mathbb{T}^3$.

For any $p \in \mathbb{T}^3$ we define an analytic function $\Delta(p, z)$ (the Fredholm determinant associated with the operator $h(p)$) in $\mathbb{C} \setminus [m(p), M(p)]$ by

$$\Delta(p, z) = u(p) - z - \frac{1}{2}\Lambda(p, z). \tag{2.3}$$

Let σ be the set of all complex numbers $z \in \mathbb{C} \setminus [m(p), M(p)]$ such that the equality $\Delta(p, z) = 0$ holds for some $p \in \mathbb{T}^3$.

Remark 2.10. *We remark that in Ref. 26 the essential spectrum of the operator H has been described by means of zeroes of the Fredholm determinant defined in (2.3) and by the spectrum $\sigma(H_{22})$ of the multiplication operator H_{22} as follows:*

$$\sigma_{\rm ess}(H) = \sigma \cup \sigma(H_{22}) \equiv \sigma \cup [m, M].$$

We point out that the equality

$$\sigma = \cup_{p\in\mathbb{T}^3}\sigma_d(h(p))$$

holds (see Lemma 4.2).

Definition 2.11. *The set* σ *resp.* $\sigma(H_{22}) \equiv [m, M]$ *is called two- resp. three-particle branch of the essential spectrum* $\sigma_{\text{ess}}(H)$ *of the operator* H, *which will be denoted by* $\sigma_{\text{two}}(H)$ *resp.* $\sigma_{\text{three}}(H)$.

Since $\Lambda(0, \cdot)$ is continuous in $z \le m$ the following finite limit exists

$$\Delta(0, m) = \lim_{z \to m-0} \Delta(0, z).$$

Definition 2.12. *Let part (i) of Assumption 2.1 be fulfilled,* $v \in \mathcal{B}(\theta, \mathbb{T}^3)$ *and* $u(0) \neq m$. *The compact operator* $h(0)$ *on* $C(\mathbb{T}^3)$ *is said to have a threshold energy resonance if the number* 1 *is an eigenvalue of the operator*

$$(\mathrm{G}\psi)(q) = \frac{v(q)}{2(u(0) - m)} \int_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{w_0(t) - m}, \quad \psi \in C(\mathbb{T}^3)$$

and the associated eigenfunction ψ *(up to a constant factor) satisfies the condition* $\psi(0) \neq 0$.

Remark 2.13. *Let part (i) of Assumption 2.1 be fulfilled and* $v \in \mathcal{B}(\theta, \mathbb{T}^3)$, $1/2 < \theta \le 1$. *(i) If* $u(0) \le m$, *then the equation* $h(0)f = mf$ *has only the trivial solution* $f \in \mathbb{C}^1 \oplus L_1(\mathbb{T}^3)$, *where* $L_1(\mathbb{T}^3)$ *is the Banach space of integrable functions. (ii) Assume that* $u(0) > m$ *and* $\Delta(0, m) = 0$. *a) If* $v(0) \neq 0$, *then the operator* $h(0)$ *has a threshold energy resonance and the vector* $f = (f_0, f_1)$, *where*

$$f_0 = const \neq 0, \ f_1(q) = -\frac{v(q) f_0}{\sqrt{2}(w_0(q) - m)} \in L_1(\mathbb{T}^3) \setminus L_2(\mathbb{T}^3), \tag{2.4}$$

obeys the equation $h(0)f = mf$ *(see Lemma 3.2). b) If* $v(0) = 0$, *then the operator* $h(0)$ *has a threshold eigenvalue and the vector* $f = (f_0, f_1)$, *where* $f_0 \in \mathbb{C}^1$ *and* $f_1 \in L_2(\mathbb{T}^3)$ *are defined by (2.4), obeys the equation* $h(0)f = mf$ *(see Lemma 3.3).*

Let us denote by $\tau_{\text{ess}}(H)$ the bottom of the essential spectrum $\sigma_{\text{ess}}(H)$ ($\tau_{\text{ess}}(H) \equiv \inf \sigma_{\text{ess}}(H)$) of the operator H and by $N(z)$ the number of eigenvalues of H lying below $z \le \tau_{\text{ess}}(H)$.

The main result of this paper is the following

Theorem 2.14. *Let Assumptions 2.1 and 2.3 be fulfilled. (i) Assume that the operator* $h(0)$ *has a threshold eigenvalue at the bottom* $z = \tau_{\text{ess}}(H)$ *and Assumption 2.6 is fulfilled. Then the operator* H *has a finite number of eigenvalues lying below* $\tau_{\text{ess}}(H)$. *(ii) Assume that the operator* $h(0)$ *has a threshold energy resonance and part (i) of Assumption 2.6 is fulfilled. Then the operator* H *has infinitely many eigenvalues lying below* $\tau_{\text{ess}}(H) = m$ *and accumulating at* $\tau_{\text{ess}}(H)$.

Moreover the function $N(\cdot)$ obeys the relation

$$\lim_{z \to m-0} \frac{N(z)}{|\log |z-m||} = \mathcal{U}_0 \, (0 < \mathcal{U}_0 < \infty). \tag{2.5}$$

Remark 2.15. *The constant $\mathcal{U}_0$ does not depend on the function v and is given as a positive function depending only on the ratio $\frac{l_1}{l_2}$ (with l_1, l_2 as in Assumption 2.1).*

Remark 2.16. *We remark that if the conditions of Theorem 2.14 are fulfilled, then $\tau_{\text{ess}}(H) = m$ (see Lemma A.3).*

Remark 2.17. *We remark that in Ref. 5 a result which is an analogue of part (ii) of Theorem 2.14, has been proven for the three-particle Schrödinger operators associated with a system of three-particles on lattices interacting via zero-range pair potentials.*

Remark 2.18. *Clearly, the infinite cardinality of the discrete spectrum of H lying on the l.h.s. of m follows automatically from the positivity of $\mathcal{U}_0$.*

3. SPECTRAL PROPERTIES OF THE OPERATORS $h(p)$, $p \in \mathbb{T}^3$

In this section we study some spectral properties of the family of Friedrichs models $h(p)$, $p \in \mathbb{T}^3$, given by (2.2), which plays a crucial role in the study of the spectral properties of H. We notice that the spectrum and resonances of the Friedrichs model have been studied in detail in Refs. 6, 14, 17, 21.

In particular, the following statement has been proven there.

Lemma 3.1. *For any $p \in \mathbb{T}^3$ the operator $h(p)$ has an eigenvalue $z \in \mathbb{C} \setminus [m(p), M(p)]$ if and only if $\Delta(p, z) = 0$.*

The following two lemmas establish in which cases the bottom of the essential spectrum is a threshold energy resonance or eigenvalue.

Lemma 3.2. *Let part (i) of Assumption 2.1 be fulfilled and $v \in \mathcal{B}(\theta, \mathbb{T}^3)$. The operator $h(0)$ has a threshold energy resonance if and only if $\Delta(0, m) = 0$ and $v(0) \neq 0$.*

Proof. "Only If Part." Suppose that the operator $h(0)$ has a threshold energy resonance. Then by Definition 2.12 the inequality $u(0) \neq m$ holds and the equation

$$\psi(q) = \frac{v(q)}{2(u(0)-m)} \int_{\mathbb{T}^3} \frac{v(t)\psi(t)dt}{w_0(t)-m}, \quad \psi \in C(\mathbb{T}^3) \tag{3.1}$$

has a nontrivial solution $\psi \in C(\mathbb{T}^3)$ which satisfies the condition $\psi(0) \neq 0$.

This solution is equal to the function v (up to a constant factor) and hence

$$\Delta(0,m) = u(0) - m - \frac{1}{2} \int_{\mathbb{T}^3} \frac{v^2(t)dt}{w_0(t)-m} = 0.$$

"If Part." Let the equality $\Delta(0,m) = 0$ hold and let $v(0) \neq 0$. Then the inequality $u(0) \neq m$ holds and the function $v \in C(\mathbb{T}^3)$ is a solution of the Eq. 3.1, that is, by Definition 2.12 the operator $h(0)$ has a threshold energy resonance. $\square$

Lemma 3.3. *Let part (i) of Assumption 2.1 be fulfilled and assume* $v \in \mathcal{B}(\theta, \mathbb{T}^3)$, $1/2 < \theta \leq 1$. *The operator* $h(0)$ *has a threshold eigenvalue if and only if* $\Delta(0,m) = 0$ *and* $v(0) = 0$.

Proof. "Only If Part." Suppose $f = (f_0, f_1)$ is an eigenvector of the operator $h(0)$ associated with the eigenvalue $z = m$. Then f_0 and f_1 satisfy the following system of equations

$$\begin{cases} (u(0)-m)f_0 + \frac{1}{\sqrt{2}} \int_{\mathbb{T}^3} v(q')f_1(q')dq' = 0 \\ \frac{1}{\sqrt{2}} v(q)f_0 + (w_0(q)-m)f_1(q) = 0. \end{cases} \tag{3.2}$$

From (3.2) we find that f_0 and f_1 are given by (2.4) and from the first equation of (3.2) we derive the equality $\Delta(0,m) = 0$.

Since $w_0(\cdot) \in C^{(3)}(\mathbb{T}^3)$ and $v(\cdot) \in \mathcal{B}(\theta, \mathbb{T}^3)$ and the function $w_0(\cdot)$ has a unique non-degenerate minimum at the origin we conclude that $f_1 \in L_2(\mathbb{T}^3)$ iff $v(0) = 0$.

"If Part." Let $v(0) = 0$ and $\Delta(0,m) = 0$. Then the vector $f = (f_0, f_1)$, where f_0 and f_1 are defined by (2.4), obeys the equation $h(0)f = mf$ and $f_1 \in L_2(\mathbb{T}^3)$. $\square$

Lemma 3.4. *Let part (i) of Assumption 2.1 and Assumptions 2.3, 2.6 be fulfilled. Let the operator* $h(0)$ *have a threshold eigenvalue. Then there exist numbers* $\delta > 0$ *and* $c > 0$ *such that*

$$|\Delta(p,m)| \geq cp^2 \quad \text{for any} \quad p \in U_\delta(0),$$

$$|\Delta(p,m)| \geq c \quad \text{for all} \quad p \in \mathbb{T}^3 \setminus U_\delta(0).$$

Proof. By Lemma 3.3 we have $\Delta(0, m) = 0$ and $v(0) = 0$. Then the function $\Delta(\cdot, m)$ can be represented in the form

$$\Delta(p, m) = u(p) - u(0) + \frac{1}{2}(\Lambda(0, m) - \Lambda(p, m)).$$

Using Assumptions 2.3 and 2.6 we complete then the proof of the lemma. $\square$

Since the function $w(\cdot, \cdot)$ has a unique non-degenerate minimum at the point $(0, 0) \in (\mathbb{T}^3)^2$ the following integral is finite

$$\int_{\mathbb{T}^3} \frac{v^2(t)\, dt}{w_p(t) - m}.$$

Lebesgue's dominated convergence theorem yields the equality

$$\Delta(0, m) = \lim_{p \to 0} \Delta(p, m)$$

and hence the function $\Delta(\cdot, m)$ is continuous on $\mathbb{T}^3$.

Set

$$\mathbb{C}_+ = \{z \in \mathbb{C} : Re z > 0\}, \quad \mathbb{R}_+ = \{x \in \mathbb{R} : x > 0\}, \quad \mathbb{R}_+^0 = \mathbb{R}_+ \cup \{0\}.$$

Let $w_0(\cdot, \cdot)$ be the function defined on $U_\delta(0) \times \mathbb{T}^3$, $\delta > 0$ sufficiently small, as

$$w_0(p, q) = w_p(q + q_0(p)) - m(p),$$

where $q_0(\cdot) \in C^{(3)}(U_\delta(0))$ and for any $p \in U_\delta(0)$ the point $q_0(p)$ is the non-degenerate minimum of the function $w_p(\cdot)$ (see Lemma A.2). Here $C^{(n)}(U_\delta(0))$ can be defined similarly to $C^{(n)}(\mathbb{T}^3)$.

For any $p \in U_\delta(0)$ we define an analytic function $D(p, \zeta)$ in $\mathbb{C}_+$ by

$$D(p, \zeta) = u(p) - m(p) + \zeta^2 - \frac{1}{2} \int_{\mathbb{T}^3} \frac{v^2(q + q_0(p))\, dq}{w_0(p, q) + \zeta^2}.$$

The following decomposition plays an important role in the proof of the main result, that is, the asymptotics (2.5).

Lemma 3.5. *Let Assumptions 2.1 and 2.3 be fulfilled. Then there exists a number $\delta > 0$ such that*

i) For any $\zeta \in \mathbb{C}_+$ the function $D(\cdot, \zeta)$ is of class $C^{(2)}(U_\delta(0))$ and the following decomposition

$$D(p, \zeta) = D(0, \zeta) + D^{res}(p, \zeta),$$

holds, where $D^{res}(p, \zeta) = O(p^2)$ as $p \to 0$ uniformly in $\zeta \in \mathbb{R}_+^0$.

ii) The right-hand derivative of $D(0,\cdot)$ at $\zeta = 0$ exists and the following decomposition

$$D(0,\zeta) = D(0,0) + 2\sqrt{2}\pi^2 v^2(0) l_1^{-\frac{3}{2}} (det W)^{-\frac{1}{2}} \zeta + D^{res}(\zeta), \ \ \zeta \in \mathbb{R}_+^0,$$

holds, where $D^{res}(\zeta) = O(\zeta^{1+\theta})$ as $\zeta \to 0$.

Remark 3.6. *An analogue of Lemma 3.5 has been proven in Ref. 5 in the case where the functions $u(\cdot)$, $v(\cdot)$ and $w(\cdot,\cdot)$ are analytic on $\mathbb{T}^3$ and $(\mathbb{T}^3)^2$, respectively.*

Proof. *i)* Since $m(\cdot) \in C^{(3)}(U_\delta(0))$ by definition of the function $D(\cdot,\cdot)$ and Assumptions 2.1 and 2.3 we obtain that the function $D(\cdot,\zeta)$ is of class $C^{(2)}(U_\delta(0))$ for any $\zeta \in \mathbb{C}_+$.

Using

$$w_0(p,q) = \frac{l_1}{2}(Wq,q) + o(|p||q|^2) + o(|q|^2) \ as \ |p|,|q| \to 0$$

we obtain that there exists $C > 0$ such that for any $\zeta \in \mathbb{R}_+^0$ and $i,j = 1,2,3$ the inequalities

$$\left| \frac{\partial^2}{\partial p_i \partial p_j} \frac{v^2(q+q_0(p))}{w_0(p,q)+\zeta^2} \right| \le \frac{C}{q^2}, \ \ p,q \in U_\delta(0) \tag{3.3}$$

and

$$\left| \frac{\partial^2}{\partial p_i \partial p_j} \frac{v^2(q+q_0(p))}{w_0(p,q)+\zeta^2} \right| \le C, \ \ p \in U_\delta(0), q \in \mathbb{T}^3 \setminus U_\delta(0) \tag{3.4}$$

hold.

Lebesgue's dominated convergence theorem implies that

$$\frac{\partial^2}{\partial p_i \partial p_j} D(p,0) = \lim_{\zeta \to 0+} \frac{\partial^2}{\partial p_i \partial p_j} D(p,\zeta).$$

Repeated application of the Hadamard lemma (see Ref. 45, v. 1, p. 512) enables us to write

$$D(p,\zeta) = D(0,\zeta) + \sum_{i=1}^{3} \frac{\partial}{\partial p_i} D(0,\zeta) p_i + \sum_{i,j=1}^{3} H_{ij}(p,\zeta) p_i p_j,$$

where for any $\zeta \in \mathbb{R}_+^0$ the functions $H_{ij}(\cdot,\zeta)$, $i,j = 1,2,3,$ are continuous in $U_\delta(0)$ and

$$H_{ij}(p,\zeta) = \frac{1}{2} \int_0^1 \int_0^1 \frac{\partial^2}{\partial p_i \partial p_j} D(x_1 x_2 p, \zeta) dx_1 dx_2.$$

The estimates (3.3) and (3.4) give

$$|H_{i,j}(p,\zeta)| \leq \frac{1}{2}\int_0^1\int_0^1 \left|\frac{\partial^2}{\partial p_i \partial p_j} D(x_1x_2p,\zeta)\right| dx_1dx_2$$
$$\leq C\left(1+\int\limits_{U_\delta(0)} \frac{v^2(q+q_0(p))dq}{q^2}\right)$$

for any $p \in U_\delta(0)$ uniformly in $\zeta \in \mathbb{R}^0_+$.

Since for any $\zeta \in \mathbb{C}_+$ the function $D(\cdot,\zeta)$ is even in $U_\delta(0)$ we have

$$\left[\frac{\partial}{\partial p_i} D(p,\zeta)\right]_{p=0} = 0, \quad i = 1,2,3.$$

ii) Now we prove that the right-hand derivative of $D(0,\cdot)$ at $\zeta = 0$ exists and the following inequalities

$$|D(0,\zeta) - D(0,0)| \leq C\zeta, \quad \zeta \in \mathbb{R}^0_+, \tag{3.5}$$

$$\left|\frac{\partial}{\partial\zeta} D(0,\zeta) - \frac{\partial}{\partial\zeta} D(0,0)\right| < C\zeta^\theta, \quad \zeta \in \mathbb{R}^0_+ \tag{3.6}$$

hold for some positive C.

Indeed, the function $D(0,\cdot)$ can be represented as

$$D(0,\zeta) = D_1(\zeta) + D_2(\zeta)$$

with

$$D_1(\zeta) = -\frac{1}{2}\int\limits_{U_\delta(0)} \frac{v^2(q)}{w_0(0,q)+\zeta^2} dq, \ \zeta \in \mathbb{C}_+$$

and

$$D_2(\zeta) = u(0) - m + \zeta^2 - \frac{1}{2}\int\limits_{\mathbb{T}^3\setminus U_\delta(0)} \frac{v^2(q)}{w_0(0,q)+\zeta^2} dq, \ \zeta \in \mathbb{C}_+.$$

Since the function $w_0(0,\cdot)$ is continuous on the compact set $\mathbb{T}^3 \setminus U_\delta(0)$ and has a unique minimum at $q = 0$ there exists $M_1 > 0$ such that $|w_0(0,q)| > M_1$ for all $q \in \mathbb{T}^3 \setminus U_\delta(0)$.

Then by $v(\cdot) \in \mathcal{B}(\theta,\mathbb{T}^3)$ we have

$$|D_2(\zeta) - D_2(0)| \leq C\zeta^2, \ \zeta \in \mathbb{R}^0_+ \tag{3.7}$$

for some $C = C(\delta) > 0$.

Applying the Morse lemma and computing some integrals we obtain that (see Lemma A.4 there exists a right-hand derivative of $D_1(\cdot)$ at $\zeta = 0$ and

$$\frac{\partial}{\partial\zeta}D_1(0) = \lim_{\zeta\to 0+}\frac{D_1(\zeta) - D_1(0)}{\zeta} = 2\sqrt{2}\pi^2 l_1^{-\frac{3}{2}} v^2(0)(det W)^{-\frac{1}{2}}$$

and hence

$$\left|D_1(\zeta) - D_1(0)\right| < C\zeta, \quad \zeta \in \mathbb{R}^0_+ \tag{3.8}$$

holds for some positive C.

Then from (3.7) and (3.8) it follows that the right-hand derivative of $D(0, \cdot)$ at $\zeta = 0$ exists and

$$\frac{\partial}{\partial\zeta}D(0, 0) = 2\sqrt{2}\pi^2 l_1^{-\frac{3}{2}} v^2(0)(det W)^{-\frac{1}{2}}.$$

Comparing (3.7) and (3.8) we obtain (3.5).

In the same way one can prove the inequality (3.6). □

Corollary 3.7. *Let the operator $h(0)$ have an m energy resonance. Then for any $p \in U_\delta(0)$, $\delta > 0$ sufficiently small, and $z \le m(p)$ the following decomposition*

$$\Delta(p, z) = 2\sqrt{2}\pi^2 v^2(0) l_1^{-\frac{3}{2}}(det W)^{-\frac{1}{2}}\sqrt{m(p) - z} + \Delta^{(02)}(m(p) - z) + \Delta^{(20)}(p, z)$$

holds, where $\Delta^{(02)}(m(p) - z)$ (resp. $\Delta^{(20)}(p, z)$) is a function behaving like $O((m(p) - z)^{\frac{1+\theta}{2}})$ (resp. $O(|p|^2)$) as $|m(p) - z| \to 0$ (resp. $p \to 0$ uniformly in $z \le m(p)$).

Proof. By Lemma 3.2 we have that $\Delta(0, m) = 0$ and $v(0) \neq 0$ and hence the proof of Corollary 3.7 immediately follows from Lemma 3.5 and the equality $\Delta(p, z) = D(p, \sqrt{m(p) - z})$, $z \le m(p)$. □

Lemma 3.8. *Let the operator $h(0)$ have an m energy resonance. Then there exist positive numbers c, C and δ such that*

$$c|p| \le \Delta(p, m) \le C|p| \quad \textit{for any} \quad p \in U_\delta(0) \tag{3.9}$$

and

$$\Delta(p, m) \ge c \quad \textit{for any} \quad p \in \mathbb{T}^3 \setminus U_\delta(0). \tag{3.10}$$

Proof. Corollary 3.7 and the asymptotics (see part (ii) of Lemma A.2)

$$m(p) = m + \frac{l_1^2 - l_2^2}{2l_1}(Wp, p) + o(p^3) \quad \text{as} \quad p \to 0 \tag{3.11}$$

yields (3.9) for some positive numbers c, C.

The inequality (3.10) follows from the positivity (see proof of Lemma A.3) and continuity of the function $\Delta(\cdot, m)$ on the compact set $\mathbb{T}^3 \setminus U_\delta(0)$. $\square$

4. THE ESSENTIAL SPECTRUM OF THE OPERATOR H

4.1. The Spectrum of the Operator $\hat{H}$

We consider the operator $\hat{H}$ acting in $\hat{\mathcal{H}} = L_2(\mathbb{T}^3) \oplus L_2((\mathbb{T}^3)^2)$ as

$$\hat{H}\begin{pmatrix} f_1(p) \\ f_2(p,q) \end{pmatrix} = \begin{pmatrix} u(p)f_1(p) + \frac{1}{\sqrt{2}}\int\limits_{\mathbb{T}^3} v(q')f_2(p,q')\,dq' \\ \frac{1}{\sqrt{2}}v(q)f_1(p) + w_p(q)f_2(p,q) \end{pmatrix}.$$

The operator $\hat{H}$ commutes with any multiplication operator U_Υ by the bounded function Υ on $\mathbb{T}^3$

$$U_\Upsilon\begin{pmatrix} f_1(p) \\ f_2(p,q) \end{pmatrix} = \Upsilon(p)\begin{pmatrix} f_1(p) \\ f_2(p,q) \end{pmatrix}, \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \hat{\mathcal{H}}.$$

Therefore the decomposition of the space $\hat{\mathcal{H}}$ into the direct integral

$$\hat{\mathcal{H}} = \int\limits_{\mathbb{T}^3} \oplus \mathcal{H}^{(2)} dp$$

yield the decomposition into the direct integral

$$\hat{H} = \int\limits_{\mathbb{T}^3} \oplus h(p)\,dp, \tag{4.1}$$

where the fiber operators $h(p)$, $p \in \mathbb{T}^3$, are defined by (2.2).

Lemma 4.1. *For the spectrum $\sigma(\hat{H})$ of $\hat{H}$ the equality*

$$\sigma(\hat{H}) \equiv \cup_{p\in\mathbb{T}^3}\sigma_d(h(p)) \cup [m, M]$$

holds.

Proof. The assertion of the lemma follows from the representation (4.1) of the operator $\hat{H}$ and the theorem on decomposable operators (see Ref. 35). $\square$

Lemma 4.2. *The essential spectrum $\sigma_{\mathrm{ess}}(H)$ of the operator H coincides with the spectrum of $\hat{H}$, that is,*

$$\sigma_{\mathrm{ess}}(H) = \sigma(\hat{H}). \tag{4.2}$$

Proof. Let σ be the set of complex numbers $z \in \mathbb{C}$ such that the equality $\Delta(p, z) = 0$ holds for some $p \in \mathbb{T}^3$. In Ref. 26 it has been proved that the essential spectrum $\sigma_{\text{ess}}(H)$ of the operator H coincides with $\sigma \cup [m, M]$. By Lemma 3.1 we have that

$$\sigma = \cup_{p \in \mathbb{T}^3} \sigma_d(h(p))$$

and hence by Lemma 4.1 we obtain (4.2). $\square$

5. THE BIRMAN-SCHWINGER PRINCIPLE

For a bounded self-adjoint operator A, we define $n(\lambda, A)$ as

$$n(\lambda, A) = \sup\{\dim F : (Au, u) > \lambda,\ u \in F,\ ||u|| = 1\}.$$

The number $n(\lambda, A)$ is equal to the infinity if λ is in the essential spectrum of A and if $n(\lambda, A)$ is finite, it is equal to the number of the eigenvalues of A bigger than λ. By the definition of $N(z)$ we have

$$N(z) = n(-z, -H),\ -z > -\tau_{\text{ess}}(H).$$

Since the function $\Delta(\cdot, \cdot)$ is positive on $\mathbb{T}^3 \times (-\infty, \tau_{\text{ess}}(H))$ the positive square root of $\Delta(p, z)$ exists for any $p \in \mathbb{T}^3$ and $z < \tau_{\text{ess}}(H)$.

In our analysis of the spectrum of H the crucial role is played by the compact operator $T(z)$, $z < \tau_{\text{ess}}(H)$ in the space $\mathcal{H}^{(2)}$ with the entries

$$(T_{00}(z) f_0)_0 = (1 - u_0 - z) f_0, \quad (T_{01}(z) f_1)_0 = -\int_{\mathbb{T}^3} \frac{v(q') f(q') dq'}{\sqrt{\Delta(q', z)}},$$

$$T_{10}(z) = T_{01}^*(z), \quad (T_{11}(z) f_1)_1(p) = \frac{v(p)}{2\sqrt{\Delta(p, z)}} \int_{\mathbb{T}^3} \frac{v(q') f(q') dq'}{\sqrt{\Delta(q', z)}(w(p, q') - z)}.$$

The following lemma is a realization of the well known Birman-Schwinger principle for the operator H (see Ref. 5, 37, 39).

Lemma 5.1. *For $z < \tau_{\text{ess}}(H)$ the operator $T(z)$ is compact and continuous in z and*

$$N(z) = n(1, T(z)). \tag{5.1}$$

Proof. The operator H can be decomposed as

$$H = \begin{pmatrix} H_{00} & 0 & 0 \\ 0 & H_{11} & 0 \\ 0 & 0 & H_{22} \end{pmatrix} + \begin{pmatrix} 0 & H_{01} & 0 \\ H_{10} & 0 & H_{12} \\ 0 & H_{21} & 0 \end{pmatrix}.$$

Denote by $I_i,\ i=0,1,2$, the identity operator on the Hilbert space $\mathcal{H}_i,\ i=0,1,2$, and by $\mathrm{I}=diag\,\{I_0,I_1\}$ resp. $\mathfrak{I}=diag\,\{I_0,I_1,I_2\}$ the identity operator on $\mathcal{H}^{(2)}$ resp. $\mathcal{H}^{(3)}$.

For any $z<\tau_{\rm ess}(H)$ the operator $H_{ii}-zI_i, i=1,2$, is positive and invertible and hence the square root $R_i^{\frac{1}{2}}(z)$ of the resolvent $R_i(z)=(H_{ii}-zI_i)^{-1}$ of $H_{ii},\ i=1,2$, exists.

Let $M(z), z<\tau_{\rm ess}(H)$ be the operator with entries

$$M_{00}(z)=(1+z)I_0-H_{00},\ \ M_{01}(z)=-H_{01}R_1^{\frac{1}{2}}(z),$$

$$M_{12}(z)=-R_1^{\frac{1}{2}}(z)H_{12}R_2^{\frac{1}{2}}(z),\quad M_{10}(z)=M_{01}^*(z),\quad M_{21}(z)=M_{12}^*(z),$$

otherwise

$$M_{\alpha\beta}(z)=0,\alpha,\beta=0,1,2.$$

One has $((H-z\mathfrak{I})f,f)<0,\ f\in\mathcal{H}$ if and only if $((M(z)-\mathfrak{I})g,g)>0,\ \ g=(g_0,g_1,g_2)$, where $g_0=f_0,\ g_i=(H_{ii}-zI_i)^{\frac{1}{2}}f_i, i=1,2$.

It follows that

$$N(z)=n(1,M(z)). \tag{5.2}$$

Let $V(z), z<\tau_{\rm ess}(H)$ be the operator in $\mathcal{H}^{(2)}$ with the entries

$$V_{11}(z)=M_{12}(z)M_{21}(z),\quad \text{otherwise}\quad V_{\alpha\beta}(z)=M_{\alpha\beta}(z),\ \alpha,\beta=0,1.$$

Denote by $F=F_0\oplus F_1\subset\mathcal{H}_0\oplus\mathcal{H}_1\equiv\mathcal{H}^{(2)}$ a subspace for which the equality

$$\dim\ F=n(1,V(z))$$

holds. Then

$$(M(z)g,g)=(V(z)f,f)\quad\text{for all}\quad f=(f_0,f_1)\in F$$
$$\text{and}\quad g=(f_0,f_1,M_{21}(z)f_1).$$

Moreover

$$((M(z)-\mathfrak{I})g,g)=((V(z)-\mathrm{I})f,f)-||f_2^{\perp}||$$

for all $f=(f_0,f_1)\in F$ and $g=(f_0,f_1,M_{21}(z)f_1+f_2^{\perp}),\ f_2^{\perp}\perp M_{21}(z)f_1$.

Therefore

$$n(1,M(z))=n(1,V(z)). \tag{5.3}$$

One has $((V(z)-\mathrm{I})\varphi,\varphi)>0,\ \ \varphi=(\varphi_0,\varphi_1)\in\mathcal{H}^{(2)}$ if and only if the inequality

$$\begin{aligned}(\psi_0,\psi_0)_0+((H_{11}-zI_1)\psi_1,\psi_1)_1&<(M_{00}(z)\psi_0,\psi_0)_0\\ &-(H_{01}\psi_1,\psi_0)_0-(H_{10}(z)\psi_0,\psi_1)_1+(H_{12}R_2(z)H_{21}\psi_1,\psi_1)_1\end{aligned} \tag{5.4}$$

holds for $\psi_0 = \varphi_0,\ \psi_1 = R_1^{\frac{1}{2}}(z)\varphi_1$. This means that

$$n(1, V(z)) = n(-z, G(z)), \tag{5.5}$$

where

$$G(z) = \begin{pmatrix} -H_{00} & -H_{01} \\ -H_{10} & H_{12}R_2(z)H_{21} - H_{11} \end{pmatrix}.$$

Now we represent the operator H_{21} as a sum of two operators $H_{21}^{(1)}$ and $H_{21}^{(2)}$ acting from $L_2(\mathbb{T}^3)$ to $L_2((\mathbb{T}^3)^2)$ as

$$\big(H_{21}^{(1)} f_1\big)(p,q) = \frac{1}{2}v(p)f_1(q),\ \big(H_{21}^{(2)} f_1\big)(p,q) = \frac{1}{2}v(q)f_1(p).$$

The operator $D(z) = H_{11} - z - H_{12}R_2(z)H_{21}^{(2)}$, $z < \tau_{\text{ess}}(H)$ is the multiplication operator by the positive function $\Delta(\cdot, z)$ defined on $\mathbb{T}^3$ by (2.3) and hence it is invertible. It is clear, that the positive square root $D^{-\frac{1}{2}}(z)$ of $D^{-1}(z)$ is the multiplication operator by the function $\Delta^{-\frac{1}{2}}(\cdot, z)$.

Thus we can conclude that $(G(z)\varphi, \varphi) > -z(\varphi, \varphi)$ holds if and only if $(T(z)\eta, \eta) > (\eta, \eta)$ holds for $\eta_0 = \varphi_0,\ \eta_1 = D^{-\frac{1}{2}}(z)\varphi_1$ and hence

$$n(-z, G(z)) = n(1, T(z)). \tag{5.6}$$

The equalities (5.2), (5.3), (5.5) and (5.6) give (5.1).

Finally we note that the operator $T(z)$, $z < \tau_{\text{ess}}(H)$ is compact and continuous in z. $\square$

6. THE FINITENESS OF THE NUMBER OF EIGENVALUES OF THE OPERATOR H

We starts the proof of the finiteness of the number of eigenvalues lying below $\tau_{\text{ess}}(H)$ (part (i) of Theorem 2.14) with the following two lemmas.

Lemma 6.1. *Let Assumption 2.1 be fulfilled. Then there exist numbers $C_1, C_2, C_3 > 0$ and $\delta > 0$ such that the following inequalities hold*

(i) $C_1(|p|^2 + |q|^2) \le w(p,q) - m \le C_2(|p|^2 + |q|^2)$ *for all* $p, q \in U_\delta(0)$,
(ii) $w(p,q) - m \ge C_3$ *for all* $(p,q) \notin U_\delta(0) \times U_\delta(0)$.

Proof. By Assumption 2.1 the point $(0,0) \in (\mathbb{T}^3)^2$ is the unique non-degenerated minimum point of the function $w(\cdot,\cdot)$. Then by (7.1) there exist positive numbers C_1, C_2, C_3 and a $\delta-$neighborhood of $p = 0 \in \mathbb{T}^3$ so that (i) and (ii) hold true. $\square$

Lemma 6.2. *Let the conditions in part (i) of Theorem 2.14 be fulfilled. Then for any $z \le m$ the operator $T(z)$ is compact and continuous from the left up to $z = m$.*

Proof. Denote by $Q(p, q; z)$ the kernel of the operator $T_{11}(z)$, $z < m$, that is,

$$Q(p, q; z) = \frac{v(p)v(q)}{2\sqrt{\Delta(p, z)}(w(p, q) - z)\sqrt{\Delta(q, z)}}.$$

Since the function $v \in C^{(2)}(\mathbb{T}^3)$ is even and $v(0) = 0$ we have $|v(p)| \le C|p|$ for some $C > 0$ and for any $p \in \mathbb{T}^3$. By virtue of Lemmas 3.4 and 6.1 the kernel $Q(p, q; z)$ is estimated by

$$C\left(\frac{\chi_\delta(p)}{|p|} + 1\right)\left(\frac{|p||q|\chi_\delta(p)\chi_\delta(q)}{p^2 + q^2} + 1\right)\left(\frac{\chi_\delta(q)}{|q|} + 1\right),$$

where $\chi_\delta(p)$ is the characteristic function of $U_\delta(0)$.

The latter function is square-integrable on $(\mathbb{T}^3)^2$ and hence for any $z \le m$ the operator $T_{11}(z)$ is Hilbert-Schmidt.

The kernel function of $T_{11}(z)$, $z < m$ is continuous in $p, q \in \mathbb{T}^3$. Therefore the continuity of the operator $T_{11}(z)$ from the left up to $z = m$ follows from Lebesgue's dominated convergence theorem.

Since for all $z \le m$ the operators $T_{00}(z)$, $T_{01}(z)$ and $T_{10}(z)$ are of rank 1 and continuous from the left up to $z = m$ one concludes that $T(z)$ is compact and continuous from the left up to $z = m$. □

We are now ready for the

Proof of (i) of Theorem 2.14. Let the conditions in part (i) of Theorem 2.14 be fulfilled. By Lemma 5.1 we have

$$N(z) = n(1, T(z)), \text{ as } z < m$$

and by Lemma 6.2 for any $\gamma \in [0, 1)$ the number $n(1 - \gamma, T(m))$ is finite. Then we have

$$n(1, T(z)) \le n(1 - \gamma, T(m)) + n(\gamma, T(z) - T(m))$$

for all $z < m$ and $\gamma \in (0, 1)$. This relation can easily be obtained by using of Weyl's inequality

$$n(\lambda_1 + \lambda_2, A_1 + A_2) \le n(\lambda_1, A_1) + n(\lambda_2, A_2)$$

for the sum of compact operators A_1 and A_2 and for any positive numbers λ_1 and λ_2.

Since $T(z)$ is continuous from the left up to $z = m$, we obtain

$$\lim_{z \to m-0} N(z) = N(m) \le n(1 - \gamma, T(m)) \text{ for all } \gamma \in (0, 1).$$

Thus

$$N(m) \leq n(1-\gamma, T(m)) < \infty. \tag{6.1}$$

The inequality (6.1) proves the assertion (i) of Theorem 2.14.

7. ASYMPTOTICS FOR THE NUMBER OF EIGENVALUES OF THE OPERATOR *H*

In this section we shall derive the asymptotics (2.5) for the number of eigenvalues of H, that is, we prove part (ii) of Theorem 2.14.

By Assumption 2.1 we get

$$\begin{aligned} w(p,q) = m &+ \frac{1}{2}(l_1(Wp,p) + 2l_2(Wp,q) + l_1(Wq,q)) \\ &+ O(|p|^{3+\theta} + |q|^{3+\theta}) \text{ as } p,q \to 0. \end{aligned} \tag{7.1}$$

By the representation (3.11) and Corollary 3.7 we get

$$\Delta(p,z) = \frac{4\pi^2 v^2(0)}{l_1^{3/2}(\det W)^{\frac{1}{2}}}\,[l(Wp,p) - 2(z-m)]^{\frac{1}{2}} + O((|p|^2 + |z-m|)^{\frac{1+\theta}{2}}) \tag{7.2}$$

as $p \to 0$, $|z-m| \to 0$, where $l = (l_1^2 - l_2^2)/l_1$.

Denote by $\hat{\chi}_\delta(\cdot)$ the characteristic function of $\hat{U}_\delta(0) = \{p \in \mathbb{T}^3 : |W^{\frac{1}{2}}p| < \delta\}$.

Let $T(\delta; |z-m|)$ be the operator in $\mathcal{H}^{(2)}$ defined by

$$T(\delta; |z-m|) = \begin{pmatrix} 0 & 0 \\ 0 & T_{11}(\delta; |z-m|) \end{pmatrix},$$

where $T_{11}(\delta; |z-m|)$ is the integral operator in $\mathcal{H}_1$ with the kernel

$$\frac{l_1^{\frac{3}{2}}(\det W)^{\frac{1}{2}}\,\hat{\chi}_\delta(p)\hat{\chi}_\delta(q)(l(Wq,q) + 2|z-m|)^{-1/4}}{2\pi^2(l(Wp,p) + 2|z-m|)^{1/4}(l_1(Wp,p) + 2l_2(Wp,q) + l_1(Wq,q) + 2|z-m|)}.$$

Lemma 7.1. *Let the conditions in part (ii) of Theorem 2.14 be fulfilled. Then the operator $T(\delta; |z-m|)$ resp. $T(z) - T(\delta; |z-m|)$ is compact and continuous in $z < m$ resp. in $z \leq m$.*

Proof. The kernel of $T(\delta; |z-m|)$, $z < m$ is square-integrable and continuous in $p, q \in \mathbb{T}^3$ and hence the operator $T(\delta; |z-m|)$ is compact and continuous in $z < m$.

Applying the asymptotics (7.1), (7.2) and Lemmas 3.8 and 6.1 one can estimate the kernel of the operator $T_{11}(z) - T_{11}(\delta; |z-m|)$ by

$$C\left(\frac{|p|^{1+\theta} + |q|^{1+\theta}}{|p|^{\frac{1}{2}}(p^2+q^2)|q|^{\frac{1}{2}}} + \frac{|m-z|^{\frac{\theta}{2}}(p^2+q^2)^{-1}}{(|p|^2+|m-z|)^{\frac{1}{4}}(|q|^2+|m-z|)^{\frac{1}{4}}} + 1\right)$$

and hence the operator $T_{11}(z) - T_{11}(\delta; |z-m|)$ belongs to the Hilbert-Schmidt class for all $z \le m$. In combination with the continuity of the kernel of the operator in $z < m$ this gives the continuity of $T_{11}(z) - T_{11}(\delta; |z-m|)$ in $z \le m$.

It is easy to see that $T_{00}(z)$, $T_{01}(z)$ and $T_{10}(z)$ are rank 1 operators and they are continuous from the left up to $z = m$. Consequently $T(z) - T(\delta; |z-m|)$ is compact and continuous in $z \le m$. $\square$

Let

$$\mathbf{S_r} \ : \ L_2((0,\mathbf{r}),\sigma_0) \to L_2((0,\mathbf{r}),\sigma_0), \ \mathbf{r} > 0,$$
$$\sigma_0 = L_2(\mathbb{S}^2), \ \mathbb{S}^2 \ \text{being the unit sphere in} \, \mathbb{R}^3,$$

be the integral operator with the kernel

$$S(t;y) = (2\pi)^{-2}\frac{l_{12}}{\cosh y + s_{12}t}, \tag{7.3}$$
$$y = x - x', \ x, x' \in (0,\mathbf{r}), \quad t =< \xi, \eta >, \ \xi, \eta \in \mathbb{S}^2,$$
$$l_{12} = \left(l_1^2/\left(l_1^2 - l_2^2\right)\right)^{\frac{1}{2}}, \ s_{12} = l_2/l_1,$$

and let

$$\hat{\mathbf{S}}(\lambda) : \ \sigma_0 \to \sigma_0, \ \ \lambda \in (-\infty, +\infty),$$

be the integral operator with the kernel

$$\hat{S}_\lambda(t) = \int_{-\infty}^{+\infty} \exp\{-i\lambda r\} S(t;r)\, dr = (2\pi)^{-1} l_{12} \frac{\sinh[\lambda(arc\cos s_{12}t)]}{\left(1 - s_{12}^2 t^2\right)^{\frac{1}{2}} \sinh(\pi\lambda)}.$$

For $\mu > 0$, define

$$U(\mu) = (4\pi)^{-1} \int_{-\infty}^{+\infty} n(\mu, \hat{\mathbf{S}}(y)) dy.$$

Lemma 7.2. *The function $U(\mu)$ is continuous in $\mu > 0$, the following limit*

$$\lim_{\mathbf{r}\to\infty} \frac{1}{2}\mathbf{r}^{-1} n(\mu, \mathbf{S_r}) = U(\mu)$$

exists and $U(1) > 0$.

Remark 7.3. *This lemma can be proven quite similarly to the corresponding results of Ref. 37. In particular, the continuity of $U(\mu)$ in $\mu > 0$ is a result of Lemma 3.2, Theorem 4.5 states the existence of the limit*

$$\lim_{\mathbf{r}\to\infty} \frac{1}{2}\mathbf{r}^{-1} n(\mu, \mathbf{S}_{\mathbf{r}}) = U(\mu)$$

and the inequality $U(1) > 0$ follows from Lemma 3.2.

Part (ii) of Theorem 2.14 will be deduced by a perturbation argument based on Lemma 4.7, which has been proven in Ref. 37. For completenees, we here reproduce the lemma.

Lemma 7.4. *Let $A(z) = A_0(z) + A_1(z)$, where $A_0(z)$ (resp.$A_1(z)$) is compact and continuous in $z < m$ (resp.$z \leq m$). Assume that for some function $f(\cdot)$, $f(z) \to 0$, $z \to m - 0$ one has*

$$\lim_{z\to m-0} f(z) n(\lambda, A_0(z)) = l(\lambda),$$

and $l(\lambda)$ is continuous in $\lambda > 0$. Then the same limit exists for $A(z)$ and

$$\lim_{z\to m-0} f(z) n(\lambda, A(z)) = l(\lambda).$$

Remark 7.5. *Since $\mathcal{U}(\cdot)$ is continuous in $\mu > 0$, according to Lemma 7.4 any perturbations of the operator $A_0(z)$ defined in Lemma 7.4, which is compact and continuous up to $z = m$ do not contribute to the asymptotics (2.5). Throughout the proof of the following theorem we shall use this fact without further comments.*

Theorem 7.6. *Let the conditions in part (ii) of Theorem 2.14 be fulfilled. Then the equality*

$$\lim_{|z-m|\to 0} |\log|z-m||^{-1} n(\mu, T(\delta; |z-m|)) = U(\mu),\ \mu > 0,$$

holds.

Proof. The space of functions having support in $\hat{U}_\delta(0)$ is an invariant subspace for the operator $T_{11}(\delta; |z-m|)$.

Let $\hat{T}_{11}^{(0)}(\delta; |z-m|)$ be the restriction of the operator $T_{11}(\delta; |z-m|)$ to the subspace $L_2(\hat{U}_\delta(0))$. By the unitary dilation

$$\mathbf{Y} : L_2(U_\delta(0)) \to L_2(\hat{U}_\delta(0)), \quad (\mathbf{Y}f)(p) = f(W^{-\frac{1}{2}}p),$$

where W is defined in Assumption 2.1, one verifies that the operator $\hat{T}_{11}^{(0)}(\delta; |z-m|)$ is unitary equivalent to the following operator $T_{11}^{(0)}(\delta; |z-m|)$ acting in $L_2(\hat{U}_\delta(0))$ as

$$\big(T_{11}^{(0)}(\delta; |z-m|)f\big)(p) = \frac{l_1^{3/2}}{2\pi^2} \int\limits_{U_\delta(0)} \frac{(lp^2+2|z-m|)^{-1/4}(lq^2+2|z-m|)^{-1/4}}{l_1p^2+2l_2(p,q)+l_1q^2+2|z-m|} f(q)\,dq.$$

The operator $T_{11}^{(0)}(\delta; |z-m|)$ is unitary equivalent to the integral operator

$$T_{11}^{(1)}(\delta; |z-m|) : L_2(U_r(0)) \to L_2(U_r(0))$$

with the kernel

$$\frac{l_1^{3/2}}{2\pi^2} \frac{(lp^2+2)^{-1/4}(lq^2+2)^{-1/4}}{l_1p^2+2l_2(p,q)+l_1q^2+2},$$

where $r = |z-m|^{-\frac{1}{2}}$ and $U_r(0) = \{p \in \mathbb{R}^3 : |p| < r\}$.

The equivalence of the operators $T_{11}^{(0)}(\delta; |z-m|)$ and $T_{11}^{(1)}(\delta; |z-m|)$ is performed by the unitary dilation

$$\mathbf{B}_r : L_2(U_\delta(0)) \to L_2(U_r(0)), \quad (\mathbf{B}_r f)(p) = \left(\frac{r}{\delta}\right)^{-3/2} f\left(\frac{\delta}{r}p\right).$$

Let $\chi_1(\cdot)$ be characteristic function of the ball $U_1(0)$. We may replace the functions

$$(lp^2+2)^{-1/4},\ (lq^2+2)^{-1/4} \quad \text{and} \quad l_1p^2+2l_2(p,q)+l_1q^2+2$$

by

$$(lp^2)^{-1/4}(1-\chi_1(p)),\ \ (lq^2)^{-1/4}(1-\chi_1(q)) \quad \text{and} \quad l_1p^2+2l_2(p,q)+l_1q^2,$$

respectively, since the error will be a Hilbert-Schmidt operator continuous up to $z = m$. Then we get the integral operator $T_{11}^{(2)}(r)$ on $L_2(U_r(0) \setminus U_1(0))$ with the kernel

$$l^{-\frac{1}{2}} \frac{l_1^{3/2}}{2\pi^2} \frac{|p|^{-1/2}|q|^{-1/2}}{l_1p^2+2l_2(p,q)+l_1q^2}.$$

By the dilation

$$\mathbf{M} : L_2(U_r(0) \setminus U_1(0)) \longrightarrow L_2((0,\mathbf{r}) \times \sigma_0),\ \mathbf{r} = 1/2|\log|z-m|,$$

where $(\mathbf{M}f)(x,w) = e^{3x/2} f(e^x w),\ x \in (0,\mathbf{r}),\ w \in \mathbb{S}^2$, one sees that the operator $T_{11}^{(2)}(r)$ is unitary equivalent with the integral operator $\mathbf{S_r}$ defined by (7.3).

The difference of the operators $\mathbf{S_r}$ and $T(\delta; |z-m|)$ is compact (up to unitarily equivalence) and hence we obtain

$$\lim_{|z-m|\to 0} |\log|z-m||^{-1} n(\mu, T(\delta; |z-m|)) = U(\mu), \ \mu > 0.$$

Theorem 7.6 is proved. □

Proof of part (ii) of Theorem 2.14. Let the conditions in part (ii) of Theorem 2.14 be fulfilled. Then the equality

$$\lim_{|z-m|\to 0} |\log|z-m||^{-1} n(1, T(z)) = U(1) > 0 \tag{7.4}$$

follows from Lemmas 7.1, 7.2, 7.4, and Theorem 7.6. Taking into account the equality (7.4) and using Lemma 5.1 we complete the proof of part (ii) of Theorem 2.14.

APPENDIX A

Lemma A.1. *Let the function v as in Assumption 2.3 and the function w be defined by (2.1) and ε be a real-analytic conditionally negative definite function on $\mathbb{T}^3$ with a unique non-degenerate minimum at the origin. Then Assumption 2.6 is fulfilled.*

Proof. It is known that the real-valued (even) conditionally negative definite function ε admits the (Lévy-Khinchin) representation (see, e.g., Refs. 4 and 9)

$$\varepsilon(p) = \varepsilon(0) + \sum_{s\in\mathbb{Z}^d\setminus\{0\}} (\cos(p,s) - 1)\hat{\varepsilon}(s), \quad p \in \mathbb{T}^3, \tag{A.1}$$

which is equivalent to the requirement that the Fourier coefficients $\hat{\varepsilon}(s)$ with $s \neq 0$ are non-positive, that is,

$$\hat{\varepsilon}(s) \le 0, \quad s \neq 0, \tag{A.2}$$

and the series $\sum_{s\in\mathbb{Z}^3\setminus\{0\}} \hat{\varepsilon}(s)$ converges absolutely.

Since w and v are even the function $\Lambda(\cdot)$ is also even. Then using the equality

$$w_0(t) - \frac{w_p(t) + w_p(-t)}{2} = \sum_{s\in\mathbb{Z}^3\setminus\{0\}} \hat{\varepsilon}(s)(1+\cos(t,s))(1-\cos(p,s))$$

we have

$$\Lambda(0,m) - \Lambda(p,m) = \frac{1}{2}\sum_{s\in\mathbb{Z}^3\setminus\{0\}} (-\hat{\varepsilon}(s))(1-\cos(p,s)) \times \int_{\mathbb{T}^3} (1+\cos(t,s))F(p,t)dt + \tilde{B}(p), \tag{A.3}$$

where

$$F(p,\cdot)=\frac{[w_p(\cdot)+w_{-p}(\cdot)-2m]}{(w_p(\cdot)-m)(w_{-p}(\cdot)-m)(w_0(\cdot)-m)}v^2(\cdot)$$

and

$$\tilde{B}(p)=\frac{1}{4}\int\limits_{\mathbb{T}^3}\frac{[w_p(t)-w_{-p}(t)]^2}{(w_p(t)-m)(w_{-p}(t)-m)(w_0(t)-m)}v^2(t)\,dt.$$

Set

$$B(p,s)=\int\limits_{\mathbb{T}^3}(1+\cos(t,s))F(p,t)dt.$$

Let $\chi_\delta(\cdot)$ be the characteristic function of $U_\delta(0)$. Choose $\delta>0$ such that

$$mes\{(\mathbb{T}^3\setminus U_\delta(0))\cap supp\ v\}>0.$$

Set $F_\delta(p,\cdot)=(1-\chi_\delta(\cdot))F(p,\cdot)$. Then for all $p\in\mathbb{T}^3$ and a.e.

$$t\in(\mathbb{T}^3\setminus U_\delta(0))\cap supp\ v(\cdot)$$

the function $F_\delta(\cdot,\cdot)$ is strictly positive. Since the function u has a unique minimum at $(0,0)$ and $v\in\mathcal{B}(\theta,\mathbb{T}^3)$ we have, for any $p\in\mathbb{T}^3$, that $F_\delta(p,\cdot)$ belongs to the Banach space $L^1(\mathbb{T}^3)$. Then for some (sufficiently large)$R>0$ and (sufficiently small) $c_1(\delta)>0$ and for all $|s|\le R$, $p\in\mathbb{T}^3$, we have the inequality

$$B_\delta^{(1)}(p,s)=\int\limits_{\mathbb{T}^3}(1+\cos(t,s))F_\delta(p,t)dt>c_1(\delta)>0.$$

The Riemann-Lebesgue lemma yields

$$\begin{aligned}B_\delta^{(1)}(p,s)\ &=\int\limits_{\mathbb{T}^3}(1+\cos(t,s))F_\delta(p,t)\,dt\\&\to\int\limits_{\mathbb{T}^3}F_\delta(p,t)dt>0,\ p\in\mathbb{T}^3\quad\text{as}\quad s\to\infty.\end{aligned}$$

The continuity of the function $\int_{\mathbb{T}^3}F_\delta(p,t)dt$ on the compact set $\mathbb{T}^3$ yields that all $p\in\mathbb{T}^3$ and $|s|>R$ the inequality $B_\delta^{(1)}(p,s)\ge c_2(\delta)$ holds.

Thus for $c(\delta)=\min\{c_1(\delta),c_2(\delta)\}$ the inequality $B_\delta^{(1)}(p,s)\ge c$ holds for all $s\in\mathbb{Z}^3$, $p\in\mathbb{T}^3$. So $B_\delta^{(2)}(p,s)\ge 0$, $s\in\mathbb{Z}^3$, $p\in\mathbb{T}^3$ yields $B(p,s)>c$, $s\in\mathbb{Z}^3$, $p\in\mathbb{T}^3$. Taking into account the inequalities $\tilde{B}(p)\ge 0$, $p\in\mathbb{T}^3$, and $\hat{\varepsilon}(s)\le 0$, $s\in\mathbb{Z}^3\setminus\{0\}$ (see (A.2)) from the representations (A.1) and (A.3) we have

$$\Lambda(0,m)-\Lambda(p,m\ge c(\varepsilon(p)-\varepsilon(0)).$$

This together with the assumptions on $\varepsilon(\cdot)$, completes the proof of Lemma A.1. $\square$

Lemma A.2. *Let Assumption 2.1 be fulfilled. Then there exists a* δ-*neighborhood* $U_\delta(0) \subset \mathbb{T}^3$ *of the point* $p = 0$ *and an odd function* $q_0(\cdot) \in C^{(2)}(U_\delta(0))$ *such that: (i) for any* $p \in U_\delta(0)$ *the point* $q_0(p)$ *is a unique non-degenerate minimum of* $w_p(\cdot)$ *and*

$$q_0(p) = -\frac{l_2}{l_1} p + O(|p|^{2+\theta}) \text{ as } p \to 0. \tag{A.4}$$

(ii) the function $m(p) = \min_{q \in \mathbb{T}^3} w(p, q) = w(p, q_0(p))$ *is even and its all the second order partial derivatives are belong to* $\mathcal{B}(\theta, \mathbb{T}^3)$. *One has the asymptotics*

$$m(p) = m + \frac{l_1^2 - l_2^2}{2l_1}(Wp, p) + O(|p|^{3+\theta}) \quad \text{as} \quad p \to 0. \tag{A.5}$$

Proof. (i) By the implicit function theorem there exist $\delta > 0$ and a function $q_0(\cdot) \in C^{(2)}(U_\delta(0))$ such that for any $p \in U_\delta(0)$ the point $q_0(p)$ is the unique non-degenerate minimum point of $w_p(\cdot)$ (see Lemma 3 in Ref. 22).

Since $w(\cdot, \cdot)$ is even with respect to $(p, q) \in (\mathbb{T}^3)^2$ for all $p \in \mathbb{T}^3$ we obtain

$$m(-p) = \min_{q \in \mathbb{T}^3} w_{-p}(q) = \min_{q \in \mathbb{T}^3} w_p(-q) = \min_{-q \in \mathbb{T}^3} w_p(q) = \min_{q \in \mathbb{T}^3} w_p(q) = m(p)$$

and hence

$$w_p(q_0(p)) = m(p) = m(-p) = w_{-p}(q_0(-p)), \quad p \in U_\delta(0), \tag{A.6}$$

since $w_p(q_0(p)) = \min_{q \in \mathbb{T}^3} w_p(q),\ p \in U_\delta(0)$.

According to the fact that $w(\cdot, \cdot)$ is even we get $w_{-p}(q_0(-p)) = w_p(-q_0(-p))$. Then from (A.6) we have

$$w_p(q_0(p)) = w_p(-q_0(-p)), \quad p \in U_\delta(0). \tag{A.7}$$

Since for each $p \in U_\delta(0)$ the point $q_0(p)$ is the unique non-degenerate minimum of the function $w_p(\cdot)$ the equality (A.7) yields $q_0(-p) = -q_0(p),\ p \in U_\delta(0)$.

The asymptotics (A.4) follows from the fact that $q_0(\cdot)$ is an odd function and its coefficient $-\frac{l_2}{l_1}$ is calculated using the identity $\nabla w(p, q_0(p)) \equiv 0,\ p \in U_\delta(0)$.

(ii) By the asymptotics (7.1), (A.4) and the equality $m(p) = w_p(q_0(p))$ we obtain the asymptotics (A.5). $\square$

Lemma A.3. *If the conditions of Theorem 2.14 are fulfilled, then the operator* $h(p)$, $p \in \mathbb{T}^3$, *has no eigenvalues lying below* m. *Therefore,* $\tau_{\mathrm{ess}}(H) = \inf \sigma_{three}(H) = \inf \sigma_{two}(H) = m$.

Proof. It suffices to prove that $\inf \sigma_{two}(H) = m$. Let the conditions of Lemma A.3 be fulfilled. Since the function $\Delta(0, \cdot)$ is decreasing on $(-\infty, m)$ and the function $u(\cdot)$ (resp. $\Lambda(\cdot)$) has a unique minimum (resp. maximum) at $p = 0$ for all $z < m$ and $p \in \mathbb{T}^3$ we have

$$\Delta(p, z) = u(p) - z - \frac{1}{2}\Lambda(p, z) > u(0) - m - \frac{1}{2}\Lambda(0, m). \tag{A.8}$$

If the operator $h(0)$ has either a threshold energy resonance or a threshold eigenvalue, then by Lemmas 3.2 and 3.3 we have $\Delta(0, m) = 0$. Hence by inequality (A.8) we conclude that $\Delta(p, z) > 0$ for all $p \in \mathbb{T}^3$ and $z < m$. By Lemma 3.1 the operator $h(p)$, $p \in \mathbb{T}^3$, has no eigenvalues lying below m.Thus, $\inf \sigma_{two}(H) = m$. □

Lemma A.4. The right-hand derivative of $D_1(\cdot)$ at $\zeta = 0$ exists and the following equality holds

$$\frac{\partial}{\partial \zeta} D_1(0) = 2\sqrt{2}\pi^2 l_1^{-\frac{3}{2}} v^2(0)(det W)^{-\frac{1}{2}}. \tag{A.9}$$

Proof. Let us consider the following difference

$$D_1(\zeta) - D_1(0) = -\frac{\zeta^2}{2} \int\limits_{U_\delta(0)} \frac{v^2(q)\, dq}{(w_0(0, q) + \zeta^2) w_0(0, q)}. \tag{A.10}$$

The function $w_0(0, \cdot)$ has a unique non-degenerate minimum at $q = 0$. Therefore, by virtue of the Morse lemma (see Ref. 16) there exists a one-to-one mapping $q = \varphi(t)$ of a certain ball $W_\gamma(0)$ of radius $\gamma > 0$ with the center at $t = 0$ to a neighborhood $\tilde{W}(0)$ of the point $q = 0$ such that:

$$w_0(0, \varphi(t)) = t^2 \tag{A.11}$$

with $\varphi(0) = 0$ and for the Jacobian $J_\varphi(t) \in \mathcal{B}(\theta, U_\delta(0))$ of the mapping $q = \varphi(t)$ the equality

$$J_\varphi(0) = \sqrt{2} l_1^{-\frac{3}{2}} (det W)^{-\frac{1}{2}}$$

holds, where $\mathcal{B}(\theta, U_\delta(0))$ can be defined similarly to $\mathcal{B}(\theta, \mathbb{T}^3)$.

In the integral in (A.10) making a change of variable $q = \varphi(t)$ and using the equality (A.11) we obtain

$$D_1(\zeta) - D_1(0) = -\frac{\zeta^2}{2} \int\limits_{W_\gamma(0)} \frac{v^2(\varphi(t)) J_\varphi(t)}{t^2(t^2 + \zeta^2)} dt. \tag{A.12}$$

Going over in the integral in (A.12) to spherical coordinates $t = r\omega$, we reduce it to the form

$$D_1(\zeta) - D_1(0) = -\frac{\zeta^2}{2} \int_0^\gamma \frac{F(r)}{r^2 + \zeta^2} dr,$$

with

$$F(r) = \int_{\mathbb{S}^2} v^2(\varphi(r\omega)) J_\varphi(r\omega) d\omega,$$

where $\mathbb{S}^2$ is the unit sphere in $\mathbb{R}^3$ and $d\omega$ is the element of the unit sphere in this space.

Using $v \in C^{(2)}(\mathbb{T}^3)$, $J_\varphi \in \mathcal{B}(\theta, U_\delta(0))$ we see that

$$|F(r) - F(0)| \leq Cr^\theta. \tag{A.13}$$

Applying the inequality (A.13) it easy to see that The function $D_1(\zeta) - D_1(0)$ can be rewritten in the form

$$\lim_{\zeta \to 0+} \frac{D_1(\zeta) - D_1(0)}{\zeta} = 2\sqrt{2}\pi l_1^{-\frac{3}{2}} v^2(0)(det W)^{-\frac{1}{2}}.$$

Hence we have that there exists a right-hand derivative of $D_1(\cdot)$ at $\zeta = 0$ and the equality (A.9) holds. □

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