The O(n) Model on the Annulus

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We use Coulomb gas methods to derive an explicit form for the scaling limit of the partition function of the critical O(n) model on an annulus, with free boundary conditions, as a function of its modulus. This correctly takes into account the magnetic charge asymmetry and the decoupling of the null states. It agrees with an earlier conjecture based on Bethe ansatz and quantum group symmetry, and with all known results for special values of n. It gives new formulae for percolation (the probability that a cluster connects the two opposite boundaries) and for self-avoiding loops (the partition function for a single loop wrapping non-trivially around the annulus.) The limit $n \rightarrow 0$ also gives explicit examples of partition functions in logarithmic conformal field theory.

1. INTRODUCTION

The Coulomb gas approach to two-dimensional critical models which can be written as loop gases, such as the O(n) model and Q-state Potts models, has been extraordinarily successful in deriving their universal bulk properties. First developed by den Nijs[1] and Nienhuis[2] in order to explain conjectured exact values for the principal bulk critical exponents, it was adapted by di Francesco, Saleur and Zuber[3] to compute the partition function on the torus, which encodes all the bulk scaling dimensions.[4]

However, this particular approach has not, so far, been successfully adapted to explain the conjectured exact values[5,6] for the *boundary* scaling dimensions in these models, even though many of them have now been derived using other methods[8] applied to related lattice models, and, more recently, rigorously using

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Schramm-Loewner evolution (SLE).[7] From the point of view of conformal field theory (CFT), both the boundary and the bulk exponents are encoded in the partition function on the annulus,[9] and therefore a direct computation of this object is of great interest. From the point of view of CFT, general values of n and Q give irrational non-unitary examples.

As we shall explain, the naive application of Coulomb gas methods to domains with boundaries fails to account either for the reduction of the central charge c from its free field value of unity, or for the elimination of the null states, which, as is known from CFT, is necessary to maintain modular invariance. In this paper we propose a resolution of these issues which provides an explicit formula for the partition function on the annulus.

Consider an annulus $(0 \le x < \ell, 0 < y < L)$, identifying $x = 0, \ell$. Introduce the conjugate moduli

$$q = e^{-\pi \ell/L}, \quad \tilde{q} = e^{-2\pi L/\ell}.$$

Note that if the annulus is conformally mapped to the region $R_1 < r < R_2$ between two circles, $\tilde{q} = \ln(R_2/R_1)$. We impose free boundary conditions on the O(n) spins on $y = 0, L$. As usual in the Coulomb gas, we introduce the parametrisation $n = \sqrt{Q} = 2 \cos \chi$, $g = 1 - \chi/\pi$, where $1 \le g \le 2$ corresponds to the dilute critical point of the O(n) model (or the tricritical point of the Q-state Potts model), and $\frac{1}{2} \le g < 1$ to the critical dense phase of the O(n) model (or the ordinary critical point of the Potts model.) Then our main result for the annulus partition function is

$$Z = q^{-\frac{c}{24}} \prod_{r=1}^{\infty}(1 - q^r)^{-1} \sum_{p \in \mathbb{Z}} \frac{\sin(p+1)\chi}{\sin \chi} \, q^{\frac{gp^2}{4} - \frac{(1-g)p}{2}}. \tag{1}$$

In terms of the conjugate modulus $\tilde{q}$ this becomes

$$Z = (2/g)^{1/2} \, \tilde{q}^{-\frac{c}{12}} \prod_{r=1}^{\infty}(1 - \tilde{q}^{2r})^{-1} \sum_{m \in \mathbb{Z}} \frac{\sin((\chi + 2m\pi)/g)}{\sin \chi} \, \tilde{q}^{\frac{(\chi+2\pi m)^2}{2\pi^2 g} - \frac{(1-g)^2}{2g}}. \tag{2}$$

This has the form expected from boundary conformal field theory[9]: (1) is $Z = q^{-c/24} \sum_{\Delta} d_{\Delta} \chi_{\Delta}(q)$, where Δ runs over the allowed set of boundary scaling dimensions, $\chi_{\Delta}(q)$ is a highest weight Virasoro character, and d_{Δ} is a degeneracy factor, which is a polynomial in n (although only integer in those cases when the theory is unitary). In this form the explicit expression (1) is not new, and indeed is originally due to Saleur and Bauer,[8] who deduced the allowed scaling dimensions from Bethe ansatz and quantum group arguments. In this context the degeneracy factor d_{Δ} is the quantum dimension. It has been used extensively in papers by Saleur and Pasquier,[10] Saleur,[11] and more recently appeared in a paper by Read and Saleur.[12] However, a direct derivation from the lattice O(n) model solely

using Coulomb gas arguments has not appeared to our knowledge, and this is the point of the present paper.

Note that (2) also has the form expected from boundary CFT,[9] namely $\tilde{q}^{-c/12} \sum_{\Delta} |b_{\Delta}|^2 \chi_{\Delta}(\tilde{q}^2)$ where now the sum is over allowed bulk scaling dimensions 2Δ, and b_{Δ} is a matrix element with a boundary state. In particular, for $\Delta = 0$ we have

$$b_0^2 = -(2/g)^{1/2} \frac{\sin(\pi/g)}{\sin \pi g}, \tag{3}$$

which gives the boundary entropy[13] $\ln b_0$.

We have explicitly written the dependence on χ in Eqs. (1), (2) because, if we wish to consider a modified partition function in which the loops which wrap non-trivially around the annulus are counted with a different weight $n' = 2 \cos \chi'$, it is simply necessary to replace $\chi \to \chi'$. This allows us to compute interesting quantities for percolation and self-avoiding loops.

For example, in critical percolation ($Q = 1$) the probability that a cluster connects the two boundaries of the annulus is

$$P = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \sum_{k \in \mathbb{Z}} \left(q^{\frac{8k^2}{3} - \frac{2k}{3}} - q^{\frac{8k^2}{3} + 2k + \frac{1}{3}} \right). \tag{4}$$

The partition function for a single self-avoiding loop which wraps non-trivially around the annulus (the number of such loops weighted by $\mu^{-\text{length}}$, where μ is the non-universal connective constant) is

$$Z_1 = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \sum_{k \in \mathbb{Z}} k(-1)^{k-1} q^{\frac{3k^2}{2} - k + \frac{1}{8}}. \tag{5}$$

In the limit $\tilde{q} \to 0$, we find $Z_1 \sim (1/6\pi)|\ln \tilde{q}|$. The form of this agrees with a rigorous result of Werner.[14]

The layout of this paper is as follows. In Sec. 2 we give a brief survey of Coulomb gas methods as applied in the plane and the cylinder, and then discuss the particular problems associated with domains with boundaries, in particular the annulus. This will lead to the proposal (1) for the partition function. As with most Coulomb gas methods, this is not wholly deductive, but relies on some heuristic reasoning. However, in Sec. 3, we show that (1) agrees with previously known results for various special cases (for example the Ising model and the 3-state Potts model.) In Sec. 4 we then derive a variety of new results, some of which have already been mentioned above.

2. COULOMB GAS ON THE CYLINDER AND THE ANNULUS

2.1. Basics

We first summarise the Coulomb gas arguments as applied to the plane and cylinder, as formulated by de Nijs[1] and Nienhuis,[2] and elaborated by Kondev.[15]

The $O(n)$ model is most easily realised on the honeycomb lattice. At each site r is an n-component spin $\mathbf{s}(r)$ (initially n is a positive integer.) The Boltzmann weight for a given configuration is

$$\prod_{r,r'}(1 + t\,\mathbf{s}(r)\cdot\mathbf{s}(r')), \tag{6}$$

where the product is over all edges (r, r') of the lattice. The partition function is the trace over these weights, a linear operation defined by $\mathrm{Tr}\,1 = 1$, $\mathrm{Tr}\,s_a(r)s_b(r) = \delta_{ab}$ and $\mathrm{Tr}\,s_a(r) = \mathrm{Tr}\,s_a(r)s_b(r)s_c(r) = 0$. Expanding (6) in powers of t gives a sum over all subsets $\mathcal{G}$ of the edges, with an associated factor $t^{|\mathcal{G}|}$. Implementing the trace operation eliminates all subgraphs which are not unions of non-intersecting closed loops (for the time being we ignore boundaries), and each of these gets counted with a weight n.

At this point we can allow n to be any positive real number. This gives a measure on the allowed subgraphs $\mathcal{G}$, called the loop gas. If t is small, the mean loop length is finite, even in the thermodynamic limit, but there is a critical value t_c at which it first diverges. This is called the dilute critical point. For $t > t_c$ a single loop contains a finite fraction of the sites: this is the dense phase.

The critical Q-state Potts model on the square lattice can also be written, via the Fortuin-Kasteleyn[16] correspondence, in terms of a loop gas, in which each closed loop carries a factor[2] $\sqrt{Q}$.

Both these loop gas models can be mapped to a model of heights $h(R)$ on the sites R of the dual lattice, by first orienting each loop, so that a configuration of N non-oriented loops corresponds to 2^N configurations of oriented loops, and then, for each edge of the lattice, assigning height differences $\Delta h = 0, \pm\pi$ between the neighbouring sites of the dual lattice according to whether the edge is contained in the oriented subgraph $\mathcal{G}$, and its orientation. The weight n (or $\sqrt{Q}$) for each non-oriented loop is distributed into a factor $e^{\pm i\chi}$ for each clockwise (anticlockwise) oriented loop, where $n = 2\cos\chi$. Although these weights are complex (a feature which lies at the heart of the difficulties associated with a rigorous treatment of the Coulomb gas approach), they have the advantage of being local, in the sense that they may be distributed so that each loop acquires a factor $e^{i\theta\chi/2\pi}$ whenever it turns through an angle θ at a vertex.

However, it should be noted that, at least for the fully packed model on the square lattice, there is a mapping to the 6-vertex model with positive Boltzmann weights: at each vertex the two loops are either oriented parallel to each other, with weight $e^{i\chi/4}\cdot e^{-i\chi/4} + e^{-i\chi/4}\cdot e^{i\chi/4} = 2$, or anti-parallel in which case the

weight is

$$e^{i\chi/4} \cdot e^{i\chi/4} + e^{-i\chi/4} \cdot e^{-i\chi/4} = 2\cos(\chi/2) = (n+2)^{1/2}.$$

The Coulomb gas method assumes that, in the continuum limit, the discrete heights become continuous and the Boltzmann weights converge to e^{-S} where S is the action of a free field theory

$$S = (g/4\pi) \int (\partial h)^2 \, dx dy.$$

The original discrete model may be recovered from this by adding a term $\lambda \sum_R \cos 2h(R)$ in the limit $\lambda \to -\infty$.

However, on a cylinder of length ℓ and circumference L, with $\ell \gg L$, this does not properly account for loops which wind around it: these can be taken into account by placing 'electric' charges $e^{\pm i(\chi/\pi)h}$ at either end. This modifies the partition function to $Z \sim e^{\pi c\ell/6L}$, identifying the total central charge

$$c = 1 - 6\frac{(\chi/\pi)^2}{g}.$$

The scaling dimensions of electric charges e^{iqh} are also modified if we put them at the ends of the cylinder as well:

$$x_q = (1/2g)((q + \chi/\pi)^2 - (\chi/\pi)^2).$$

Note that $x_q \neq x_{-q}$: this an example of the electric charge asymmetry introduced by this construction.

g is fixed in terms of χ by requiring[15] that $\cos 2h$ be marginal in the sense of the renormalisation group, i.e. $x_2 = 2$. This fixes

$$g = 1 \pm (\chi/\pi),$$

with the sign depending on whether we choose x_2 or x_{-2}. In fact this ambiguity is to be expected: for each value of n, χ is only defined up to a sign (actually we can add multiples of 2π as well, but these give less relevant operators) and these correspond to the dilute ($g > 1$) and dense ($g < 1$) cases of the critical O(n) model. In the following we take the lower sign by convention.

Note that these ideas are easy to extend to the partition function on the torus, correctly taking into account loops which wrap around some combination of the two cycles.[3]

Now consider the case of the annulus. Throughout this paper we assume free boundary conditions on the O(n) spins, which means that there are only closed loops in the loop gas representation. (Partial results using Coulomb gas methods were found for the case of fixed boundary conditions in Ref. 17 for the special case $n = 1$ in the dense phase.)

First consider the case when $\ell \gg L$, where we expect

$$Z \sim e^{\pi c \ell / 24 L} \sim q^{-c/24},$$

with c given as above. In this limit there is no contribution of loops wrapping around the annulus, so we expect that that $h(y = L) = h(y = 0)$. Naively then, we get a free field theory with (equal) Dirichlet boundary conditions, which gives $c = 1$.

Where does the correction to c come from? One (incorrect) possibility is as follows: looking back at the lattice construction, we see that there are extra factors of $e^{\pm i \chi / 2}$ whenever a loop is next to the boundary, which are not properly taken into account in the the bulk Boltzmann weights of the height model. The sign is determined by whether the height at a site next to the boundary is $\pm \pi$. In the continuum limit these would lead to boundary terms in the action proportional to $i \chi \int \partial_\perp h \, dl$ where $\partial_\perp$ is along the inward pointing normal to the boundary and dl is a line element. For the annulus these would give something proportional to

$$i \chi \int (\partial_y h(x, y = 0) - \partial_y h(x, y = L)) \, dx \tag{7}$$

However, an explicit calculation (see Appendix) shows that such a combination does not contribute to c. In fact, if we add to the action a general boundary term

$$\int (\alpha_1 \partial_y h(x, y = 0) + \alpha_2 \partial_y h(x, y = L)) \, dx,$$

we find that the effective central charge is

$$c = 1 - (24/g)(\alpha_1 + \alpha_2)^2.$$

Thus not only is there no contribution if $\alpha_1 = -\alpha_2$, as in (7), we must also have $\alpha_1 + \alpha_2$ real, rather than pure imaginary.

An equivalent, and easier, way of getting the same modification to c is to assume that the correct boundary conditions, even when $\ell \gg L$, are

$$h(y = L) - h(y = 0) = \pi m_0 \neq 0.$$

In that case we can write $h = \pi m_0 y / L + \tilde{h}$, where $\tilde{h}$ vanishes on both $y = 0$ and $y = L$. The functional integral over $\tilde{h}$ gives $c = 1$ as before, and the modification to the partition function is $\sim \exp(-(g/4\pi)(\pi m_0)^2 \ell / L)$. So if we take

$$m_0 = \pm \chi / \pi g, \tag{8}$$

we get the correct c. Note that, for a long rectangular strip rather than an annulus, this is like adding magnetic charges, or vortices, at the ends.

Thus it would appear that there should be a spontaneous average magnetic flux around the annulus, if $g \neq 1$. This may be understood heuristically in terms of the preferred parallel, rather than antiparallel, alignment of neighbouring loops:
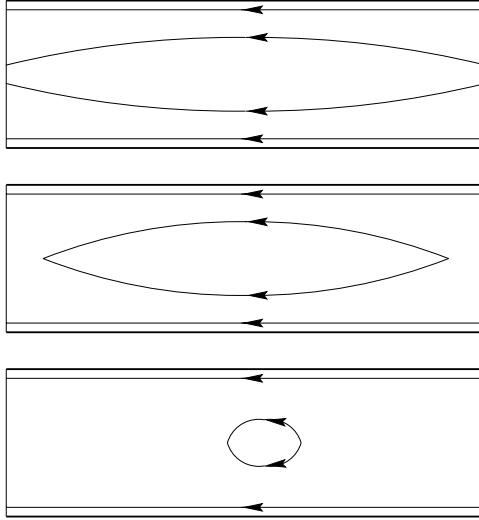
Fig. 1. The screening of magnetic flux in a long rectangle. If the background flux m is too large, vortex pairs of strength ± 2 can be shed from either end of the rectangle and will annihilate in order to reduce the free energy. If m is too small, the opposite effect occurs.

this effect should be enhanced near a boundary, since loops are geometrically constrained to lie approximately parallel or anti-parallel to the boundary.

The actual value of m_0 may be fixed by the following argument. Consider first the geometry of a long rectangle with $\ell \gg L$, where the loops are allowed to end on the boundaries at $x = 0, \ell$, but, as before, not on $y = 0, L$. In that case the total charge m flowing along the rectangle is not fixed, and we can ask the question what is its mean value m_0 in the state of lowest free energy. A flux m corresponds to vortices of strengths $\pm m$ at the ends of the rectangle. However this can increase or decrease in units of 2 by shedding vortices from either end of the rectangle (see Fig. 1). The additional free energy for creating such a pair of vortices is $(g/4\pi)((m+2)^2 - m^2)(\pi/L)^2(\ell L)$ scaling dimension of a vortex of strength ± 2, from which we read off the scaling dimension

$$\Delta_2 = (g/4)((m+2)^2 - m^2)$$

If m is too large, $\Delta_2 > 1$, which means that the corresponding renormalisation group $y_2 = 1 - \Delta_2 < 0$. (Note that since we are in the boundary, rather than the bulk, situation, the eigenvalue is $1 - \Delta$ rather than $2 - \Delta$.) This implies that such ± 2 vortex pairs are closely bound. Thus any such pairs shed from the boundaries at $x = 0, \ell$ will annihilate to reduce their free energy. On the other hand, if m is too small, $y_2 > 0$, which means that any such vortex pairs will unbind. This will act to increase the effective value of m.

This screening effect implies that the mean value m_0 of the magnetic flux which minimises the free energy corresponds to $y_2 = 0$, that is

$$(g/4)\big((m_0 + 2)^2 - m_0^2\big) = 1,$$

so that

$$m_0 = (1 - g)/g = \chi/\pi g,$$

which is the same as found above in (8). There is a similar minimum free energy solution with $m = -m_0$. Note that the above argument is analogous to the earlier argument which fixed g, where we demanded that electric ± 2 charges should be marginal.

If we now go the annulus, we expect a total average magnetic flux $\pm m_0$ to spontaneously form, even when $\ell \gg L$. Now suppose that ℓ/L is not so large, so we can have extra loops wrapping around the x-cycle. We can orient these as before. If the total number of up arrows minus down arrows (the additional magnetic flux flowing along the annulus) is p, then we get $h(y = L) - h(y = 0) = \pi(p \pm m_0)$. As for the cylinder, in order to count them correctly we need to put in a factor $\exp(i(p \pm m_0)\chi)$. Thus we get the following first guess for the partition function on the annulus:

$$\widetilde{Z} = Z_0 \sum_{p \in \mathbb{Z}} e^{i(p+m_0)\chi} e^{-(g/4)(\pi p + m_0)^2(\pi \ell/L)} + (m_0 \to -m_0),$$

where $Z_0 = q^{-1/24} \prod_{r=1}^{\infty}(1 - q^r)^{-1}$ is the partition function from $\tilde{h}$. Note that we should sum over both possible signs for m_0. If we let $p \to -p$ in the first term this simplifies to

$$\widetilde{Z} = q^{-c/24} \prod_{r=1}^{\infty}(1 - q^r)^{-1} \sum_{p \in \mathbb{Z}} \cos((p - m_0)\chi)\, q^{(g/4)p^2 - (1-g)p/2}. \qquad (9)$$

Note that if we want to count loops wrapping around the annulus with a different weight $n' = 2 \cos \chi'$, we just change $\chi \to \chi'$ in the above (keeping g the same.)

Equation (9) has some good features and some bad ones. In general we expect that Z can be written as a sum of terms q^h where h runs over all the allowed scaling dimensions of the allowed boundary operators. (For a unitary theory the coefficients should be non-negative integers, but this doesn't have to hold for general n.) We see in (9) for $p = N \geq 1$ the scaling dimensions of the boundary N-leg operators, as first conjectured by Saleur and Duplantier.[6]

However, for the dilute case with $g > 1$, $p = -1$ actually gives the next-to-leading term as $q \to 0$. This doesn't make sense: we expect this to come from $p = N = 1$. More seriously, (9) fails to account for the fact that the scaling dimensions of the boundary N-leg operators correspond to those of the degenerate cases $h_{1,N+1}$ of the Kac table: in general these operators correspond to highest weight states

whose Virasoro representations are reducible: they have a null descendent state (which corresponds to a term in the expansion of $\prod_r (1 - q^r)^{-1}$) of dimension $h_{1,N+1} + N + 1$. For a unitary theory, or more generally a minimal model, we know that such states (and all their descendents) should be subtracted out of the partition function. For a non-unitary theory this is not necessary: however we shall show later that if they are retained, the behaviour as $\tilde{q} \to 0$ is incorrect. This leads to the conclusion that each term in (9) should be modified according to

$$q^{(g/4)p^2 - (1-g)p/2} \to q^{(g/4)p^2 - (1-g)p/2} (1 - q^{p+1}) = q^{h_{1,p+1}} - q^{h_{1,-p-1}}.$$

Note that this has the feature of automatically eliminating the 'rogue' state at $p = -1$.

   We now give a physical argument for this subtraction. Once again it is useful to think about the rectangle geometry where magnetic flux can be created or destroyed at the boundaries at $x = 0, \ell$. It is also useful to think in terms of the energy eigenstates of the hamiltonian $(\pi/L)(L_0 - c/24)$ which generates translations in x. In general, each configuration with total magnetic flux p will be accompanied by excitations of the $\tilde{h}$ field, which correspond to the Virasoro descendents. If the energy of these is correct they can resonate with the original highest weight state plus a number of pairs of marginally bound ± 2 magnetic charges, each of which has energy π/L. If the excitation energy is $(p + 1)\pi/L$, exactly $p + 1$ vortex pairs can be shed from the boundaries at $x = 0, \ell$ (see Fig. 2). These states, however, are identical to those with total magnetic charge $p - 2(p + 1) = -2 - p$, and should therefore not be doubly counted.

   The effect of this subtraction is therefore to modify the sum in (1) to

$$\sum_{p \in \mathbb{Z}} \cos((p - m_0)\chi)\left(q^{h_{1,p+1}} - q^{h_{1,-p-1}}\right).$$

Relabelling $p \to -2 - p$ in the second term has the effect of modifying

$$\cos((p - m_0)\chi) \to \cos((p - m_0)\chi) - \cos((p + 2 + m_0))\chi) \propto \sin((p + 1)\chi),$$

which finally leads to the conjecture (1) for the annulus partition function, after normalising so that the coefficient of the $p = 0$ term (the contribution of the identity operator) is unity:

$$Z = q^{-\frac{c}{24}} \prod_{r=1}^{\infty}(1 - q^r)^{-1} \sum_{p \in \mathbb{Z}} \frac{\sin(p + 1)\chi}{\sin \chi} \, q^{\frac{gp^2}{4} - \frac{(1-g)p}{2}}.$$

   Note that all the coefficients in (1) are polynomials in $n = 2 \cos \chi$ as we expect. For $p = 1$ we get exactly n (the degeneracy factor for a single loop wrapping around the annulus), for $p = 2$ we get $n^2 - 1$, and so on.
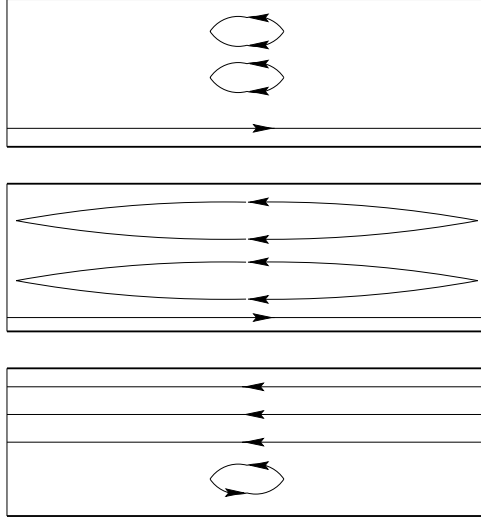
Fig. 2. Mechanism for the appearance of null states. In this example, an excited state in the $p = 1$ sector has just sufficient energy $2\pi/L$ to allow two marginally bound pairs of ± 2 vortices to form. These can then move to the ends of the rectangle, and one of them can then annihilate with the original flux line. The state is therefore equivalent to the ground state in the $p = -3$ sector, which has already been counted in (9) and which therefore must be subtracted off.

2.2. Modular Properties

Now we express (1) (for general χ') in terms of the conjugate modulus $\tilde{q} = e^{-2\pi L/\ell}$. Setting $q = e^{-\delta}$ we have

$$Z = Z_0 \, q^{\frac{1-c}{24}} (\sin \chi')^{-1} \operatorname{Im} e^{i\chi'} \sum_p e^{ip\chi'} e^{-\delta(\frac{gp^2}{4} - \frac{(1-g)p}{2})}.$$

Using the Poisson sum formula $\sum_p \to \sum_m \int dp \, e^{2\pi imp}$, the integral is

$$\int e^{-\frac{\delta g p^2}{4} + [\frac{(1-g)\delta}{2} + i\chi' + 2\pi im]p} dp$$

$$= (4\pi/\delta g)^{1/2} e^{-\frac{1}{\delta g}[\chi' + 2\pi m - i\frac{(1-g)\delta}{2}]^2}$$

$$= (4\pi/\delta g)^{1/2} e^{-\frac{1}{\delta g}(\chi' + 2\pi m)^2 + i(\chi' + 2\pi m)\frac{(1-g)}{g} + \frac{(1-g)^2 \delta}{2g}}.$$

The last term in the exponential cancels the $q^{(1-c)/24}$. Under a modular transformation

$$Z_0 = (\delta/2\pi)^{1/2} \tilde{q}^{-\frac{1}{12}} \prod_r (1 - \tilde{q}^{2r})^{-1},$$

so we end up with

$$Z = (2/g)^{1/2} \tilde{q}^{-\frac{c}{12}} \prod_r (1 - \tilde{q}^{2r})^{-1} \sum_m \frac{\sin((2m\chi + \chi')/g)}{\sin \chi'} \tilde{q}^{\frac{(\chi'+2\pi m)^2 - \chi^2}{2\pi^2 g}},$$

which finally simplifies to (2). Note that if we had not subtracted off the null states, as in (9), we would find $\widetilde{Z} \sim \tilde{q}^{-1/12}$ rather than $\tilde{q}^{-c/12}$. This is an example of how the null state subtraction is necessary to maintain the correct modular properties.[4]

The leading term as $\tilde{q} \to 0$, with $m = 0$, agrees with the exponent found in Ref. 18 for the case of loops wrapping around a long cylinder counted with weight $n' = 2 \cos \chi'$. We now have also the prefactor:

$$Z \sim (2/g)^{1/2} \frac{\sin(\chi'/g)}{\sin \chi'} \tilde{q}^{\frac{\chi'^2 - \chi^2}{2\pi^2 g}}. \tag{10}$$

If we set $\chi' = \chi$, the other exponents in (2) are those of *even* electric charge operators x_{2m}. For general g these are not in the Kac table, consistent with the fact that there is no explicit substraction of null states in (2), so the characters are simply given by the infinite product.

We now check (1) against some known cases.

3. COMPARISON WITH KNOWN RESULTS

3.1. *n* = 0 in the Dilute Regime

In this case $g = 3/2$, $\chi = -\pi/2$, for which we expect $Z = 1$, since loops wrapping round the annulus should all get a weight $n = 0$. (1) gives

$$Z = \prod_r (1 - q^r)^{-1} \sum_p \sin((p + 1)\pi/2) q^{\frac{3p^2}{8} + \frac{p}{4}}.$$

The prefactor is $+1$ if $p \equiv 0 \pmod 4$, -1 if $p \equiv 2 \pmod 4$, and zero otherwise. After a little algebra we get

$$Z = \frac{\sum_{k \in \mathbb{Z}} (q^{6k^2 + k} - q^{6k^2 + 5mk + 1})}{\prod_r (1 - q^r)}.$$

This is identically equal to 1 by Euler's pentagonal identity.

3.2. *n* = 1, Dilute Phase

This should correspond to the unitary CFT which describes the scaling limit of the critical Ising model. Now $\chi = -\pi/3$, $g = 4/3$. The numerator in (1) is

$$\sum_p \frac{\sin \frac{(p+1)\pi}{3}}{\sin \frac{\pi}{3}} q^{\frac{p^2}{3} + \frac{p}{6}}.$$

Now

$$p = 6k \text{ gives } q^{12k^2+k}$$

$$p = 6k - 2 : -q^{12k^2-7k+1}$$

$$p = 6k + 1 : q^{12k^2+5k+\frac{1}{2}}$$

$$p = 6k + 3 : -q^{12k^2+13k+\frac{7}{2}}.$$

Using the Rocha–Caridi character formula[19]

$$\chi_{r,s}(q) = \prod_r (1 - q^r)^{-1} \sum_{k\in\mathbb{Z}} \left(q^{\frac{(24k+4r-3s)^2-1}{48}} - \{s \to -s\} \right),$$

for the case $c = \frac{1}{2}$, we see that the first 2 terms give $\chi_{1,1}$ and the second pair give $\chi_{1,3}$. This then agrees with the result[9] for the Ising model with free boundary conditions

$$Z = \chi_{1,1} + \chi_{1,3}.$$

Alternatively we can look at the dual spins, which are fixed on the boundary. If they are fixed into the same state on both boundaries we must have p even, so that $Z = \chi_{1,1}$, and if they are fixed into opposite states p must be odd, so $Z = \chi_{1,3}$. These also agree with Ref. 9.

3.3. $n = 2$

In this case $\chi = 0$ and $g = 1$, so $\sin(p + 1)\chi / \sin \chi \to p + 1$. The numerator in (1) becomes

$$\sum_p (p + 1)q^{\frac{p^2}{4}} = \sum_{p\in\mathbb{Z}} q^{\frac{p^2}{4}}.$$

This agrees with the interpretation as the XY model at the Kosterlitz-Thouless transition: the terms with $p \neq 0$ correspond to a total vorticity $\pm p$ along the annulus.

3.4. $Q = 3$ Potts Model

Now $\chi = \pi/6$ and $g = \frac{5}{6}$. The numerator in (1) is

$$\sum_p \frac{\sin((p + 1)\pi/6)}{\sin \pi/6} q^{\frac{5p^2}{24} - \frac{p}{12}}.$$

If we take free boundary conditions on both boundaries we should restrict p to be even. Then

$$p = 12k : q^{30k^2-k}$$

$$p = 12k + 2 : 2q^{30k^2+9k+\frac{2}{3}}$$

$$p = 12k + 4 : q^{30k^2+19k+3}$$

$$p = 12k + 6 : -q^{30k^2+29k+7}$$

$$p = 12k - 4 : -2q^{30k^2-21k+frac113}$$

$$p = 12k - 2 : -q^{30k^2-11k+1}.$$

These pair up as follows: $((1, 6), (2, 5), (3, 4))$ to give

$$Z = \chi_{1,1} + 2\chi_{1,3} + \chi_{1,5},$$

which agrees with Ref. 9.

Note that if we choose free boundary conditions on one edge and fixed on the other, p is restricted to be odd, and the leading term as $q \to 0$ comes from $p = 1$, and is

$$Z \sim \sqrt{3}\, q^{\frac{1}{8}}.$$

The $\sqrt{3}$ is to be expected, because in the Fortuin-Kasteleyn representation each closed loop carries a factor $\sqrt{Q}$.

3.5. $n = Q = 1$, Dense Phase

In this case $\chi = \pi/3$, $g = 2/3$. The numerator in (1) is

$$\sum_p \frac{\sin((p + 1)\pi/3)}{\sin(\pi/3)}\, q^{\frac{p^2}{6}-\frac{p}{6}}.$$

We get a non-zero contribution in the following cases:

$$p = 6r : q^{6r^2-r}$$

$$p = 6r - 2 : -q^{6r^2-5r+1}$$

$$p = 6r + 1 : q^{6r^2+r}$$

$$p = 6r + 3 : -q^{6r^2+5r+1}.$$

Using Euler's identity again we see that $Z = 2$, consistent with the dual interpretation as the Ising model at zero temperature. (The factor 2 is due to the global spin reversal.) On the other hand this model can be interpreted as the $Q = 1$ Potts

model (percolation). Choosing the sites on both boundaries to be in the same Potts state enforces p to be even, and then we get $Z = 1$ as expected.

3.6. $n = Q = 0$, Dense Phase

Now $\chi = \pi/2$, $g = \frac{1}{2}$. The numerator in (1) is

$$\sum_p \sin(p+1)\pi/2 \, q^{\frac{p^2}{8} - \frac{p}{4}},$$

so p is even. For

$$p = 4k : q^{2k^2 - k}$$
$$p = 4k + 2 : -q^{2k^2 + k},$$

so

$$Z = \sum_k \left(q^{2k^2 - k} - q^{2k^2 + k} \right) = 0.$$

This is correct, since in this case there is just one macroscopic loop (or spanning tree) which is counted with weight $n = 0$.

4. SOME NEW RESULTS

4.1. Percolation

By setting $\cos \chi = 0$ in (1) with $g = \frac{2}{3}$ we suppress all other contributions with a non-zero number of loops wrapping around the annulus. In terms of percolation, this happens if and only if there exists a cluster connecting the two boundaries. This crossing probability is therefore

$$P = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \sum_p \sin((p+1)\pi/2) q^{\frac{p^2}{6} - \frac{p}{6}}$$

$$= \prod_{r=1}^{\infty} (1 - q^r)^{-1} \sum_{k \in \mathbb{Z}} \left(q^{\frac{8k^2}{3} - \frac{2k}{3}} - q^{\frac{8k^2}{3} + 2k + \frac{1}{3}} \right),$$

so that $1 - P \sim q^{1/3}$ as $q \to 0$. Using the Jacobi triple product formula this can be written in terms of the Dedekind function $\eta(\tau) \equiv q^{1/24} \prod_{r=1}^{\infty} (1 - q^r)$ with $q = e^{-2\pi i/\tau}$ as

$$P = \frac{\eta(-1/3\tau)\eta(-4/3\tau)}{\eta(-1/\tau)\eta(-2/3\eta)} = (3/2)^{1/2} \frac{\eta(3\tau)\eta(3\tau/4)}{\eta(\tau)\eta(3\eta/2)}.$$

In the opposite limit, using (11) or the above, we have

$$P \sim (3/2)^{1/2} \tilde{q}^{\frac{5}{48}},$$

as $\tilde{q} \to 0$, which is the well-known 'magnetic' exponent[1] for the $Q = 1$ Potts model (also known as the 1-arm exponent[20] in the SLE literature.) Note that this result is different from, and much larger than, the result found in Ref. 17. This is because in that paper crossing clusters which also wrap around the annulus were disallowed. It would be interesting to compare the above result with the implicit formula derived by Dubédat[21] using SLE methods.

Note that, in principle, one can solve for $e^{i\chi'}$ as a function of n' and substitute in (1), hence obtaining the complete generating function for the probabilities that a given number of clusters wrap around the annulus.

4.2. Self-Avoiding Loop: Dilute Case

If we take the $O(n')$ term in (1) with $g = \frac{3}{2}$ we obtain the partition function Z_1 for a single self-avoiding loop which wraps around the annulus. From (1) we need

$$\frac{\partial}{\partial n'} \left. \frac{\sin((p+1)\chi')}{\sin \chi'} \right|_{\chi'=-\pi/2} = -\frac{1}{2}(p+1)\cos((p+1)\pi/2).$$

So we get a non-zero result only when p is odd, say $p = 2k - 1$, whence, after a little algebra,

$$Z_1 = \prod_{r=1}^{\infty} (1 - q^r)^{-1} \sum_k k \in \mathbb{Z} k(-1)^{k-1} q^{\frac{3k^2}{2} - k + \frac{1}{8}}. \tag{11}$$

The leading behaviour as $q \to 0$ comes from $k = 1$ and is

$$Z_1 \sim q^{\frac{5}{8}},$$

as expected.

In the opposite limit we can use (10). In this case the leading behaviour comes from differentiating the exponent:

$$Z_1 \sim \frac{1}{2}(2/g)^{1/2} \frac{\sin(\chi/g)}{\sin \chi} \frac{\chi}{g\pi^2} \ln \tilde{q} = \frac{1}{6\pi} |\ln \tilde{q}|.$$

If the annulus is mapped into the region between two circles radii r_1 and $r_2 > r_1$, the last factor is just $\ln(r_2/r_1)$.

4.3. Self-Avoiding Loop: Dense Phase

In this case

$$Z_1 = -q^{-c/24} \prod_r (1 - q^r)^{-1} \sum_p \frac{p+1}{2} \cos((p+1)\pi/2) \, q^{\frac{p^2}{8} - \frac{p}{4}}.$$

If we let $p \to 2 - p$ we get the same expression except $(p + 1) \to (3 - p)$. So the sum is

$$-\sum_p \cos((p + 1)\pi/2) \, q^{\frac{p^2}{8} - \frac{p}{4}},$$

and finally (since $c = -2$)

$$Z_1 = q^{1/12} \prod_r (1 - q^r)^{-1} \sum_k \left(q^{2k^2 - \frac{1}{8}} - q^{2k^2 - 2k + \frac{3}{8}} \right).$$

Using the Jacobi triple product formula this can be rewritten as

$$Z_1 = q^{-\frac{1}{24}} \prod_{m=1}^{\infty} \left(1 - q^{m - \frac{1}{2}} \right)^2.$$

The leading term as $q \to 0$ is

$$Z_1 \sim q^{-\frac{1}{24}}.$$

This is reasonable since the loop is weighted by a factor $\mu^{-\text{length}}$ so the contribution grows exponentially with ℓ.

4.4. Logarithmic Cases

Logarithmic CFTs have been studied for some time, although the question of how they satisfy modular invariance has not been resolved in general. [22] The limit $n \to 0$ of the O(n) model affords an example in both the dilute and dense regimes; other examples have been discussed in Ref. 12. If we differentiate the whole expression for Z wrt n at $n = 0$, we get 3 kinds of contribution: the first comes from differentiating wrt χ': this gives the partition function for loops wrapping round the annulus as found above, and is a regular series in q. The second comes from differentiating $q^{-c/24}$ and gives $-(c'(0)/24)\ln q$ times the usual partition function at $n = 0$. The third comes from differentiating the exponents: using

$$\frac{\partial}{\partial g} \left(\frac{g p^2}{4} + \frac{(g-1)p}{2} \right) = \frac{p^2}{4} + \frac{p}{2},$$

the result, in the dilute case when $g = \frac{3}{2}$, is proportional to

$$(\ln q) \sum_p (p^2 + 2p) \sin((p+1)\pi/2) \, q^{\frac{3p^2}{8} + \frac{p}{4}}.$$

As before, the only contributions come from p even, and proceeding as before we get

$$\ln q \, \sum_k \left(k(2k+1)q^{6k^2+k} - k(2k-1)q^{6k^2-5k+1} \right),$$

or

$$Z_{\log} \propto \ln q \, \sum_k k(2k+1)\left(q^{6k^2+k} - q^{6k^2+5k+1} \right).$$

Note that we still get the null state structure in this logarithmic sector.

In the dense phase, the partition function vanishes. The non-logarithmic terms have already been evaluated. The logarithmic term is very similar to the above:

$$\ln q \, \sum_k \left(k(2k+1)q^{2k^2+k} - k(2k-1)q^{2k^2-3k+1} \right),$$

or

$$Z_{\log} \propto \ln q \, \sum_k k(2k+1)\left(q^{2k^2+k} - q^{2k^2+3k+1} \right).$$

once again showing the null states. The leading term as $q \to 0$ gives the contribution of single dense loops (or spanning trees) which do not wrap around the annulus: note that this is much smaller than the $O(q^{-1/24})$ contribution from those which do.

5. SUMMARY AND FURTHER REMARKS

In this paper we have presented an explicit result for scaling limit of the partition function of the critical O(*n*) and *Q*-state Potts models on the annulus. Our formalism makes it simple to count loops which wrap around the annulus with different weights, leading potentially to many new formula for crossing probabilities in percolation and for self-avoiding loops, some of which have been presented here.

The electric-magnetic dual of our arguments for a long ($\ell \gg L$) annulus can in fact be applied to a long ($L \gg \ell$) cylinder, to give an alternative derivation of the usual relation between g, χ and n. For a long annulus we argued that ± 2 magnetic charges (vortices) would rearrange themselves in such a way as to induce mean magnetic charges $\pm m_0 = \pm(1-g)/g$ at the ends of a long rectangle, leading to a net magnetic flux around the annulus. For a long cylinder, one may similarly

argue that the ± 2 electric charges in the model also rearrange themselves to give net electric charges $\pm e_0 = \pm(1 - g)$ at either end. These can then be interpreted as counting loops going around the cylinder with the weight $n = 2 \cos \pi e_0$.

Like all Coulomb gas methods, however, the arguments are somewhat heuristic, although they lead to completely explicit formulae, and, because of the complex weights, it seems hard to make them rigorous. In particular, although we have computed the partition function, it is by no means clear that the same ensemble can be used to compute correlation functions in the original model. It would be nice to see a direct connection with the other 'Coulomb gas' approach which has been employed in CFT, namely that originally developed by Dotsenko and Fateev.[23] This is essentially a way of constructing holomorphic conformal blocks using modified free field theory. However, it has many features in common with the Coulomb gas construction used here: the background charge m_0, and the marginal screening operators, which in our case are the ± 2 vortices. In the bulk, it is still necessary, in the Dotsenko-Fateev approach, to sew together the holomorphic and anti-holomorphic blocks in a consistent way to obtain correlation functions, but in boundary CFT the correlation functions are linear combinations of the conformal blocks (specialised to real values of their arguments), and so the correpondence between the two approaches should be more direct. It would, of course, be important to establish any of these results rigorously, for example by using SLE methods.

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A. BOUNDARY TERMS IN THE GAUSSIAN MODEL

Consider the action

$$S = S_0 + S_1 = \frac{g}{4\pi} \int (\nabla h)^2 dx\, dy + \alpha_1 \int \partial_y h(x, y = 0)\, dx$$

$$+ \alpha_2 \int \partial_y h(x, y = L)\, dx.$$

We wish to compute the regularised free energy, or equivalently the ground state energy E_0 of the associated hamiltonian. Let

$$h(x, y) = \sum_{n=1}^{\infty} \frac{f_n(x)}{(gL/4\pi)^{1/2}} \sin \frac{n\pi y}{L}.$$

Then

$$S_0 = \int dx \sum_n \left(\frac{1}{2} \dot{f}_n^2 + \frac{1}{2} (n\pi/L)^2 f_n^2 \right),$$

from which we read off the ground state energy

$$E_0 = \frac{1}{2} \sum_n \omega(n\pi/L),$$

where $\omega(k) = k$. We can regularise this sum either by modifying the dispersion relation (eg using a lattice, in which case $\omega(k) = 2\sin(k/2)$) and using the Euler-Maclaurin formula, or using zeta-function, in which case we get the standard result

$$E_0 = \frac{\pi}{2L} \zeta(-1) = -\frac{\pi}{24L},$$

corresponding to $c = 1$.

Now add in

$$S_1 = (4\pi/gL)^{1/2} \sum_n (n\pi/L)(\alpha_1 + (-1)^n \alpha_2) f_n.$$

Completing the square, the contribution of the nth mode can be written

$$\frac{1}{2}((n\pi/L)f_n + (4\pi/gL)^{1/2}(\alpha_1 + (-1)^n \alpha_2))^2 - \frac{2\pi}{gL}(\alpha_1 + (-1)^n \alpha_2)^2,$$

so the change in the ground state energy is

$$E_1 = -(2\pi/gL) \left(\sum_{n \text{odd}} (\alpha_1 - \alpha_2)^2 + \sum_{n \text{even}} (\alpha_1 + \alpha_2)^2 \right).$$

If we use zeta function regularisation we have

$$\sum_{n \text{odd}} 1 = \lim_{s \to 0} \left(\sum_n n^{-s} - \sum_n (2n)^{-s} \right) = \lim_{s \to 0} (1 - 2^{-s}) \zeta(s) = 0$$

$$\sum_{n \text{even}} 1 = \lim_{s \to 0} \sum_n (2n)^{-s} = \zeta(0) = -\frac{1}{2},$$

so

$$E_1 = \frac{\pi}{gL}(\alpha_1 + \alpha_2)^2. \tag{1}$$

If we use a lattice dispersion relation and also replace the boundary derivatives by finite differences, the sum becomes

$$\sum_{n=1}^{L-1} \frac{(\alpha_1 + (-1)^n \alpha_2)^2}{\cos^2(n\pi/2L)},$$

on which we use the formulae

$$\sum_{n\,\mathrm{odd}} f(n/L) = \frac{1}{2}L \int_0^1 f(x)\,dx + O(L^{-2}),$$

$$\sum_{n\,\mathrm{even}} f(n/L) = \frac{1}{2}L \int_0^1 f(x)\,dx - \frac{1}{2L}f(0) + O(L^{-2}),$$

giving the same result. It can also be verified by writing, for a general position dependent α

$$Z = Z_0 \left\langle \exp\left(\int \alpha(l)\partial_\perp h(l)dl\right)\right\rangle = Z_0 \exp\left(\frac{1}{2}\iint \alpha(l)\alpha(l')\partial_\perp\partial'_\perp G(l,l')dl dl'\right),$$

where G is the Green's function for the free field with Dirichlet boundary conditions.

(2) leads to a modification to the effective central charge

$$c = 1 - \frac{24}{g}(\alpha_1 + \alpha_2)^2,$$

as claimed in Sec. 2.

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