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Propagation of Correlations in Quantum Lattice Systems

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We provide a simple proof of the Lieb-Robinson bound and use it to prove the existence of the dynamics for interactions with polynomial decay. We then use our results to demonstrate that there is an upper bound on the rate at which correlations between observables with separated support can accumulate as a consequence of the dynamics.

KEY WORDS: Lieb-Robinson bounds, quantum spin systems, correlations

1. INTRODUCTION

Recently, there has been increasing interest in understanding correlations in quantum lattice systems prompted by applications in quantum information theory and computation(2,3,4,10) and the study of complex networks(5). The questions that arise in the context of quantum information and computation are sufficiently close to typical problems in statistical mechanics that the methods developed in one framework are often relevant in the other. The bound on the group velocity in quantum spin dynamics generated by a short-range Hamiltonian, which was proved by Lieb and Robinson more than three decades ago(8), is a case in point. For example, as explained in Ref. 2, the Lieb-Robinson bound provides an upper bound on the speed of information transmission through channels modeled by a quantum lattice systems with short-range interactions.

The Lieb-Robinson bound plays a crucial role in the derivation of several recent results. For some of these results it was useful, indeed necessary, to generalize

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and sharpen these bounds. Several such improvements have recently appeared[9,6]. In this paper we provide a new proof of the Lieb-Robinson bound (Theorem 2.1) and other estimates based on a norm-preserving property of the dynamics (see Lemma A.1). We apply this result to give upper bounds on the rate at which correlations can be established between two separated regions in the lattice for a general class of models (Theorem 2.2). Moreover, our bounds allow us to prove the existence of the dynamics (Theorem 2.2), in the sense of a strongly continuous group of automorphisms on the algebra of quasi-local observables for a larger class of interactions than was previously known[1,11,7]. In particular, the new condition (see (1.7) with $a = 0$) does not include an exponential penalty on multi-body interactions. It is also of interest to note that existence of the dynamics immediately implies the equivalence of different conditions for thermal equilibrium for this larger class of interactions, such as the KMS condition and the variational principle (see (Ref. 1 Theorem 6.2.36)) and the KMS condition and the auto-correlation lower bounds, also called entropy-energy inequalities (see (Ref. 1 Theorem 5.3.15)).

1.1. The Set Up

We will be considering quantum spins systems defined over a set of vertices Λ equipped with a metric d. A finite dimensional Hilbert space $\mathcal{H}_x$ is assigned to each vertex $x \in \Lambda$. In the most common cases Λ is a graph, and the metric is given by the graph distance, $d(x, y)$, which may be the length of the shortest path of edges connecting x and y in the graph.

For any finite subset $X \subset \Lambda$, the Hilbert space associated with X is the tensor product $\mathcal{H}_X = \bigotimes_{x\in X} \mathcal{H}_x$, and the set of corresponding observables supported in X is denoted by $\mathcal{A}_X = \mathcal{B}(\mathcal{H}_X)$, the bounded linear operators over $\mathcal{H}_X$. These local observables form an algebra, and with the natural embedding of $\mathcal{A}_{X_1}$ in $\mathcal{A}_{X_2}$ for any $X_1 \subset X_2$, one can define the C^*-algebra of all observables, $\mathcal{A}$, as the norm completion of the union of all local observable algebras $\mathcal{A}_X$ for finite $X \subset \Lambda$.

An interaction is a map Φ from the set of subsets of Λ to $\mathcal{A}$ with the property that $\Phi(X) \in \mathcal{A}_X$ and $\Phi(X) = \Phi(X)^*$ for all finite $X \subset \Lambda$. A quantum spin model is then defined to be the Hamiltonian, expressed in terms of its interaction, given by

$$H_\Phi := \sum_{X\subset\Lambda} \Phi(X). \tag{1.1}$$

For notational convenience, we will often drop the dependence of H_Φ on Φ.

The dynamics, or time evolution, of a quantum spin model is the one-parameter group of automorphisms, $\{\tau_t\}_{t\in\mathbb{R}}$, defined by

$$\tau_t(A) = e^{itH} A e^{-itH}, \quad A \in \mathcal{A}, \tag{1.2}$$

which is always well defined for finite sets Λ. In the context of infinite systems, a boundedness condition on the interaction is required in order for the finite-volume dynamics to converge to a strongly continuous one-parameter group of automorphisms on $\mathcal{A}$.

To describe the interactions we wish to consider in this article, we first put a condition on the set Λ; which is only relevant in the event that Λ is infinite. We assume that there exists a non-increasing function $F\colon [0, \infty) \to (0, \infty)$ for which:

i) F is uniformly integrable over Λ, i.e.,

$$\| F \| \; := \; \sup_{x \in \Lambda} \sum_{y \in \Lambda} F(d(x, y)) \; < \; \infty, \tag{1.3}$$

and

ii) F satisfies

$$C \; := \; \sup_{x, y \in \Lambda} \sum_{z \in \Lambda} \frac{F\,(d(x, z))\; F\,(d(z, y))}{F\,(d(x, y))} \; < \; \infty. \tag{1.4}$$

Given a set Λ equipped with a metric d, it is easy to see that if F satisfies i) and ii) above, then for any $a \geq 0$ the function

$$F_a(x) := e^{-ax} \, F(x), \tag{1.5}$$

also satisfies i) and ii) with $\|F_a\| \leq \|F\|$ and $C_a \leq C$.

As a concrete example, take $\Lambda = \mathbb{Z}^d$ and $d(x, y) = |x - y|$. In this case, one may take the function $F(x) = (1 + x)^{-d-\epsilon}$ for any $\epsilon > 0$. Clearly, (1.3) is satisfied, and a short calculation demonstrates that (1.4) holds with

$$C \; \leq \; 2^{d+\epsilon+1} \sum_{n \in \mathbb{Z}^d} \frac{1}{(1 + |n|)^{d+\epsilon}}. \tag{1.6}$$

We also observe that, although the *purely exponential* function $G(x) = e^{-ax}$, is integrable for $a > 0$, i.e., it satisfies i), it does not satisfy ii). This is evident from the fact that the cardinality of the set $\{z \in \mathbb{Z}^d : |x - z| + |z - y| - |x - y| = 0\}$ is proportional to $|x - y|$, and therefore, there exists no constant C uniform in $|x - y|$.

To any set Λ for which there exists a function F satisfying i) and ii) above, we define the set $\mathcal{B}_a(\Lambda)$ to be those interactions Φ on Λ which satisfy

$$\|\Phi\|_a \; := \; \sup_{x, y \in \Lambda} \sum_{X \ni x, y} \frac{\|\Phi(X)\|}{F_a\,(d(x, y))} \; < \; \infty. \tag{1.7}$$

2. LIEB-ROBINSON ESTIMATES AND EXISTENCE THE DYNAMICS

2.1. Lieb-Robinson Bounds

We present a variant of the Lieb-Robinson result which was first proven in Refs. 6 and 9. An important difference of the new bound with respect to the original bound in Refs. 1 and 8 and the one in Ref. 9 is that the norm on the interaction previously used required a uniform bound on the dimension of the single-site Hilbert space. Another derivation of this result can be found in [(6)]. For the convenience of the reader we provide a proof here.

Theorem 2.1. (Lieb-Robinson Bound) *Let $a \geq 0$ and take $\Lambda_1 \subset \Lambda$ a finite subset. Denote by $\tau_t^{\Lambda_1}$ the time evolution corresponding to a Hamiltonian*

$$H := \sum_{X \subset \Lambda_1} \Phi(X) \tag{2.1}$$

defined in terms of an interaction $\Phi \in \mathcal{B}_a(\Lambda)$. There exists a function $g_a : \mathbb{R} \to [0, \infty)$ with the property that, given any pair of local observable $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $X, Y \subset \Lambda_1$, one may estimate

$$\left\| \left[\tau_t^{\Lambda_1}(A), B \right] \right\| \leq \frac{2\,\|A\|\,\|B\|}{C_a} g_a(t) \sum_{x \in X} \sum_{y \in Y} F_a\left(d(x, y)\right), \tag{2.2}$$

for any $t \in \mathbb{R}$. Here the function

$$g_a(t) = \begin{cases} (e^{2\,\|\Phi\|_a\,C_a\,|t|} - 1) & \text{if } d(X, Y) > 0, \\ e^{2\,\|\Phi\|_a\,C_a\,|t|} & \text{otherwise.} \end{cases} \tag{2.3}$$

Proof: Consider the function $f : \mathbb{R} \to \mathcal{A}$ defined by

$$f(t) := \left[\tau_t^{\Lambda_1}(A), B \right]. \tag{1.4}$$

Clearly, f satisfies the following differential equation

$$f'(t) = i \left[f(t), \tau_t^{\Lambda_1}(H_X) \right] + i \left[\tau_t^{\Lambda_1}(A), \left[\tau_t^{\Lambda_1}(H_X), B \right] \right], \tag{1.5}$$

where we have used the notation

$$H_Y = \sum_{\substack{Z \subset \Lambda_1: \\ Z \cap Y \neq \emptyset}} \Phi(Z), \tag{1.6}$$

for any subset $Y \subset \Lambda_1$. The first term in (1.5) above is norm-preserving, and therefore the inequality

$$\| \left[\tau_t^{\Lambda_1}(A), B \right] \| \leq \|[A, B]\| + 2\|A\| \int_0^{|t|} \| \left[\tau_s^{\Lambda_1}(H_X), B \right] \| \, ds \tag{2.7}$$

follows immediately from Lemma A.1 and the automorphism property of $\tau_t^{\Lambda_1}$. If we further define the quantity

$$C_B(X,t) := \sup_{A\in\mathcal{A}_X} \frac{\left\|\left[\tau_t^{\Lambda_1}(A),\, B\right]\right\|}{\|A\|}, \tag{2.8}$$

then (2.7) implies that

$$C_B(X,t) \leq C_B(X,0) + 2 \sum_{\substack{Z\subset\Lambda_1:\\ Z\cap X\neq\emptyset}} \|\Phi(Z)\| \int_0^{|t|} C_B(Z,s)ds. \tag{2.9}$$

Clearly, one has that

$$C_B(Z,0) \ \leq\ 2\,\|B\|\,\delta_Y(Z), \tag{2.10}$$

where $\delta_Y(Z) = 0$ if $Z\cap Y = \emptyset$ and $\delta_Y(Z) = 1$ otherwise. Using this fact, one may iterate (2.9) and find that

$$C_B(X,t) \ \leq\ 2\|B\| \sum_{n=0}^{\infty} \frac{(2|t|)^n}{n!} a_n, \tag{2.11}$$

where

$$a_n \ =\ \sum_{\substack{Z_1\subset\Lambda_1:\\ Z_1\cap X\neq\emptyset}} \ \sum_{\substack{Z_2\subset\Lambda_1:\\ Z_2\cap Z_1\neq\emptyset}} \cdots \sum_{\substack{Z_n\subset\Lambda_1:\\ Z_n\cap Z_{n-1}\neq\emptyset}} \prod_{i=1}^{n} \|\Phi(Z_i)\|\,\delta_Y(Z_n). \tag{2.12}$$

For an interaction $\Phi\in\mathcal{B}_a(\Lambda)$, one may estimate that

$$a_1 \ \leq\ \sum_{x\in X}\sum_{y\in Y}\sum_{Z\ni x,y} \|\Phi(Z)\| \ \leq\ \|\Phi\|_a \sum_{x\in X}\sum_{y\in Y} F_a\left(d(x,y)\right). \tag{2.13}$$

In addition,

$$\begin{aligned} a_2 &\leq \sum_{x\in X}\sum_{y\in Y}\sum_{z\in\Lambda_1} \sum_{\substack{Z_1\subset\Lambda_1:\\ Z_1\ni x,z}} \|\Phi(Z_1)\| \sum_{\substack{Z_2\subset\Lambda_1:\\ Z_2\ni z,y}} \|\Phi(Z_2)\| \\ &\leq \|\Phi\|_a^2 \sum_{x\in X}\sum_{y\in Y}\sum_{z\in\Lambda} F_a\left(d(x,z)\right)\ F_a\left(d(z,y)\right) \\ &\leq \|\Phi\|_a^2\, C_a \sum_{x\in X}\sum_{y\in Y} F_a\left(d(x,y)\right), \end{aligned} \tag{2.14}$$

using (1.4). With analogous arguments, one finds that

$$a_n \ \leq\ \|\Phi\|_a^n\, C_a^{n-1} \sum_{x\in X}\sum_{y\in Y} F_a\left(d(x,y)\right). \tag{2.15}$$

Inserting (2.15) into (2.11) we see that

$$C_B(X,t) \leq \frac{2\,\|B\|}{C_a} \exp\left[2\,\|\Phi\|_a\, C_a\, |t|\right] \sum_{x \in X} \sum_{y \in Y} F_a\left(d(x,y)\right), \tag{2.16}$$

from which (2.2) immediately follows.

In the event that $d(X,Y) > 0$, one has that $C_B(X,0) = 0$. For this reason the term corresponding to $a_0 = 0$, and therefore, the bound derived in (2.16) above holds with $e^{2\|\Phi\|_a C_a |t|}$ replaced by $e^{2\|\Phi\|_a C_a |t|} - 1$. □

We note that, for fixed local observables A and B, the bounds above are independent of the volume $\Lambda_1 \subset \Lambda$.

In the event that $\Phi \in \mathcal{B}_a(\Lambda)$ for some $a > 0$, then the bound in (2.2) implies that

$$\left\| \left[\tau_t^{\Lambda_1}(A), B \right] \right\| \;\leq\; \frac{2\,\|A\|\,\|B\|}{C_a}\, \|F\| \,\min(|X|,|Y|)\, e^{-a\left[d(X,Y) - \frac{2\|\Phi\|_a C_a}{a}|t|\right]}, \tag{2.17}$$

which corresponds to a velocity of propagation given by

$$V_\Phi := \inf_{a>0} \frac{2\|\Phi\|_a C_a}{a}. \tag{2.18}$$

We further note that the bounds in (2.2) and (2.17) above only require that one of the observables have finite support; in particular, if $|X| < \infty$ and $d(X,Y) > 0$, then the bounds are valid irrespective of the support of B.

One can also view the Lieb-Robinson bound as a means of localizing the dynamics. Let Λ be finite and take $X \subset \Lambda$. Denote by $X^c = \Lambda \setminus X$. For any observable $A \in \mathcal{A}_\Lambda$ set

$$\langle A \rangle_{X^c} := \int_{\mathcal{U}(X^c)} U^* A U \,\mu(dU), \tag{2.19}$$

where $\mathcal{U}(X^c)$ denotes the group of unitary operators over the Hilbert space $\mathcal{H}_{X^c}$ and μ is the associated normalized Haar measure. It is easy to see that for any $A \in \mathcal{A}_\Lambda$, the quantity $\langle A \rangle_{X^c} \in \mathcal{A}_X$ and the difference

$$\langle A \rangle_{X^c} \;-\; A \;=\; \int_{\mathcal{U}(X^c)} U^* \left[A, U\right] \,\mu(dU). \tag{2.20}$$

We can now combine these observations with the Lieb-Robinson bounds we have proven. Let $A \in \mathcal{A}_X$ be a local observable, and choose $\epsilon \geq 0$, $a > 0$, and an interaction $\Phi \in \mathcal{B}_a(\Lambda)$. We will denote by

$$B_t(\epsilon) \;=\; B(A,t,\epsilon) \;:=\; \left\{ x \in \Lambda \;:\; d(x,X) \;\leq\; \frac{2\|\Phi\|_a C_a}{a}\,|t| \;+\; \epsilon \right\}, \tag{2.21}$$

the ball centered at X with radius as specified above. For any $U \in \mathcal{U}(B_t^c(\epsilon))$, we clearly have that

$$d\,(X, \mathrm{supp}(U)) \;\geq\; \frac{2\|\Phi\|_a C_a}{a}\,|t| \,+\epsilon, \tag{2.22}$$

and therefore, using (2.20) above, we immediately conclude that

$$\begin{aligned} \left\| \,\tau_t(A) \,-\, \langle \tau_t(A) \rangle_{B_t^c(\epsilon)} \,\right\| &\leq \int_{\mathcal{U}(B_t^c(\epsilon))} \| \,[\tau_t(A), U]\,\| \;\mu(dU) \\ &\leq \frac{2\,\|A\|\,|X|}{C_a}\,\|F\|\, e^{-a\epsilon}, \end{aligned} \tag{2.23}$$

where for the final estimate we used (2.17).

2.2. Existence of the Dynamics

As is demonstrated in Ref. 1, one can use a Lieb-Robinson bound to establish the existence of the dynamics for interactions $\Phi \in \mathcal{B}_a(\Lambda)$. In the following we consider the thermodynamic limit over a increasing exhausting sequence of finite subsets $\Lambda_n \subset \Lambda$.

Theorem 2.2. *Let $a \geq 0$, and $\Phi \in \mathcal{B}_a(\Lambda)$. The dynamics $\{\tau_t\}_{t\in\mathbb{R}}$ corresponding to Φ exists as a strongly continuous, one-parameter group of automorphisms on $\mathcal{A}$. In particular,*

$$\lim_{n\to\infty} \left\| \tau_t^{\Lambda_n}(A) - \tau_t(A) \right\| = 0 \tag{2.24}$$

for all $A \in \mathcal{A}$. The convergence is uniform for t in compact sets and independent of the choice of exhausting sequence $\{\Lambda_n\}$.

Proof: Let $n > m$. Then, $\Lambda_m \subset \Lambda_n$. It is easy to verify that for any local observable $A \in \mathcal{A}_Y$,

$$\tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \;=\; \int_0^t \frac{d}{ds}\left(\tau_s^{\Lambda_n} \tau_{t-s}^{\Lambda_m}(A) \right) ds, \tag{2.25}$$

and therefore

$$\left\| \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \right\| \;\leq\; \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X\ni x} \int_0^{|t|} \left\| \left[\, \Phi(X),\; \tau_s^{\Lambda_m}(A) \right] \right\| \,ds. \tag{2.26}$$

Applying Theorem 2.1, we see that the right hand side of (2.26) is bounded from above by

$$2\,\|A\| \int_0^{|t|} g_a(s)ds \sum_{x\in\Lambda_n\setminus\Lambda_m} \sum_{X\ni x} \|\Phi(X)\| \sum_{z\in X} \sum_{y\in Y} F_a\left(d(z,y)\right). \tag{2.27}$$

Rewriting the sum on $X \ni x$ and $y \in X$ as the sum on $y \in \Lambda$ and $X \ni x, y$, one finds that

$$\left\| \tau_t^{\Lambda_n}(A) - \tau_t^{\Lambda_m}(A) \right\| \leq 2 \, \|A\| \, \|\Phi\|_a \, C_a \int_0^{|t|} g_a(s) ds \sum_{x \in \Lambda_n \setminus \Lambda_m} \sum_{z \in Y} F_a\left(d(x, z)\right)$$
$$\leq 2 \, \|A\| \, \|\Phi\|_a \, C_a \int_0^{|t|} g_a(s) ds \, |Y| \sup_{z \in Y} \sum_{x \in \Lambda_n \setminus \Lambda_m} F_a\left(d(x, z)\right). \tag{2.28}$$

As $m, n \to \infty$, the above sum goes to zero. This proves that the sequence is Cauchy and hence convergent. The remaining claims follow as in Theorem 6.2.11 of Ref. 1. □

3. GROWTH OF SPATIAL CORRELATIONS

The goal of this section is to prove Theorem 3.1 below which bounds the rate at which correlations can accumulate, under the influence of the dynamics, starting from a product state.

3.1. The Main Result

Let Ω be a normalized product state, i.e. $\Omega = \bigotimes_{x \in \Lambda} \Omega_x$, where for each x, Ω_x is a state (not necessarily pure) for the systems at site x. We will denote by $\langle \cdot \rangle$ the expectation with respect to Ω, and prove

Theorem 3.1. *Let $a \geq 0$, $\Phi \in \mathcal{B}_a(\Lambda)$, and take Ω to be a normalized product state as described above. Given $X, Y \subset \Lambda$ with $d(X, Y) > 0$ and local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, one has that*

$$| \langle \tau_t(AB) \rangle - \langle \tau_t(A) \rangle \langle \tau_t(B) \rangle | \leq 4 \, \|A\| \, \|B\| \, \|F\| \, (|X| + |Y|) \, G_a(t) \, e^{-ad(X,Y)}, \tag{3.1}$$

Here

$$G_a(t) = \frac{C_a + \|F_a\|}{C_a} \|\Phi\|_a \int_0^{|t|} g_a(s) \, ds, \tag{3.2}$$

and g_a is the function which arises in the Lieb-Robinson estimate Theorem 2.1.

In the event that $a = 0$, the bound above does not decay. However, the estimate (3.24) below, which does decay, is valid. Moreover, a straight forward application of the techniques used below also provides estimates on the increase of correlations, due to the dynamics, for non-product states.

We begin by writing the interaction Φ as the sum of two terms, one of which decouples the interactions between observables supported near X and Y.

3.1.1. Decoupling the Interaction

Consider two separated local observables, i.e., $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $d(X, Y) > 0$. Let

$$S_{A,B} := \left\{ y \in \Lambda \, : \, d(y, X) \leq \frac{d(X, Y)}{2} \right\}, \tag{3.3}$$

denote the ball centered at X with distance $d(X, Y)/2$ from Y. For any $\Phi \in \mathcal{B}_a(\Lambda)$, write

$$\Phi = \Phi(1 - \chi_{A,B}) + \Phi\chi_{A,B} =: \Phi_1 + \Phi_2, \tag{3.4}$$

where for any $Z \subset \Lambda$

$$\chi_{A,B}(Z) := \begin{cases} 1 \text{ if } Z \cap S_{A,B} \neq \emptyset \text{ and } Z \cap S_{A,B}^c \neq \emptyset, \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{3.5}$$

In this case, one has

Lemma 3.2 *Let $a \geq 0$, $\Phi \in \mathcal{B}_a(\Lambda)$, and consider any two separated local observables $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$ with $d(X, Y) > 0$. Writing $\Phi = \Phi_1 + \Phi_2$, as in (3.4), one may show that*

$$\int_0^{|t|} \| [H_2, \tau_s^{(1)}(O)] \| \, ds \leq 2 \, \|O\| \, G_a(t) \sum_{o \in \mathrm{supp}(O)} \sum_{\substack{x \in \Lambda: \\ 2d(x,o) \geq d(X,Y)}} F_a\left(d(x, o)\right), \tag{3.6}$$

is valid for observables $O \in \{A, B\}$. One may take

$$G_a(t) = \frac{C_a + \|F_a\|}{C_a} \, \|\Phi\|_a \int_0^{|t|} g_a(s) \, ds, \tag{3.7}$$

where g_a is the function from Theorem 2.1.

Proof: For $O \in \{A, B\}$ and $s > 0$,

$$\| [H_2, \tau_s^{(1)}(O)] \| \leq \sum_{\substack{Z \subset \Lambda: \\ Z \cap S_{A,B} \neq \emptyset, Z \cap S_{A,B}^c \neq \emptyset}} \| [\Phi(Z), \tau_s^{(1)}(O)] \|, \tag{3.8}$$

as is clear from the definition of $\chi_{A,B}$; see (3.5). Applying Theorem 2.1 to each term above, we find that

$$\| \left[\Phi(Z), \tau_s^{(1)}(O) \right] \| \leq \frac{2 g_a(s) \|O\| \|\Phi(Z)\|}{C_a} \sum_{z \in Z} \sum_{o \in \text{supp}(O)} F_a(d(z,o)). \quad (3.9)$$

One may estimate the sums which appear above as follows:

$$\begin{aligned} \sum_{\substack{Z \subset \Lambda: \\ Z \cap S_{A,B} \neq \emptyset, Z \cap S_{A,B}^c \neq \emptyset}} \sum_{z \in Z} &= \sum_{\substack{Z \subset \Lambda: \\ Z \cap S_{A,B} \neq \emptyset, Z \cap S_{A,B}^c \neq \emptyset}} \left(\sum_{\substack{z \in Z: \\ z \in S_{A,B}}} + \sum_{\substack{z \in Z: \\ z \in S_{A,B}^c}} \right) \\ &\leq \sum_{z \in S_{A,B}} \sum_{x \in S_{A,B}^c} \sum_{Z \ni z,x} + \sum_{z \in S_{A,B}^c} \sum_{x \in S_{A,B}} \sum_{Z \ni z,x}, \end{aligned} \quad (3.10)$$

and therefore, we have the bound

$$\int_0^{|t|} \| \left[H_2, \tau_s^{(1)}(O) \right] \| \, ds \leq \frac{2\|O\|}{C_a} (S_1 + S_2) \int_0^{|t|} g_a(s) ds, \quad (3.11)$$

where

$$S_1 = \sum_{z \in S_{A,B}} \sum_{x \in S_{A,B}^c} \sum_{Z \ni z,x} \|\Phi(Z)\| \sum_{o \in \text{supp}(O)} F_a(d(z,o)) \quad (3.12)$$

and

$$S_2 = \sum_{z \in S_{A,B}^c} \sum_{x \in S_{A,B}} \sum_{Z \ni z,x} \|\Phi(Z)\| \sum_{o \in \text{supp}(O)} F_a(d(z,o)). \quad (3.13)$$

In the event that the observable $O = A$, then one may bound S_1 by

$$\begin{aligned} S_1 &\leq \|\Phi\|_a \sum_{z \in S_{A,B}} \sum_{x \in S_{A,B}^c} F_a(d(z,x)) \sum_{y \in X} F_a(d(z,y)) \\ &\leq C_a \|\Phi\|_a \sum_{x \in S_{A,B}^c} \sum_{y \in X} F_a(d(x,y)) \end{aligned} \quad (3.14)$$

and similarly,

$$\begin{aligned} S_2 &\leq \|\Phi\|_a \sum_{z \in S_{A,B}^c} \sum_{x \in S_{A,B}} F_a(d(z,x)) \sum_{y \in X} F_a(d(z,y)) \\ &\leq \|F_a\| \|\Phi\|_a \sum_{z \in S_{A,B}^c} \sum_{y \in X} F_a(d(z,y)) \end{aligned} \quad (3.15)$$

An analogous bound holds in the case that $O = B$. We have proven (3.6). □

3.1.2. Proof of Theorem 3.1

To prove Theorem 3.1, we will first provide an estimate which measures the effect on the dynamics resulting from dropping certain interaction terms.

Lemma 3.3 *Let $\Phi_0 = \Phi_1 + \Phi_2$ be an interaction on Λ for which each of the dynamics $\{\tau_t^{(i)}\}_{t\in\mathbb{R}}$, for $i \in \{0, 1, 2\}$, exists as a strongly continuous group of $*$-automorphisms on $\mathcal{A}$. Let $\{A_t\}_{t\in\mathbb{R}}$ be a differentiable family of quasi-local observables on $\mathcal{A}$. The estimate*

$$\| \tau_t^{(0)}(A_t) - \tau_t^{(1)}(A_t) \| \leq \int_0^{|t|} \left\| \left[H_2, \tau_s^{(1)}(A_s) \right] \right\| + \left\| \tau_s^{(0)}(\partial_s A_s) - \tau_s^{(1)}(\partial_s A_s) \right\| \, ds, \tag{3.16}$$

holds for all $t \in \mathbb{R}$. Here, for each $i \in \{0, 1, 2\}$, we denote by H_i the Hamiltonian corresponding to Φ_i.

Proof: Define the function $f : \mathbb{R} \to \mathcal{A}$ by

$$f(t) := \tau_t^{(0)}(A_t) - \tau_t^{(1)}(A_t). \tag{3.17}$$

A simple calculation shows that f satisfies the following differential equation:

$$f'(t) = i\,[H_0, f(t)] + i\left[H_2, \tau_t^{(1)}(A_t)\right] + \tau_t^{(0)}(\partial_t A_t) - \tau_t^{(1)}(\partial_t A_t), \tag{3.18}$$

subject to the boundary condition $f(0) = 0$. The first term appearing on the right hand side of (3.18) above is norm preserving, and therefore, Lemma A.1 implies that

$$\| f(t) \| \leq \int_0^{|t|} \left\| \left[H_2, \tau_s^{(1)}(A_s) \right] \right\| + \left\| \tau_s^{(0)}(\partial_s A_s) - \tau_s^{(1)}(\partial_s A_s) \right\| \, ds, \tag{3.19}$$

as claimed. □

We will now prove Theorem 3.1. Denote by $B_t := B - \langle \tau_t(B) \rangle$, and observe that proving (3.1) is equivalent to bounding $|\langle \tau_t(AB_t) \rangle|$. Write $\Phi = \Phi_1 + \Phi_2$, as is done in (3.4). One easily sees that Φ_1 decouples A from B, i.e.,

$$\left\langle \tau_t^{(1)}(AB) \right\rangle = \left\langle \tau_t^{(1)}(A) \right\rangle \left\langle \tau_t^{(1)}(B) \right\rangle. \tag{3.20}$$

Here, again, we have denoted by $\tau_t^{(1)}$ the time evolution corresponding to Φ_1. It is clear that

$$\begin{aligned} |\langle \tau_t(AB_t) \rangle| &\leq \left| \left\langle \tau_t^{(1)}(AB_t) \right\rangle \right| + \left| \left\langle \tau_t(AB_t) - \tau_t^{(1)}(AB_t) \right\rangle \right| \\ &\leq \|A\| \left\| \tau_t(B) - \tau_t^{(1)}(B) \right\| + \left\| \tau_t(AB_t) - \tau_t^{(1)}(AB_t) \right\| . \end{aligned} \tag{3.21}$$

Moreover, the second term on the right hand side above can be further estimated by

$$\left\| \tau_t(AB_t) - \tau_t^{(1)}(AB_t) \right\| \leq 2\|B\| \left\| \tau_t(A) - \tau_t^{(1)}(A) \right\| + \|A\| \left\| \tau_t(B_t) - \tau_t^{(1)}(B_t) \right\| . \tag{3.22}$$

Applying Lemma 3.3 to the bounds we have found in (3.21) and (3.22) yields

$$|\langle \tau_t(AB_t) \rangle| \leq 2\|A\| \int_0^{|t|} \left\| \left[H_2, \tau_s^{(1)}(B) \right] \right\| \, ds + 2\|B\| \int_0^{|t|} \left\| \left[H_2, \tau_s^{(1)}(A) \right] \right\| \, ds. \tag{3.23}$$

In fact, we are only using (3.16) in trivial situations where the second term, i.e., $\tau_s(\partial_s A_s) - \tau_s^{(1)}(\partial_s A_s)$ is identically zero. Finally, using Lemma 3.2, we find an upper bound on $|\langle \tau_t(AB_t) \rangle|$ of the form

$$4 \, \|A\| \, \|B\| \, G_a(t) \left(\sum_{x \in X} \sum_{\substack{y \in \Lambda: \\ 2d(x,y) \geq d(X,Y)}} F_a\left(d(x,y)\right) + \sum_{y \in Y} \sum_{\substack{x \in \Lambda: \\ 2d(x,y) \geq d(X,Y)}} F_a\left(d(x,y)\right) \right). \tag{3.24}$$

Theorem 3.1 readily follows from (3.24) above.

APPENDIX A

In this appendix, we recall a basic lemma about the growth of the solutions of first order, inhomogeneous differential equations.

Let $\mathcal{B}$ be a Banach space. For each $t \in \mathbb{R}$, let $A(t) : \mathcal{B} \to \mathcal{B}$ be a linear operator, and denote by $X(t)$ the solution of the differential equation

$$\partial_t X(t) = A(t) X(t) \tag{A.1}$$

with boundary condition $X(0) = x_0 \in \mathcal{B}$.We say that the family of operators $A(t)$ is *norm-preserving* if for every $x_0 \in \mathcal{B}$, the mapping $\gamma_t : \mathcal{B} \to \mathcal{B}$ which associates $x_0 \to X(t)$, i.e., $\gamma_t(x_0) = X(t)$, satisfies

$$\| \gamma_t(x_0) \| = \| x_0 \| \quad \text{for all } t \in \mathbb{R}. \tag{A.2}$$

Some obvious examples are the case where $\mathcal{B}$ is a Hilbert space and $A(t)$ is anti-hermitian for each t, or when $\mathcal{B}$ is an $*$-algebra of operators on a Hilbert space with a spectral norm and, for each t, $A(t)$ is a derivation commuting with the $*$-operation.

Lemma A.1 *Let $A(t)$, for $t \in \mathbb{R}$, be a family of norm preserving opeartors in some Banach space $\mathcal{B}$. For any function $B : \mathbb{R} \to \mathcal{B}$, the solution of*

$$\partial_t Y(t) = A(t)Y(t) + B(t), \tag{A.3}$$

with boundary condition $Y(0) = y_0$, satisfies the bound

$$\| Y(t) - \gamma_t(y_0) \| \leq \int_0^t \| B(t') \| \, dt'. \tag{A.4}$$

Proof: For any $t \in \mathbb{R}$, let $X(t)$ be the solution of

$$\partial_t X(t) = A(t) X(t) \tag{A.5}$$

with boundary condition $X(0) = x_0$, and let γ_t be the linear mapping which takes x_0 to $X(t)$. By variation of constants, the solution of the inhomogeneous equation (A.3) may be expressed as

$$Y(t) = \gamma_t \left(y_0 + \int_0^t (\gamma_s)^{-1} (B(s)) \, ds \right). \tag{A.6}$$

The estimate (A.4) follows from (A.6) as $A(t)$ is norm preserving. □

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