Journal of Statistical Physics, Vol. 119, Nos. 5/6, June 2005 (© 2005)
DOI: 10.1007/s10955-005-4650-4

Brownian Motion with Dry Friction

P.-G. de Gennes[1]

Received February 15, 2005; accepted March 17, 2005

A small object (Solid or droplet) is placed on a horizontally vibrating plate, imposing an acceleration $\gamma(t)$ in the form of a white noise. The object experiences dry friction (due to soild/solid interaction, or to contact angle hysteresis in the case of a droplet). The object is driven by a force

$$\gamma(t) - \Delta\sigma(t)$$

where $\sigma(t), = \pm 1$, depending on the sign of the velocity. We discuss the motion at two levels: (i) in terms of simple scaling laws, (ii) by a propagator technique.

(a) When Δ is below a certain crossover value Δ^*, we expect an unperturbed (Langevin) Brownian motion.

(b) When $\Delta > \Delta^*$, we expect a reduced diffusion coefficient proportional to Δ^{-4} for small Δ.

KEY WORDS: Brownian motion; Friction; Tribology; Random walks; Acoustic noise.

1. PRINCIPLES

Dry friction of a solid on a solid was studied first by L. da Vinci.[(1)] His results were rediscovered by Amontons.[(1)] The basic laws have been understood – in terms of rubbing asperities – in the 20th century, mainly thanks to the British school.[(1)] When an object lies on a horizontal plate, it does not start to move until the horizontal force f reaches a certain threshold. For macroscopic objects, the threshold is proportional to the weight of the object. For mesoscopic objects (of micrometer size), the same sort of threshold may still exist, provided that the solid solid contact involves a large number of asperities.

[1]Institut Curie Recherche, 11 rue P. et M. Curie, F-75231 Paris Cedex 05; France; e-mail: pgg@curie.fr

Another example involves liquid droplets (of size $\lesssim 1\ mm$). It was found recently by Daniel and Chaudury.[2] that a droplet deposited on a flat surface and exposed to a *non symmetrical* horizontal vibration of the plate, moves at a well defined speed. This has been interpreted in terms of hysteresis in the contact angle.[2] The droplet problem is slightly more complex than the solid/solid problem, because it involves two degrees of freedom: the center of gravity of the droplet, and also the center of the contact surface. This is of practical interest: if the vibration frequency is close to the resonance frequency of a pinned droplet, the effects are enhanced. But the general principles are very similar.

The driven motion of drops may be of some interest in microfluidics.

We now also have observations and theoretical predictions for the driven motion for the solid/solid case.[3] This led us naturally to consider another case of interest, where the vibration is a random noise, and the solid performs a new form of Brownian motion (Fig. 1).

In the reference frame of the vibrating plate, we assume a one dimensional equation of motion for the solid velocity $V(t) = \frac{dx}{dt}$ of the form

$$\frac{dV}{dt} + \frac{1}{\tau} V = \gamma(t) - \sigma(V)\Delta \tag{1}$$

where τ is a Langevin relaxation time,[4] and $-\gamma(t)$ is the horizontal acceleration of the plate in the laboratory frame, corresponding to a white noise.

$$\langle \gamma(t_1)\gamma(t_2) \rangle = K\delta(t_1 - t_2) \tag{2}$$

In equation (1), $\sigma(t) = V(t)/|V(t)|$ is the sign of the velocity (with $\sigma(0) = 0$), and Δ defines the acceleration threshold.

It is important to note that, even when $|\gamma(t)| < \Delta$, we may still have some motion. But, if $|\gamma| < \Delta$ and $V = 0$ simultaneously, then the particle stops, and starts again only when, at some later time t', we find $|\gamma(t')| > \Delta$.

To discuss the magnitude of γ compared with Δ, Eq. (2) is oversimplified: with a strict white noise, the probability of having $\gamma < \Delta$ in any

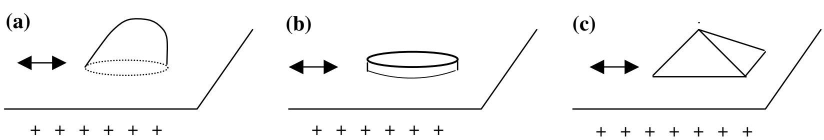

Fig. 1. Three possible examples of Brownian motion induced by random vibrations of a supporting plate, (a) A liquid drople,[2] (b) A macroscopic coin,[3] (c) A microcrystal.

time interval (however small) is zero. A crude but simple way of overcoming this is to divide the time axis in short intervals (duration τ_c).

(a) Inside each interval we assume a constant γ with a gaussian distribution and a mean square F^2.

(b) The values of γ in different intervals are uncorrelated.

We ultimately write instead of Eq. (2):

$$\begin{aligned} \langle\gamma(t_1)\,\gamma(t_2)\rangle &= F^2 \qquad |t_1 - t_2| < \tau_c \\ \langle\gamma(t_1)\,\gamma(t_2)\rangle &= 0 \qquad\; |t_1 - t_2| > \tau_c \end{aligned} \tag{3}$$

The white noise limit corresponds to $\tau_c \ll \tau$, and the constant K in Eq. (1) is given by

$$K = F^2 \tau_c \tag{4}$$

We shall analyse the motion qualitatively in Section 3. But, to make the discussion more concrete, we first comment upon the various control parameters which appear in Eq. (1).

2. RELATION TO OBSERVABLE QUANTITIES

Our object may be a coin moving on a plastic plate, as studied recently by A. Buguin.[3]

(1) We can in principle measure Δ by a static experiment with a tilted plate (tilt angle θ). At a critical angle θ_c the object starts to move, and we have

$$\Delta = \mathrm{g}\,\tan\theta_c \tag{5}$$

(2) We can measure the Langevin time τ on a horizontal plate by kicking the object (in the absence of any noise) and imposing a prescribed initial velocity V_0. Solving Eq. (1) with $\gamma \equiv 0$, we find that the particle trajectory is defined by

$$V(t) = (V_0 + \Delta\tau)e^{-t/\tau} - \Delta\tau \tag{6}$$

and the particle stops at a time t_1

$$t_1 = \tau \ln\left[1 + \frac{V_0}{\Delta\tau}\right] \tag{7}$$

Thus, from t_1 we can go back to τ. In practical cases, with a coin on a plastic plate, a typical value of τ is a few milliseconds.

(3) This implies that, to reach the white noise limit, we must have a correlation time of the vibrations τ_c which is much smaller than τ and thus less than 10^{-4} sec.

(4) Consider now a case with random noise, but no friction – i.e. a pure Langevin problem ($\Delta = 0$).[4]

Then the velocity correlations have the classical structure

$$\langle V(0)\, V(t)\rangle = \langle V_{\mathrm{L}}^2\rangle e^{-t/\tau} \tag{8}$$

$$\langle V_{\mathrm{L}}^2\rangle = \frac{1}{2} K\tau = \frac{1}{2} F^2 \tau_c \tau \tag{9}$$

(The index L stands for Langevin).

We may qualitatively describe this Langevin motion as bursts of velocity V_{L} lasting for a time τ (although, of course, there is a distribution of amplitudes and durations).

We may possibly determine V_{L}, on a surface without any dry friction (measuring also the Langevin time τ for this surface) and check Eq. (9). Note that if the noise had a purely thermal origin (no forced vibration) we would have

$$K = \frac{2kT}{M\tau}$$

where M is the particle mass and kT the thermal energy. This could be relevant for micron-size particles on a solid.

3. A SCALING DISCUSSION OF THE MOTION

We now consider the effects of dry friction ($\Delta \neq 0$). Of course, if Δ is comparable to the typical fluctuations F defined in Eq.(3), the particle is completely stuck. The interesting case corresponds to the opposite limit, $\Delta \ll F$.

(1) We can define a crossover point $\Delta = \Delta^*$ such that for $\Delta \leqslant \Delta^*$ we return to the Langevin problem (Eqs. 8, 9). If we examine Eq. (1) we see that, at crossover, the Δ term should be comparable to V/τ, where V is equal to the unperturbed rms velocity V_{L} (Eq. 9).

$$\Delta^* = \frac{V_{\mathrm{L}}}{\tau} \tag{10}$$

(2) The interesting regime is above Δ^*

$$\Delta^* < \Delta < F \tag{11}$$

and it can exist provided that $\Delta^* \ll F$ or $F\tau \gg V_{\mathrm{L}}$. Returning to Eq. (9) we see that this inequality is equivalent to $\tau \gg \tau_c$, and is thus satisfied.

The condition defined by the inequalities Eq. (11) defines what we call *the partly stuck regime*. From now on we concentrate on this regime. We assume (as in section 2.4) that we can describe the typical motions in terms of

– a velocity amplitude V_Δ

– a correlation time for the velocities τ_Δ.

(a) Keeping only the Δ term on the right of Eq. (1), and omitting now the V/τ term on the left, we find

$$V_\Delta \cong \Delta \tau_\Delta \tag{12}$$

(b) Since the velocity correlation time is now τ_Δ rather than τ, Eq. (9) is replaced by

$$\langle V_\Delta^2 \rangle \cong \frac{1}{2} K \tau_\Delta \cong K \tau_\Delta \tag{13}$$

(dropping all numerical coefficients).

We can solve Eqs. (12, 13) for

$$V_\Delta \cong \frac{K}{\Delta} \cong V_{\mathrm{L}} \frac{\Delta^*}{\Delta} \tag{14}$$

$$\tau_\Delta \cong \frac{K}{\Delta^2} \cong \tau \left(\frac{\Delta^*}{\Delta} \right)^2 \tag{15}$$

Ultimately we can construct a scalling low for the diffusion coefficient

$$D \cong V_\Delta^2 \tau_\Delta \sim D_{\mathrm{L}} \left(\frac{\Delta^*}{\Delta} \right)^4 \quad (\Delta^* < \Delta < F) \tag{16}$$

where $D_{\mathrm{L}} \cong V_\tau^2$ is the Langevin (unperturbed) value in the absence of dry friction.

4. PARTLY STUCK REGIME: DETAILS OF THE CORRELATION FUNCTION

We now assume the Δ satisfies the inequalities (Eq. 11) and omit the V/τ term in Eq. (1). The velocity correlation function can be obtained through a standard method based on the propagator $\Gamma_t(W|V)$: this is the statistical weight for a trajectory where the velocity starts from a value W at time 0, and reaches a value V at time t.

To set up the transport equation for Γ, we think of a random walk (along the V axis) where the position is V at a time t, and the probability distribution is $\Gamma_t(W|V)$. From Eq. (1) the current (along the V axis) is the sum of a dispersion term and a drift term

$$J = -K\frac{\partial \Gamma}{\partial V} - \Delta\sigma(V)\Gamma \tag{17}$$

$$J = -K\left[\left(\frac{\partial \Gamma}{\partial V}\right) + 2p\Gamma\right] \tag{18}$$

In Eq. (8) we assumed $V > 0$ to simplify the notation, and we introduced

$$p = \frac{\Delta}{2K} = \frac{1}{2}V_0^{-1} \tag{19}$$

The transport equation for Γ is then

$$\frac{\partial \Gamma}{\partial t} = -\frac{\partial J}{\partial V} \tag{20}$$

$$\frac{\partial \Gamma}{\partial t} = K\left[-\frac{\partial^2 \Gamma}{\partial V^2} - 2p\frac{\partial \Gamma}{\partial V}\right] \tag{21}$$

The bounding condition of Γ at time < 0 is

$$\Gamma_{t=0}(V|W) = \delta(V - W)Q(V) \tag{22}$$

where $Q(V)$ is the stationary distribution of velocities. We can obtain an Eq. for $Q(V)$ by setting $\frac{\partial}{\partial t} = 0$ in Eq. (20)

$$\frac{\partial^2 Q(V)}{\partial V^2} + 2p\frac{\partial Q}{\partial V} = 0 \tag{23}$$

The (normalised) adequate solution is

$$Q(V) = pe^{-2p|V|} \tag{24}$$

We shall now construct the propagators Γ in terms of an eigenfunction expansion. Because the operator in Eq. (21) is not self adjoint, we must first perform a transformation, writing

$$\Gamma_t(W/V) = Q^{\frac{1}{2}}(W)Q^{\frac{1}{2}}(V)\sum_k U_k(W)U_k(V)e^{-E_k t} \tag{25}$$

where the $U_k(V)$ are eigenfunctions of the following self adjoint equation

$$E_k U_k(V) = K\left[-\frac{\partial^2}{\partial V^2}U_k(V) + p^2 U_k\right] \tag{26}$$

Equation (26) must be supplemented by a boundary condition at $V = 0$, which is the equivalent of the cusp singularity showing up in Eq. (24)

$$\frac{1}{U_k}\frac{dU_k}{dV}|V_{=+0} = -p \tag{27}$$

The eigenmodes of Eqs. (25, 26) correspond to a one dimensional Schrodinger equation where the particle position is V, and the potential contains a constant term (Kp^2) plus an attractive delta function potential at the origin (equivalent to the boundary condition (27). Thus there is a bound state.

$$U_0(V) = p^{\frac{1}{2}}e^{-pV} = Q^{\frac{1}{2}}(V) \tag{28}$$

with energy $E_0 = 0$, and a continuum of states

$$U_k = \left(\frac{2}{V_m}\right)^{\frac{1}{2}}\cos(kV + \varphi_k) \tag{29}$$

(where we have normalised the eigenfunctions in a large box of size V_m). The phase shift angle φ_k is deduced from Eq. (27) and is defined by

$$\tan\varphi_k = \frac{p}{k} \tag{30}$$

The eigenvalue E_k is equal to $K(p^2+k^2)$.

The set (U_p, U_k) is orthonormal and complete: we can check on Eq. (25) that the boundary condition (22) on Γ is satisfied.

We can now proceed to a calculation of the velocity-velocity correlation function

$$\langle V(0)V(t)\rangle = \int dW\,dV\,\Gamma_t(W|V)VW \tag{31}$$

Using the fact that U_0 is identical to $Q^{\frac{1}{2}}$ we can ultimately express this in terms of matrix elements

$$(0|V|k) = \int_{-\infty}^{\infty} U_0(V)VU_k(V)dV \tag{32}$$

$$\langle V(0)V(t)\rangle = \sum_k e^{-E_k t}|(0|V|k)|^2 \tag{33}$$

$$\langle V(0)V(t)\rangle = \frac{4}{\pi}\int_0^{\infty} dk \frac{pk^2}{(k^2+p^2)^3}\exp\left[-K(p^2+k^2)t\right] \tag{34}$$

Note first that Eq. (34) justifies the scaling ansatz of Section 4 (where $V_\Delta = p^{-l}$ and $\tau_\Delta^{-1} = Kp^2$).

More precisely:

(a) the mean square velocity is

$$\langle V^2\rangle = \frac{1}{4p^2} \tag{35}$$

(b) the diffusion constant is

$$D = \int_0^{\infty}\langle V(0)V(t)\rangle\,dt = \frac{1}{8}K^{-l}p^{-4} \tag{36}$$

It is also of interest to examine the decay of the velocity correlation function at long times ($t > \tau_\Delta$). From Eq. (34) we find:

$$\langle V(0)V(t)\rangle \cong \exp\left(-\frac{t}{\tau_\Delta}\right)\left(\frac{\tau\Delta}{t}\right)^{\frac{3}{2}} \tag{37}$$

5. CONCLUSIONS

A – Brownian motion limited by dry friction could be observable in (at least) three categories of physical systems.

(a) Micron-size solid particles under thermal noise

(b) Macroscopic particles (eg, a coin on a vibrated solid sheet) as studied recently (under non symmetric vibrations) by A. Buguin[(3)]

(c) Droplets.

B – The main difficulty to be met in those experiments is (probably) the effect of *inhomogeneities*: the frictional properties, summarised here by the parameter Δ, may slightly when one explores different regions of the supporting plate. In an "island", where $\Delta > F$ (on an area comparable to the contact area of the solid particle), the particle will become stuck for ever.

C – An amusing question (raised by Y. Tsori) is the relation between *mobility* and diffusion coefficient. The mobility μ is defined by applying a small force $M\bar{\gamma}$ (for instance by tilting the plate) and measuring the resulting drift velocity $\langle V \rangle = \mu M \bar{\gamma}$.

(a) For the Langevin case ($\Delta \equiv 0$) we have the Einstein result

$$\frac{D}{\mu} = M \langle V^2 \rangle \tag{38}$$

(b) When $\Delta \neq 0$ in the partly stuck regime ($\Delta > \Delta^*$) we can get a scaling estimate for D/μ, starting from Eq. (1) where we add a constant $\bar{\gamma}$ on the right hand side. This gives

$$\langle V \rangle = \int_{-\infty}^{t} dt' \{ \langle \gamma(t') \rangle + \bar{\gamma} - \Delta \langle \sigma(V(t')) \rangle \} \exp\left\{ -\frac{t - t'}{\tau} \right\} \tag{39}$$

$$\langle V \rangle = \tau [\bar{\gamma} - \Delta \langle \sigma \rangle] \tag{40}$$

We estimate $\langle \sigma \rangle$ as follows (for small $\bar{\gamma}$):

$$\langle \sigma \rangle = \frac{k \langle V \rangle}{V_\Delta} \tag{41}$$

where k is a numerical constant.

Solving self consistently for V we find

$$\begin{aligned} \mu &= \frac{\tau}{M}\left[1+k\Delta^2/(\Delta^*)^2\right]^{-1} \\ \frac{D}{\mu} &= MV_\Delta^2[1+k\Delta^2/(\Delta^*)^2] \end{aligned} \tag{42}$$

Thus we lose the simplicity of Eq. (38).

ACKNOWLEDGMENTS

I have benefited from many exchanges with A. Buguin, F. Brochard, M. Chaudhury, E. Raphaël, Y. Tsori, J. Villain, and T. Vilmin.

Remarks. Another preprint on the same subject has recently been produced (March 5, 2005) by Dr. Hisao Hayakawa (arXiv:cond-mat/0407789 v1 Jul 2004, to be published in Physica D). This paper discusses mainly the steady state distribution $Q(V)$. On this matter it is more complete than the present work, because it covers all values of Δ, while the present analysis leading to Eq. (24) is restricted to $\Delta > \Delta^*$.

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