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Inverse scheduling with maximum lateness objective

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Abstract We study a range of counterparts of the single-machine scheduling problem with the maximum lateness criterion that arise in the context of inverse optimization. While in the forward scheduling problem all parameters are given and the objective is to find the optimal job sequence for which the value of the maximum lateness is minimum, in inverse scheduling the exact values of processing times or due dates are unknown, and they should be determined so that a prespecified solution becomes optimal. We perform a fairly complete classification of the corresponding inverse models under different types of norms that measure the deviation of adjusted parameters from their given estimates.

Keywords Single-machine scheduling · Maximum lateness · Inverse optimization

1 Introduction

In recent years, the interest in *inverse optimization* has increased dramatically. Unlike traditional optimization models for which all parameters are given and the objective is to find the best solution that satisfies specific constraints, in inverse optimization the exact values of some parameters are unknown, and they should be determined so that a prespecified solution becomes optimal. While inverse optimization has attracted much attention of researchers in different areas of combinatorial optimization, scheduling problems have not yet been studied in terms of inverse optimization (see, e.g., surveys Ahuja and Orlin 2001; Heuberger 2004). In this paper, we study the inverse counterparts of the single-machine scheduling problem $1\|L_{\max}$ with the maximum lateness criterion.

In the *forward scheduling problem* $1\|L_{\max}$, a set of jobs $N=\{1,2,\ldots,n\}$ should be processed by a single machine without preemption. All jobs are available at time 0. The processing time of a job $j\in N$ is given by p_j, and it should be completed by a given due date d_j. We denote a due date vector $(d_1,\ldots,d_n)$ by $\mathbf{d}$. A schedule is uniquely defined by a job permutation π, which induces completion times $C_j(\pi)$ for the jobs scheduled one after another without idle time. For a schedule given by permutation π, the lateness of job j is determined as

$$L_j(\pi,\mathbf{d})=C_j(\pi)-d_j,$$

and the overall performance of a schedule is measured in terms of the maximum lateness

$$L_{\max}(\pi,\mathbf{d})=\max_{j\in N}\big\{L_j(\pi,\mathbf{d})\big\}.$$

The objective of the forward problem $1\|L_{\max}$ is to find a permutation π^* for which $L_{\max}(\pi,\mathbf{d})$ achieves its minimum value:

$$L_{\max}\big(\pi^*,\mathbf{d}\big)\le L_{\max}(\pi,\mathbf{d})\quad \text{for any job permutation } \pi.$$

In what follows we do not use π and $\mathbf{d}$ in the notation if no ambiguity arises.

In the *inverse scheduling problem*, the typical processing times p_j and due dates d_j are given together with a

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target job sequence π. Permutation π may not be optimal for the given values of p_j and d_j, $j \in N$. The objective is to modify the parameters within certain limits to be as close to the typical ones as possible so that the target job sequence becomes optimal. In what follows we assume that the target job permutation is given by $\pi = (1, 2, \dots, n)$; otherwise the jobs can be renumbered. If processing times are fixed and due dates are adjustable, then the inverse problem is denoted by $1|\text{adjustable } d_j,\ \pi|L_{\max}$. In this problem, for each job $i \in N$, its typical due date d_j is given together with its variability interval $[\underline{d}_j, \overline{d}_j]$, $d_j \in [\underline{d}_j, \overline{d}_j]$. The adjusted due dates $\hat{\mathbf{d}} = (\hat{d}_1, \hat{d}_2, \dots, \hat{d}_n)$ should be selected within given boundaries $\hat{d}_j \in [\underline{d}_j, \overline{d}_j]$, $j \in N$, so that the deviation $\|\hat{\mathbf{d}} - \mathbf{d}\|$ from the original due dates is minimum and the target job permutation π is optimal:

$$\begin{aligned} \min \quad & \|\hat{\mathbf{d}} - \mathbf{d}\| \\ \text{s.t.} \quad & L_{\max}(\pi, \hat{\mathbf{d}}) \le L_{\max}(\sigma, \hat{\mathbf{d}}) \text{ for any job permutation } \sigma, \\ & \underline{d}_j \le \hat{d}_j \le \overline{d}_j, \quad j \in N. \end{aligned}$$

The typical norms that are considered in inverse optimization have different costs for positive and negative deviations:

ℓ_1 (Manhattan):

$$\begin{aligned} &\|\hat{d} - d\|_{1,\alpha,\beta} \\ &\quad = \sum_{j=1}^{n} \big[\alpha_j \max\{\hat{d}_j - d_j, 0\} \\ &\qquad + \beta_j \max\{d_j - \hat{d}_j, 0\}\big], \end{aligned}$$

ℓ_2 (Euclidean):

$$\begin{aligned} &\|\hat{d} - d\|_{2,\alpha,\beta} \\ &\quad = \sqrt{\sum_{j=1}^{n} \big[\alpha_j (\max\{\hat{d}_j - d_j, 0\})^2 + \beta_j (\max\{d_j - \hat{d}_j, 0\})^2\big]}, \end{aligned}$$

ℓ_∞:

$$\begin{aligned} &\|\hat{d} - d\|_{\infty,\alpha,\beta} \\ &\quad = \max_{j=1,\dots,n} \big[\alpha_j \max\{\hat{d}_j - d_j, 0\} \\ &\qquad + \beta_j \max\{d_j - \hat{d}_j, 0\}\big], \end{aligned}$$

ℓ_H^{Σ} (Hamming):

$$\begin{aligned} &\|\hat{d} - d\|_{H,\alpha,\beta}^{\Sigma} \\ &\quad = \sum_{j=1}^{n} \big[\alpha_j \operatorname{sgn}(\max\{\hat{d}_j - d_j, 0\}) \\ &\qquad + \beta_j \operatorname{sgn}(\max\{d_j - \hat{d}_j, 0\})\big], \end{aligned}$$

$\ell_H^{\max}$ (Hamming):

$$\begin{aligned} &\|\hat{d} - d\|_{H,\alpha,\beta}^{\max} \\ &\quad = \max_{j=1,\dots,n} \big[\alpha_j \operatorname{sgn}(\max\{\hat{d}_j - d_j, 0\}) \\ &\qquad + \beta_j \operatorname{sgn}(\max\{d_j - \hat{d}_j, 0\})\big]. \end{aligned}$$

Here all costs α_j and β_j are nonnegative. Observe that we consider the generalized form of the two Hamming norms; the inverse optimization problems with the symmetric versions of these two norms when $\alpha_j = \beta_j$ are studied in Duin and Volgenant (2006), Liu and Zhang (2006).

In addition to the inverse problem $1|\text{adjustable } d_j,\ \pi|L_{\max}$, we introduce a reverse problem following the classification due to Heuberger (2004). In a *reverse* problem, instead of a target permutation π, a target value L^* of the objective function $L_{\max}$ is given, and the due dates should be adjusted to achieve that value:

$$\begin{aligned} \min \quad & \|\hat{\mathbf{d}} - \mathbf{d}\| \\ \text{s.t.} \quad & L_{\max}(\sigma, \hat{\mathbf{d}}) \le L^*, \quad \text{for some permutation } \sigma, \\ & \underline{d}_j \le \hat{d}_j \le \overline{d}_j, \quad j \in N. \end{aligned}$$

We denote this problem by $1|\text{adjustable } d_j,\ L^*|L_{\max}$.

The inverse and reverse problems formulated above consider fixed processing times and adjustable due dates. Similar formulations can be introduced for fixed due dates and adjustable processing times $\hat{p}_j$ which should be selected within given boundaries $\underline{p}_j \le \hat{p}_j \le \overline{p}_j$, $j \in N$, so that the deviation from the original processing times $\|\hat{\mathbf{p}} - \mathbf{p}\|$ is minimum. In the corresponding notation, we simply replace "adjustable d_j" by "adjustable p_j."

We describe the possible scenarios that involve inverse and reverse scheduling. In real-life situations, the interests of customers and producers are often in conflict. Inverse and reverse models may be used as a negotiation tool to resolve such conflicts.

In a scenario corresponding to the inverse scheduling problem with adjustable processing times, a producer may have a preferred job sequence π predetermined by some estimates of processing times and due dates and by technical restrictions. If the actual values of parameters appear to be quite different from the estimates so that the customers' due dates cannot be met if the preferred production sequence is used, then the producer may identify a few jobs that can be produced faster at an additional cost giving an opportunity to complete all jobs in time following the fixed sequence. Adjusted processing times must be such that π is the best possible sequence for the producer and that the customers' due dates are respected.

In a scenario corresponding to the reverse scheduling problem with adjustable due dates, the producer aims to complete the jobs either on time or within an admittable limit L^* from the due dates, but he is not restricted by some preferred job sequences. If adhering to the claimed quality of service measured by L^* is not possible, the producer may offer the customers some compensation to override their due dates slightly, so that the claimed quality of service is achieved for the modified due dates. The bargained due dates must be such that the quality level L^* is met and the costs incurred are minimal.

In this paper, we consider the problems with adjustable due dates first and the counterparts with adjustable processing times next.

2 Preliminaries

It is known (Brucker 2004; Jackson 1955) that the forward problem $1\|L_{\max}$ can be solved by sequencing the jobs in nondecreasing order of their due dates, which is often called the *earliest due date* (EDD) order. The EDD order is sufficient but not necessary for a job permutation to be optimal.

The necessary and sufficient conditions for optimality of a given job sequence for problem $1\|L_{\max}$ were formulated and proved by Lin and Wang (2007). These conditions are essentially used in solving the inverse and reverse problems in the subsequent sections.

Theorem 1 *The job sequence $\pi = (1, 2, \ldots, n)$ is optimal for problem $1\|L_{\max}$ if and only if there exists a job k such that the following two conditions are satisfied*:

$$C_k - d_k \ge C_j - d_j \quad \textit{for } 1 \le j \le n, \tag{1}$$

$$d_j \le d_k \quad \textit{for } 1 \le j \le k-1. \tag{2}$$

Observe that there may exist several jobs satisfying condition (1) with the same value $C_k - d_k$. In what follows we call the job(s) satisfying condition (1) *critical* for d and π.

3 Adjustable due dates

In this section we assume that the processing times p_j are fixed for all jobs $j \in N$, while the due dates d_j should be adjusted to guarantee that the target job sequence $\pi = (1, 2, \ldots, n)$ is optimal or a target value L^* of the maximum lateness $L_{\max}$ is achieved. The inverse problem with the target permutation π is considered first (Sect. 3.1), and the reverse problem with the target objective value L^* next (Sect. 3.2).

3.1 Inverse problem $1|\text{adjustable } d_j,\ \pi|L_{\max}$

The objective of the inverse problem $1\|L_{\max}$ is to find the adjusted due dates $\hat{\mathbf{d}}$ within the given boundaries $\hat{d}_j \in [\underline{d}_j, \overline{d}_j]$, $j \in N$, so that the deviation $\|\hat{\mathbf{d}} - \mathbf{d}\|$ from the original due dates is minimum and the target job permutation $\pi = (1, 2, \ldots, n)$ is optimal.

First we prove that a job that is critical for the initial due dates $\mathbf{d}$ remains critical for the optimum adjusted due dates $\hat{\mathbf{d}}$. Then we demonstrate how the optimum adjusted due dates $\hat{\mathbf{d}} = (\hat{d}_1, \ldots, \hat{d}_n)$ can be found.

Lemma 1 *Let h be a critical job for initial due dates $\mathbf{d}$ and a target job sequence $\pi = (1, 2, \ldots, n)$. If the inverse problem $1|\text{adjustable } d_j,\ \pi|L_{\max}$ is feasible, then there exists an optimal solution $\hat{\mathbf{d}}$ such that the same job h is critical for the adjusted due dates $\hat{\mathbf{d}}$ and job sequence π.*

Observe that problem $1|\text{adjustable } d_j,\ \pi|L_{\max}$ is infeasible if the due dates cannot be adjusted within their boundaries to make permutation π optimal.

The proof of the lemma appears in the Appendix.

It follows from Lemma 1 that in order to find the optimum adjusted due dates $\hat{\mathbf{d}}$, we can limit our consideration to a class of schedules with a fixed critical job h, which is defined as a critical job for the initial due dates $\mathbf{d}$.

If conditions (1)–(2) of Theorem 1 are satisfied for the target permutation π with the critical job h for initial due dates $\mathbf{d}$, then no further action is required; the current schedule is optimal, and the deviation $\|\hat{\mathbf{d}} - \mathbf{d}\|$ is 0. Otherwise we consider different values $\hat{d}_h \in [\underline{d}_h, \overline{d}_h]$ and define the adjusted due dates $\hat{d}_j$ for all other jobs $j \in N \backslash \{h\}$ depending on $\hat{d}_h$.

In order to derive the formulas for due date adjustments, we split the interval $[\underline{d}_h, \overline{d}_h]$ into subintervals in such a way that in each subinterval the same subset of jobs is subject to adjustment. Considering the subintervals one by one, we perform parametric analysis of the whole interval $[\underline{d}_h, \overline{d}_h]$. In each subinterval, we find an optimum due date $\hat{d}_h$ and corresponding due dates $\hat{d}_j$ for $j \in N \backslash \{h\}$ to ensure that the necessary and sufficient conditions from Theorem 1 are satisfied and the deviation $\|\hat{\mathbf{d}} - \mathbf{d}\|$ is minimum. The solution to the problem is found by considering the solutions for all subintervals and selecting the one with the smallest deviation $\|\hat{\mathbf{d}} - \mathbf{d}\|$.

The interval $[\underline{d}_h, \overline{d}_h]$ is split by the different values from $\{C_h - L_1, C_h - L_2, \ldots, C_h - L_n\} \cup \{d_1, d_2, \ldots, d_h\}$ that belong to that interval. Introduce the ordered sequence of the above values:

$$\underline{d}_h = t_{k_1} < t_{k_2} < \cdots < t_{k_z} = \overline{d}_h. \tag{3}$$

Observe that for job h, the two values $C_h - L_h$ and d_h coincide, so that $z \le n + h - 1$.

Suppose that the adjusted due date $\hat{d}_h$ belongs to the subinterval $[t_{k_g}, t_{k_{g+1}}]$, $1 \le g \le z-1$. Consider condition (1) of Theorem 1. For any job $j \in N\backslash\{h\}$, the value $C_h - L_j$ satisfies one of the conditions:

$$C_h - L_j \le t_{k_g} \le \hat{d}_h \tag{4}$$

or

$$C_h - L_j \ge t_{k_{g+1}} \ge \hat{d}_h. \tag{5}$$

The jobs $j \in N\backslash\{h\}$ that satisfy (4) violate condition (1) of Theorem 1 for any $\hat{d}_h \in [t_{k_g}, t_{k_{g+1}}]$, and those that satisfy (5) do not.

Consider now condition (2) of Theorem 1. For any job $j \in \{1, 2, \dots, h-1\}$, the value d_j satisfies one of the conditions:

$$d_j \le t_{k_g} \le \hat{d}_h \tag{6}$$

or

$$d_j \ge t_{k_{g+1}} \ge \hat{d}_h. \tag{7}$$

The jobs that satisfy (6) do not violate condition (2) of Theorem 1, and those that satisfy (7) violate it for any $\hat{d}_h \in [t_{k_g}, t_{k_{g+1}}]$.

Thus we can define two subsets of jobs for which due dates should be adjusted in order to achieve the necessary and sufficient conditions of Theorem 1 for the target permutation π with the critical job h and $\hat{d}_h \in [t_{k_g}, t_{k_{g+1}}]$:

$U_g = \{u \mid u \in N\backslash\{h\} \text{ and } C_h - L_u \le t_{k_g}\}$—the jobs that violate condition (1),

$V_g = \{v \mid v \in \{1, \dots, h-1\} \text{ and } d_v \ge t_{k_{g+1}}\}$—the jobs that violate condition (2).

Clearly the due dates of the jobs from U_g should be increased, while those of the jobs from V_g should be decreased.

Observe that $U_g \cap V_g = \emptyset$: for any job $v \in V_g$,

$$C_v < C_h$$

and

$$d_v \ge t_{k_{g+1}} > t_{k_g},$$

so that

$$L_v = C_v - d_v < C_h - t_{k_g},$$

and condition (4) which characterizes U_g does not hold.

The subsets U_g and V_g defined for the subinterval $[t_{k_g}, t_{k_{g+1}}]$ may differ from U_{g+1} and V_{g+1} defined for the next subinterval $[t_{k_{g+1}}, t_{k_{g+2}}]$. In particular, for two consecutive intervals, $U_g \subseteq U_{g+1}$ and $V_g \supseteq V_{g+1}$.

We start the solution process with the adjusted due date $\hat{d}_h$ belonging to the leftmost interval $[t_{k_1}, t_{k_2}]$ and then proceed with next intervals $[t_{k_g}, t_{k_{g+1}}]$, $g = 2, 3, \dots, z$, considering them one by one. For each interval $[t_{k_g}, t_{k_{g+1}}]$, we denote the adjusted due dates of the jobs from U_g and V_g by

$$\hat{d}_u = d_u + x_u, \quad u \in U_g,$$

$$\hat{d}_v = d_v - y_v, \quad v \in V_g,$$

and define the associated problem as follows:

$$\begin{array}{ll} \min & F(\hat{d}_h, \mathbf{x}, \mathbf{y}) \\ \text{s.t.} & t_{k_g} \le \hat{d}_h \le t_{k_{g+1}}, \\ & C_u - (d_u + x_u) \le C_h - \hat{d}_h, \quad u \in U_g, \\ & d_v - y_v \le \hat{d}_h, \quad v \in V_g, \\ & 0 \le x_u \le \overline{d}_u - d_u, \quad u \in U_g, \\ & 0 \le y_v \le d_v - \underline{d}_v, \quad v \in V_g, \end{array} \tag{8}$$

where the objective function F is of the form

$$F(\hat{d}_h, \mathbf{x}, \mathbf{y}) = \begin{cases} \alpha_h(\hat{d}_h - d_h) + \sum_{u \in U_g} \alpha_u x_u + \sum_{v \in V_g} \beta_v y_v \\ \quad \text{for } \ell_{1,\alpha,\beta} \text{ norm}, \\ \alpha_h(\hat{d}_h - d_h)^2 + \sum_{u \in U_g} \alpha_u x_u^2 + \sum_{v \in V_g} \beta_v y_v^2 \\ \quad \text{for } \ell_{2,\alpha,\beta} \text{ norm}, \\ \max\{\alpha_h(\hat{d}_h - d_h), \max_{u \in U_g} \alpha_u x_u, \max_{v \in V_g} \beta_v y_v\} \\ \quad \text{for } \ell_{\infty,\alpha,\beta} \text{ norm}, \\ \alpha_h \operatorname{sgn}(\hat{d}_h - d_h) + \sum_{u \in U_g} \alpha_u \operatorname{sgn} x_u \\ \quad + \sum_{v \in V_g} \beta_v \operatorname{sgn} y_v \quad \text{for } \ell^{\Sigma}_{H,\alpha,\beta} \text{ norm}, \\ \max\{\alpha_h \operatorname{sgn}(\hat{d}_h - d_h), \\ \quad \max_{u \in U_g} \alpha_u \operatorname{sgn} x_u, \max_{v \in V_g} \beta_v \operatorname{sgn} y_v\} \\ \quad \text{for } \ell^{\max}_{H,\alpha,\beta} \text{ norm}. \end{cases}$$

The cost of these adjustments is minimum if conditions (1) and (2) hold as equalities for the adjusted due dates:

$$C_u - (d_u + x_u) = C_h - \hat{d}_h, \quad u \in U_g,$$

$$d_v - y_v = \hat{d}_h, \quad v \in V_g.$$

Finding the expressions for x_u and y_v from the above conditions and substituting them into (8), we obtain a problem with one variable $\hat{d}_h$:

$$\begin{aligned} \min \quad & F(\hat{d}_h) \\ \text{s.t.} \quad & \max\Big\{t_{k_g}, \max_{u\in U_g}\{A_{uh}\}, \max_{v\in V_g}\{\underline{d}_v\}\Big\} \\ & \le \hat{d}_h \le \min\Big\{t_{k_{g+1}}, \min_{u\in U_g}\{\overline{A}_{uh}\}, \min_{v\in V_g}\{d_v\}\Big\}, \end{aligned} \tag{9}$$

where

$$F(\hat{d}_h) = \begin{cases} \alpha_h(\hat{d}_h - d_h) + \sum_{u\in U_g}\alpha_u(\hat{d}_h - A_{uh}) \\ \quad + \sum_{v\in V_g}\beta_v(-\hat{d}_h + d_v) \quad \text{for } \ell_{1,\alpha,\beta}, \\ \alpha_h(\hat{d}_h - d_h)^2 + \sum_{u\in U_g}\alpha_u(\hat{d}_h - A_{uh})^2 \\ \quad + \sum_{v\in V_g}\beta_v(-\hat{d}_h + d_v)^2 \quad \text{for } \ell_{2,\alpha,\beta}, \\ \max\Big\{\alpha_h(\hat{d}_h - d_h), \max_{u\in U_g}\alpha_u(\hat{d}_h - A_{uh}), \\ \quad \max_{v\in V_g}\beta_v(-\hat{d}_h + d_v)\Big\} \quad \text{for } \ell_{\infty,\alpha,\beta}, \\ \alpha_h + \sum_{u\in U_g}\alpha_u + \sum_{v\in V_g}\beta_v \quad \text{for } \ell^{\Sigma}_{H,\alpha,\beta}, \\ \max\Big\{\alpha_h, \max_{u\in U_g}\alpha_u, \max_{v\in V_g}\beta_v\Big\} \quad \text{for } \ell^{\max}_{H,\alpha,\beta}, \end{cases}$$

and the constants A_{uh} and $\overline{A}_{uh}$ are defined for all $u \in U_g$ as

$$A_{uh} = d_u - C_u + C_h,$$
$$\overline{A}_{uh} = \overline{d}_u - C_u + C_h.$$

Observe that in the case of the Hamming norm $\ell^{\Sigma}_{H,\alpha,\beta}$ and $\ell^{\max}_{H,\alpha,\beta}$, the function $F(\hat{d}_h)$ is constant and does not depend on $\hat{d}_h$.

The constraint of problem (9) may be infeasible with the left-hand side larger than the right-hand side. In such a case no adjusted value $\hat{d}_h \in [t_{k_g}, t_{k_{g+1}}]$ exists such that permutation π is optimal. If this happens for all intervals $[t_{k_g}, t_{k_{g+1}}]$, $g = 1, 2, \dots, z-1$, then the inverse problem $1|\text{adjustable } d_j, \ \pi|L_{\max}$ does not have a solution. For example, if the due dates of all jobs are fixed, i.e., $\underline{d}_j = \overline{d}_j$, $j \in N$, and a given permutation π is not optimal, then no adjustments are possible and no solution to inverse problem exists.

We estimate the time complexity of the described approach. For the first problem with $\hat{d}_h \in [t_{k_1}, t_{k_2}]$, the sets V_1 and U_1 can be constructed in $O(n)$ time. The objective function $F(\hat{d}_h)$ and the box constraint can be obtained in $O(n)$ time.

Consider the transition from the problem with $\hat{d}_h \in [t_{k_{g-1}}, t_{k_g}]$ to the problem with $\hat{d}_h \in [t_{k_g}, t_{k_{g+1}}]$, $2 \le g \le z-1$. Each additional job that joins the U-set and each redundant job that is removed from the V-set can be found in $O(1)$ time, and the new formulation (9) is solvable in $O(1)$ time for any type of the norm. Repeating this process, we can find the optimal adjusted due date $\hat{d}_h$ in $O(n)$ time since $z \le n+h-1$. Taking into account that the ordered sequence (3) can be found in $O(n\log n)$ time, the overall time complexity of solving the inverse problem is $O(n\log n)$ for any type of the norm.

3.2 Reverse problem $1|\text{adjustable } d_j, L^*|L_{\max}$

Suppose that the target value L^* of the maximum lateness $L_{\max}$ is given and the objective is to find the adjusted due dates $\hat{d}_j$, $\hat{d}_j \in [\underline{d}_j, \overline{d}_j]$, such that $\|\hat{\mathbf{d}} - \mathbf{d}\|$ is minimum and the target value L^* is achieved.

Since the smallest value of $L_{\max}$ can be guaranteed by sequencing the jobs in the *earliest due date* order (EDD) (Jackson 1955), we can limit our search to the class of EDD-schedules.

We start with the EDD-schedule with the original due dates. If the value of $L_{\max}$ for it is no larger than L^*, then no further action is required. Otherwise the due dates of some jobs should be increased.

Let $\mathcal{H} = \{h_i\}$ be the set of critical jobs (the notion of a critical job was introduced in Sect. 2), and let L be the value of the maximum lateness, $L = L_{\max}$. If $L > L^*$, then the due dates of the jobs from $\mathcal{H}$ should be increased. Clearly, the increment amount should be the same for all jobs from $\mathcal{H}$,

$$\hat{d}_j = d_j + x, \quad x \ge 0, \ j \in \mathcal{H},$$

and the due date boundaries should be observed:

$$x \le \min_{j\in\mathcal{H}}\{\bar{d}_j - d_j\}.$$

First we discuss how the ties should be broken in the EDD-sequence if several jobs have equal due dates. If none of them is critical, then their order is immaterial. Otherwise, only the last job among those with equal due dates is critical and requires adjustment. Depending on the type of the norm, the deviation $\|\hat{\mathbf{d}} - \mathbf{d}\|$ is calculated by

$$\|\hat{\mathbf{d}} - \mathbf{d}\| = \begin{cases} \sum_{h\in\mathcal{H}}\alpha_h x & \text{for } \ell_{1,\alpha,\beta} \text{ norm}, \\ \sum_{h\in\mathcal{H}}\alpha_h x^2 & \text{for } \ell_{2,\alpha,\beta} \text{ norm}, \\ \max_{h\in\mathcal{H}}\{\alpha_h\} \times x & \text{for } \ell_{\infty,\alpha,\beta} \text{ norm}, \\ \sum_{h\in\mathcal{H}}\alpha_h & \text{for } \ell^{\Sigma}_{H,\alpha,\beta} \text{ norm}, \\ \max_{h\in\mathcal{H}}\{\alpha_h\} & \text{for } \ell^{\max}_{H,\alpha,\beta} \text{ norm}. \end{cases}$$

In the class of EDD-schedules, the value of $\|\hat{\mathbf{d}} - \mathbf{d}\|$ is minimum for each of the above norms, if the jobs with equal due dates are sequenced in nonincreasing order of α_j.

We introduce a notion of the *main* permutation. Permutation σ is called the main permutation if the jobs are sorted in nondecreasing order of d_j and the jobs with equal due dates are additionally sorted in nonincreasing order of costs α_j.

The due date adjustment of the critical jobs may violate the EDD order, and/or new critical jobs may appear. In order to maintain the main permutation σ and keep track of the critical jobs $\mathcal{H}$, the due date adjustment should be performed iteratively. At each iteration we assume that the jobs are numbered in accordance with the main permutation. Increasing due date d_{h_i} of job $h_i \in \mathcal{H}$ may cause a structural change that corresponds to one of the following three events. We assume that h_i is the kth job in permutation σ, i.e., $h_i = \sigma(k)$.

Event A: the due date of job $\sigma(k)$ reaches the due date of the next job $\sigma(k+1)$ of the main permutation.
Event B: a job $j \in N \setminus \mathcal{H}$ becomes critical.
Event C: the target value L^* of the maximum lateness is achieved.
Event D: the due date of job $\sigma(k)$ reaches its upper bound $\overline{d}_{\sigma(k)}$.

It is straightforward to verify that increasing $d_{\sigma(k)}$ by the amount

$$x^A_{\sigma(k)} = d_{\sigma(k+1)} - d_{\sigma(k)},$$

$$x^B_{\sigma(k)} = L - \max_{j \in N \setminus \mathcal{H}} \{C_j - d_j\},$$

$$x^C_{\sigma(k)} = L - L^*,$$

or

$$x^D_{\sigma(k)} = \overline{d}_{\sigma(k)} - d_{\sigma(k)}$$

leads to Event A, B, C, or D, respectively.

The value of $L_{\max}$ decreases if the due dates of all jobs from $\mathcal{H}$ are increased by the same amount x until the earliest event A, B, C, or D occurs. Hence x is defined as

$$x = \min\Big[\min_{\sigma(k)\in\mathcal{H}} \{d_{\sigma(k+1)} - d_{\sigma(k)}\}, L - \max_{j\in N\setminus\mathcal{H}} \{C_j - d_j\}, L - L^*, \min_{\sigma(k)\in\mathcal{H}} \{\overline{d}_{\sigma(k)} - d_{\sigma(k)}\}\Big]. \tag{10}$$

If Event A occurs and the due dates of jobs $\sigma(k)$ and $\sigma(k+1)$ become equal, then renumbering and updating the current permutation may be required so that the jobs with equal due dates are sequenced in nonincreasing order of α_j. If Event B occurs, then the set $\mathcal{H}$ should be updated. In both cases the current value L of the maximum lateness should be decreased by x. Increasing the due dates of the set $\mathcal{H}$ continues iteratively until one of the Events C or D occurs. In the case of Event C the target value L^* is achieved; the resulting solution is optimal since in each iteration the EDD permutation is considered and among the jobs with equal due dates the one with the smallest value of α_j is selected for due date adjustment. In the case of Event D the due date of at least one critical job cannot be increased any more so that the target value L^* cannot be achieved.

The initial job sequence can be constructed in $O(n \log n)$ time. In each iteration, the adjustment amount x is calculated in $O(n)$ time. Events A and B occur no more than n times each, while Event C or D occurs once. Thus the overall time complexity of solving the reverse problem is $O(n^2)$. Thus we have proved the following result.

Theorem 2 *The reverse problem* $1|\text{adjustable } d_j,\ L^*|L_{\max}$ *is solvable in* $O(n^2)$ *time for any type of the norm by decompressing all critical jobs iteratively by the same amount* x *defined by* (10). *If no solution exists, then this can be verified also in* $O(n^2)$ *time using the same approach.*

4 Adjustable processing times

In this section we assume that the due dates d_j are fixed for all jobs $j \in N$, while the processing times p_j should be adjusted to guarantee that the target job sequence $\pi = (1, 2, \dots, n)$ is optimal or the target value L^* is achieved.

4.1 Inverse problem $1|\text{adjustable } p_j, \pi|L_{\max}$

The objective of the inverse problem $1|\text{adjustable } p_j, \pi|L_{\max}$ is to find the adjusted processing times $\hat{\mathbf{p}}$ within the given boundaries so that $\hat{p}_j \in [\underline{p}_j,\ \overline{p}_j]$, $j \in N$, the deviation from the original processing times $\|\hat{\mathbf{p}} - \mathbf{p}\|$ is minimum, and the target job permutation $\pi = (1, 2, \dots, n)$ is optimal.

Consider a schedule defined by permutation π with initial processing times $\mathbf{p}$. Let $\mathcal{H}$ be a set of critical jobs, $|\mathcal{H}| \geq 1$. If there exists at least one critical job which satisfies the necessary and sufficient conditions of optimality of permutation π, then processing times $\mathbf{p}$ are optimal, and no further action is required. Otherwise condition (1) of Theorem 1 is satisfied, while condition (2) is violated for each critical job from $\mathcal{H}$ and cannot be repaired by adjusting processing times. This means that in an optimal solution to the inverse problem a new job h should be critical, $h \notin \mathcal{H}$. Since potentially any job $h \notin \mathcal{H}$ can be critical, we consider different classes of schedules with the fixed critical job h, $1 \leq h \leq n$. Observe that for the inverse problem $1|\text{adjustable } d_j,\ \pi|L_{\max}$ studied in Sect. 3.1, Lemma 1 justifies that only one class of problems can be considered.

Let h, $1 \leq h \leq n$, be a selected job which should become critical for adjusted processing times $\hat{\mathbf{p}}$. If the due dates do not satisfy relations (2) for the target permutation π and selected job h, then no adjustment of processing times can make permutation π optimal for the critical job h. If relations (2) are satisfied, then adjusting processing times changes job completions times C_j so that relations (1) can be achieved. Taking into account that

$$C_j = \sum_{i=1}^{j} \hat{p}_i,$$

inequalities (1) reduce to the following relations:

$$\sum_{i=j+1}^{h} \hat{p}_i \geq d_h - d_j \quad \text{for } 1 \leq j \leq h-1,$$

$$\sum_{i=h+1}^{j} \hat{p}_i \leq d_j - d_h \quad \text{for } h+1 \leq j \leq n.$$

Thus the problem can be formulated as follows:

$$\begin{aligned} \min \quad & \|\hat{\mathbf{p}} - \mathbf{p}\| \\ \text{s.t.} \quad & \sum_{i=j}^{h} \hat{p}_i \geq d_h - d_{j-1}, \quad 2 \leq j \leq h, \\ & \sum_{i=h+1}^{j} \hat{p}_i \leq d_j - d_h, \quad h+1 \leq j \leq n, \\ & \underline{p}_j \leq \hat{p}_j \leq \overline{p}_j, \quad 1 \leq j \leq n. \end{aligned} \tag{11}$$

Clearly, if the original processing times $\mathbf{p}$ do not satisfy the constraints of problem (11), then the processing times of some jobs from $N_1 = \{2, 3, \dots, h\}$ should be increased, and those of the jobs from $N_2 = \{h+1, h+2, \dots, n\}$ should be decreased. It follows that the adjustments required can be represented in the form

$$\hat{p}_j = p_j + x_j, \quad j \in N_1,$$
$$\hat{p}_j = p_j - y_j, \quad j \in N_2,$$

and the deviations are within the boundaries:

$$0 \leq x_j \leq \overline{p}_j - p_j, \quad j \in N_1,$$
$$0 \leq y_j \leq p_j - \underline{p}_j, \quad j \in N_2.$$

We introduce constants P_j, Q_j and A_j, B_j by

$$P_j = (d_h - d_{j-1}) - \sum_{i=j}^{h} p_i, \quad j \in N_1,$$

$$Q_j = \sum_{i=h+1}^{j} p_i - (d_j - d_h), \quad j \in N_2,$$

$$A_j = \overline{p}_j - p_j, \quad j \in N,$$
$$B_j = p_j - \underline{p}_j, \quad j \in N,$$

and rewrite formulation (11) as follows:

$$\begin{aligned} \min \quad & F(\mathbf{x}, \mathbf{y}) \\ \text{s.t.} \quad & \sum_{i=j}^{h} x_i \geq P_j, \quad j \in N_1, \\ & \sum_{i=h+1}^{j} y_i \geq Q_j, \quad j \in N_2, \\ & 0 \leq x_j \leq A_j, \quad j \in N_1, \\ & 0 \leq y_j \leq B_j, \quad j \in N_2, \end{aligned} \tag{12}$$

where the objective function F is of the form

$$F(\mathbf{x}, \mathbf{y}) = \begin{cases} \sum_{j \in N_1} \alpha_j x_j + \sum_{j \in N_2} \beta_j y_j & \text{for } \ell_{1,\alpha,\beta}, \\ \sum_{j \in N_1} \alpha_j x_j^2 + \sum_{j \in N_2} \beta_j y_j^2 & \text{for } \ell_{2,\alpha,\beta}, \\ \max\left\{\max_{j \in N_1}\{\alpha_j x_j\}, \max_{j \in N_2}\{\beta_j y_j\}\right\} & \text{for } \ell_{\infty,\alpha,\beta}, \\ \sum_{j \in N_1} \alpha_j \operatorname{sgn} x_j + \sum_{j \in N_2} \beta_j \operatorname{sgn} y_j & \text{for } \ell^{\Sigma}_{H,\alpha,\beta}, \\ \max\left\{\max_{j \in N_1}\{\alpha_j \operatorname{sgn} x_j\}, \max_{j \in N_2}\{\beta_j \operatorname{sgn} y_j\}\right\} & \\ \quad \text{for } \ell^{\max}_{H,\alpha,\beta}. & \end{cases}$$

The solution to problem (12) defines an optimal solution to the inverse problem in a class of schedules with the critical job h. Observe that in some classes no solution may exist. The solution to the inverse problem $1|\text{adjustable } p_j, \pi|L_{\max}$ can be found by enumerating all classes for which a solution exists and selecting the one with the smallest value of F.

In what follows we study problem (12) for different types of the norms.

4.1.1 Norms ℓ_1 and ℓ_2

Consider first the norms $\ell_{1,\alpha,\beta}$ and $\ell_{2,\alpha,\beta}$. It is convenient to rewrite formulation (12) using complementary variables u_j and v_j,

$$\begin{aligned} u_j &= \overline{p}_j - \hat{p}_j = A_j - x_j, \quad j \in N_1, \\ v_j &= \hat{p}_j - \underline{p}_j = B_j - y_j, \quad j \in N_2, \end{aligned} \tag{13}$$

which are bounded by A_j and B_j:

$$\begin{aligned} 0 &\leq u_j \leq A_j, \quad j \in N_1, \\ 0 &\leq v_j \leq B_j, \quad j \in N_2. \end{aligned} \tag{14}$$

Then formulation (12) can be rewritten as

$$\begin{aligned}
\min \quad & F_{\Sigma}(\mathbf{u},\mathbf{v}) \\
\text{s.t.} \quad & \sum_{i=j}^{h} u_i \le \sum_{i=j}^{h} A_i - P_j, \quad j \in N_1, \\
& \sum_{i=h+1}^{j} v_i \le \sum_{i=h+1}^{j} B_i - Q_j, \quad j \in N_2, \\
& 0 \le u_j \le A_j, \quad j \in N_1, \\
& 0 \le v_j \le B_j, \quad j \in N_2,
\end{aligned} \tag{15}$$

where

$$F_{\Sigma}(\mathbf{u},\mathbf{v}) = \begin{cases} \sum_{j\in N_1} \alpha_j (A_j - u_j) + \sum_{j \in N_2} \beta_j (B_j - v_j) \\ \quad \text{for } \ell_{1,\alpha,\beta}, \\ \sum_{j\in N_1} \alpha_j (A_j - u_j)^2 + \sum_{j \in N_2} \beta_j (B_j - v_j)^2 \\ \quad \text{for } \ell_{2,\alpha,\beta}. \end{cases}$$

Since the objective function $F_{\Sigma}(\mathbf{u},\mathbf{v})$ is separable, problem (15) can be decomposed into two subproblems with nested constraints defined: for variables N_1,

$$\begin{aligned}
\min \quad & \sum_{j\in N_1} \alpha_j (A_j - u_j) \quad \text{for norm } \ell_1 \text{ or} \\
& \sum_{j\in N_1} \alpha_j (A_j - u_j)^2 \quad \text{for norm } \ell_2 \\
\text{s.t.} \quad & \sum_{i=j}^{h} u_i \le \sum_{i=j}^{h} A_i - P_j, \quad j \in N_1, \\
& 0 \le u_j \le A_j, \quad j \in N_1,
\end{aligned}$$

and, for variables N_2,

$$\begin{aligned}
\min \quad & \sum_{j\in N_2} \beta_j (B_j - v_j) \quad \text{for norm } \ell_1 \text{ or} \\
& \sum_{j\in N_2} \beta_j (B_j - v_j)^2 \quad \text{for norm } \ell_2 \\
\text{s.t.} \quad & \sum_{i=h+1}^{j} v_i \le \sum_{i=h+1}^{j} B_i - Q_j, \quad j \in N_2, \\
& 0 \le v_j \le B_j, \quad j \in N_2.
\end{aligned}$$

The above problems can be classified as *Resource Allocation Problems with Nested Constraints* with continuous variables, which are solvable in $O(n \log n)$ time for linear and quadratic objective functions by the algorithm from Hochbaum and Hong (1995) and Katoh and Ibaraki (1998, p. 219). Hence in the class of schedules with the critical job h, the problem can be solved in $O(n \log n)$ time, and the overall time complexity for considering n classes of schedules is $O(n^2 \log n)$.

4.1.2 Norm ℓ_∞

Consider the norm ℓ_∞ and formulation (12). The objective function to be minimized is $\max\{\max_{j\in N_1}\{\alpha_j x_j\}, \max_{j\in N_2}\{\beta_j y_j\}\}$. We introduce an auxiliary variable θ for the value of the objective function and rewrite problem (12) accordingly:

$$\min \quad \theta \tag{16}$$

$$\text{s.t.} \quad \alpha_j x_j \le \theta, \quad j \in N_1, \tag{17}$$

$$\beta_j y_j \le \theta, \quad j \in N_2, \tag{18}$$

$$\sum_{i=j}^{h} x_i \ge P_j, \quad j \in N_1, \tag{19}$$

$$\sum_{i=h+1}^{j} y_i \ge Q_j, \quad j \in N_2, \tag{20}$$

$$0 \le x_j \le A_j, \quad j \in N_1, \tag{21}$$

$$0 \le y_j \le B_j, \quad j \in N_2. \tag{22}$$

It is easy to see that if the optimal value $\theta^* = \max\{\max_{j\in N_1}\{\alpha_j x_j\}, \max_{j\in N_2}\{\beta_j y_j\}\}$ of the objective function were known, then all variables x_j and y_j could be set equal to their largest possible values:

$$x_i^* = \min\{A_i, \theta^*/\alpha_i\}, \quad i \in N_1,$$
$$y_k^* = \min\{B_k, \theta^*/\beta_k\}, \quad k \in N_2.$$

Clearly, the optimum value θ^* should be as small as possible so that all constraints (19)–(22) are satisfied. We start with $\theta = 0$ and increase it iteratively. For each current value of θ, the x- and y-values are given by

$$x_i(\theta) = \min\{A_i, \theta/\alpha_i\}, \quad i \in N_1,$$
$$y_k(\theta) = \min\{B_k, \theta/\beta_k\}, \quad k \in N_2.$$

Let $\overline{N}_1 \subseteq N_1$ and $\overline{N}_2 \subseteq N_2$ be the subsets of variables which have already reached their upper bounds:

$$x_i(\theta) = A_i, \quad i \in \overline{N}_1,$$
$$y_k(\theta) = B_k, \quad k \in \overline{N}_2.$$

We calculate the deficits D_j for constraints (19)–(20):

$$D_j = \max\left\{P_j - \sum_{i=j}^{h} x_i(\theta), 0\right\} \quad \text{if } j \in N_1,$$

$$D_j = \max\left\{Q_j - \sum_{i=h+1}^{j} y_i(\theta), 0\right\} \quad \text{if } j \in N_2.$$

If $D_j > 0$, then the corresponding inequality is violated, and the variables involved should be increased by the total amount of D_j. That increase is only possible if θ is increased.

Suppose that the current value of θ is increased by $\delta > 0$. Then each variable $x_i(\theta)$, $i \in N_1 \backslash \overline{N}_1$, is increased by δ/α_i, and each variable $y_k(\theta)$, $k \in N_2 \backslash \overline{N}_2$, is increased by δ/β_k. As a result, the left-hand side of each inequality j from (19) is increased by $\delta \sum_{i\in\{j,\dots,h\}\backslash \overline{N}_1} 1/\alpha_i$ and that of each inequality j from (20) is increased by $\delta \sum_{i\in\{h+1,\dots,j\}\backslash \overline{N}_2} 1/\beta_i$.

Select δ to be the smallest value such that either all inequalities from (19)–(20) are satisfied or at least one of the variables x_i, $i \in N_1 \backslash \overline{N}_1$, or y_k, $k \in N_2 \backslash \overline{N}_2$, reaches its upper bound A_i or B_k:

$$\delta = \min\left\{\min_{j\in N_1}\left\{\frac{D_j}{\sum_{i\in\{j,\dots,h\}\backslash \overline{N}_1} 1/\alpha_i}, \max\{A_j - x_j(\theta), 0\}\right\}, \min_{j\in N_2}\left\{\frac{D_j}{\sum_{i\in\{h+1,\dots,j\}\backslash \overline{N}_2} 1/\beta_i}, \max\{B_j - y_j(\theta), 0\}\right\}\right\}.$$

We set $\theta := \theta + \delta$ and modify the subsets of jobs $\overline{N}_1$, $\overline{N}_2$. If there are still violated constraints from (19)–(20), then we find the next δ-increment and continue increasing θ.

Since each time at least one variable reaches its upper bound A_i or B_k, there are no more than n iterations that involve increasing θ. Each value of an increment δ can be found in $O(n)$ time. Hence in the class of schedules with the critical job h, the problem can be solved in $O(n^2)$ time, and the overall time complexity is $O(n^3)$.

4.1.3 Norm ℓ_H^Σ

Consider now the Hamming norm $\ell_{H,\alpha,\beta}^\Sigma$. Similar to the case of norms ℓ_1 and ℓ_2, the objective function for norm $\ell_{H,\alpha,\beta}^\Sigma$ is separable, and the corresponding problem can be decomposed into two subproblems:

$$\begin{aligned} \min\quad & \sum_{j\in N_1} \alpha_j \operatorname{sgn}(A_j - u_j) \\ \text{s.t.}\quad & \sum_{i=j}^{h} u_i \le \sum_{i=j}^{h} A_i - P_j, \quad j \in N_1, \\ & 0 \le u_j \le A_j, \quad j \in N_1, \end{aligned}$$

and

$$\begin{aligned} \min\quad & \sum_{j\in N_2} \beta_j \operatorname{sgn}(B_j - v_j) \\ \text{s.t.}\quad & \sum_{i=h+1}^{j} v_i \le \sum_{i=h+1}^{j} B_i - Q_j, \quad j \in N_2, \\ & 0 \le v_j \le B_j, \quad j \in N_2. \end{aligned}$$

It is easy to see that there always exists an optimal solution to the above problems such that $u_j \in \{0, A_j\}$, $j \in N_1$, and $v_j \in \{0, B_j\}$, $j \in N_2$. Thus we can introduce new variables $u'_j = u_j/A_j$ and $v'_j = v_j/B_j$ and reformulate the above two subproblems as follows:

$$\begin{aligned} \min\quad & \sum_{j\in N_1} \alpha_j \operatorname{sgn}(1 - u'_j) \\ \text{s.t.}\quad & \sum_{i=j}^{h} A_i u'_i \le \sum_{i=j}^{h} A_i - P_j, \quad j \in N_1, \\ & u'_j \in \{0, 1\}, \quad j \in N_1, \end{aligned} \tag{23}$$

and

$$\begin{aligned} \min\quad & \sum_{j\in N_2} \beta_j \operatorname{sgn}(1 - v'_j) \\ \text{s.t.}\quad & \sum_{i=h+1}^{j} B_i v'_i \le \sum_{i=h+1}^{j} B_i - Q_j, \quad j \in N_2, \\ & v_j \in \{0, 1\}, \quad j \in N_2. \end{aligned} \tag{24}$$

Consider the special case of problem (23) with

$$\overline{p}_j \le d_j - d_{j-1} \quad \text{for all } j \in N_1. \tag{25}$$

We show that the specified special case of problem (23) is NP-hard. Indeed, inequalities (25) imply

$$\sum_{i=j}^{h} \overline{p}_i - (d_h - d_{j-1}) \le \sum_{i=j+1}^{h} \overline{p}_i - (d_h - d_j),$$

or equivalently

$$\sum_{i=j}^{h} A_i - P_j \le \sum_{i=j+1}^{h} A_i - P_{j+1},$$

so that the nested inequalities of problem (23) are redundant for all $j \in N_1$ except for the first inequality with $j = 2$. Since minimizing $\operatorname{sgn}(1 - u'_j)$ for $u'_j \in \{0, 1\}$ is equivalent to maximizing u'_j, the special case of problem (23) is equivalent to the knapsack problem, which is known to be NP-hard:

$$\begin{aligned} \max \quad & \sum_{i=2}^{h} \alpha_i u_i' \\ \text{s.t.} \quad & \sum_{i=2}^{h} A_i u_i' \le \sum_{i=2}^{h} A_i - P_2, \\ & u_i' \in \{0,1\}, \quad 2 \le i \le h. \end{aligned}$$

Using similar arguments, one can demonstrate that problem (24) is NP-hard as well. Thus the inverse problem $1|\text{adjustable } p_j, \pi|L_{\max}$ under the Hamming norm $\ell^{\Sigma}_{H,\alpha,\beta}$ is NP-hard.

4.1.4 Norm $\ell^{\max}_H$

The problem with the norm $\ell^{\max}_{H,\alpha,\beta}$ is similar to that with the norm ℓ_∞ studied in Sect. 4.1.2 and can be represented in the form

$$\begin{aligned} \min \quad & \max\Big\{\max_{j\in N_1}\{\alpha_j \operatorname{sgn} x_j\}, \max_{j\in N_2}\{\beta_j \operatorname{sgn} y_j\}\Big\} \\ \text{s.t.} \quad & \text{constraints (19)–(22).} \end{aligned}$$

Due to the type of the objective function, the solution can be found by considering the variables x_j and y_j one by one in the nondecreasing order of the corresponding penalties α_j and β_j and increasing them to their maximum values A_j and B_j, respectively, until all constraints (19) and (20) become satisfied.

Thus given the sequenced penalties α_j and β_j, the inverse problem can be solved in $O(n)$ time in the class of schedules with the critical job h and in $O(n^2)$ time in all classes.

4.2 Reverse problem $1|\text{adjustable } p_j,\ L^*|L_{\max}$

Suppose that the optimal value of $L_{\max}$ for the given values p_j and d_j is L, a target value of the maximum lateness is L^*, $L > L^*$, and the objective is to find adjusted processing times $\hat{p}_j$, $\hat{p}_j \in [\underline{p}_j, \overline{p}_j]$, so that $\|\hat{\mathbf{p}} - \mathbf{p}\|$ is minimum and the target value L^* is achieved. This means that the processing times of some jobs should be compressed at the minimum cost.

The target value L^* induces the deadlines $\overline{d}_j = d_j + L^*$ for the jobs $j \in N$. Thus the reverse problem $1|\text{adjustable } p_j,$ $L^*|L_{\max}$ reduces to the single-machine problem with controllable processing times $1|p_j\ contr,\ d_j \le \overline{d}_j|K$ defined as follows. In that problem, job processing times can be compressed within given boundaries $[\underline{p}_j, p_j]$. Compression of the processing time of job j from the maximum value p_j by the amount y_j, $0 \le y_j \le p_j - \underline{p}_j$, incurs the cost

$$K(y_1,\dots,y_n) = \begin{cases} \sum_{j=1}^{n} \beta_j y_j & \text{for } \ell_1\text{-norm}, \\ \sum_{j=1}^{n} \beta_j y_j^2 & \text{for } \ell_2\text{-norm}, \\ \max\{\beta_j y_j\} & \text{for } \ell_\infty\text{-norm}, \\ \sum_{j=1}^{n} \beta_j \operatorname{sgn} y_j & \text{for } \ell^{\Sigma}_H\text{-norm}, \\ \max\{\beta_j \operatorname{sgn} y_j\} & \text{for } \ell^{\max}_H\text{-norm}. \end{cases} \quad (26)$$

The objective is to find the compressed processing times $\hat{p}_j = p_j - y_j$ for all jobs $j \in N$ such that the jobs meet their deadlines $\overline{d}_j$ and the compression cost K is minimum.

4.2.1 Norms ℓ_1 and ℓ_2

Consider first the norms $\ell_{1,\alpha,\beta}$ and $\ell_{2,\alpha,\beta}$. The problem with controllable processing times $1|p_j\ contr,\ d_j \le \overline{d}_j|K$ can be represented in the form

$$\begin{aligned} \min \quad & K \\ \text{s.t.} \quad & \sum_{i=1}^{j} (p_i - y_i) \le \overline{d}_j, \quad j \in N, \\ & 0 \le y_j \le B_j, \quad j \in N, \end{aligned} \quad (27)$$

where $B_j = p_j - \underline{p}_j$, and K is linear or quadratic, see (26). The latter problem corresponds to the well-known *Resource Allocation Problem with Nested Constraints* and can be solved $O(n \log n)$ time by an algorithm described in Hochbaum and Hong (1995) and Katoh and Ibaraki (1998, p. 219) for linear and quadratic objective function K.

4.2.2 Norm ℓ_∞

Consider the norm $\ell_{\infty,\alpha,\beta}$. The reverse problem $1|\text{adjustable }$ $p_j,\ L^*|L_{\max}$ reduces to the single-machine problem with controllable processing times $1|p_j\ contr,\ d_j \le \overline{d}_j|$ $\max\{\beta_j y_j\}$ with the min–max compression cost function. For this problem, Choi et al. (1998) have suggested an algorithm of time complexity $O(n \log n + cn)$, where c is a constant depending on $\log[\max_{j\in N}\{\alpha_j(p_j - \underline{p}_j)\}]$.

4.2.3 Norm ℓ^{Σ}_H

Consider now the Hamming norm $\ell^{\Sigma}_{H,\alpha,\beta}$. Since the smallest value of $L_{\max}$ can be guaranteed by sequencing the jobs in the *earliest due date* order (EDD) (Jackson 1955), we can limit our search to the class of EDD-schedules. Renumber the jobs in the EDD order and sequence them in the order of their numbering. Define job completion times using their initial processing times $p_1, p_2, \dots, p_n$:

$$C_i = \sum_{j=1}^{i} p_j.$$

Suppose that all jobs meet their deadlines except for the last job n:

$$\sum_{j=1}^{n} p_j > \overline{d}_n.$$

We demonstrate that finding a set of jobs the processing times of which should be compressed is NP-hard even if there is only one job n that does not meet its deadline $\overline{d}_n$. We assume that all jobs $j \in N$ are compressible, i.e., $p_j - \underline{p}_j > 0$.

The problem of finding the optimum job compressions is similar to problem (27), but the objective function corresponds to the Hamming norm $\ell^{\Sigma}_{H,\alpha,\beta}$:

$$\begin{aligned} \min \quad & \sum_{j \in N} \beta_j \operatorname{sgn} y_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} (p_j - y_j) \le \overline{d}_n, \\ & 0 \le y_j \le B_j, \quad 1 \le j \le n, \end{aligned}$$

where $B_j = p_j - \underline{p}_j$.

It is easy to see that there always exists an optimal solution to the above problem such that $y_j \in \{0, B_j\}$. Thus we can introduce new variables $y'_j = y_j / B_j$ and reformulate the above problem as follows:

$$\begin{aligned} \min \quad & \sum_{j \in N} \beta_j y'_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} (p_j - B_j y'_j) \le \overline{d}_n, \\ & y'_j \in \{0, 1\}, \quad 1 \le j \le n. \end{aligned}$$

To demonstrate that the latter problem is equivalent to the knapsack problem, we rewrite the problem using the new variables $z_j = 1 - y'_j$:

$$\begin{aligned} \max \quad & \sum_{j \in N} \beta_j z_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} B_j z_j \le \overline{d}_n - \sum_{j=1}^{n} (p_j - B_j), \\ & z_j \in \{0, 1\}, \quad 1 \le j \le n. \end{aligned}$$

Since the knapsack problem is known to be NP-hard, the reverse problem $1|\text{adjustable } p_j,\ L^*|L_{\max}$ under the Hamming norm $\ell^{\Sigma}_{H,\alpha,\beta}$ is NP-hard as well.

4.2.4 *Norm* $\ell^{\max}_{H}$

Consider now the norm $\ell^{\max}_{H,\alpha,\beta}$. Since the smallest value of $L_{\max}$ can be guaranteed by sequencing the jobs in the *earliest due date* order (EDD) (Jackson 1955), we can limit our search to the class of EDD-schedules. Renumber the jobs in the EDD order; break ties by giving priority to smaller weights β_j. Sequence the jobs N in the order of their numbering using their normal processing times $p_1, p_2, \dots, p_n$. If deadlines $\overline{d}_j$ are met for every job $j \in N$, then no further action is required. Otherwise the cost $K(y_1, \dots, y_n) = \max\{\beta_j \operatorname{sgn} y_j\}$ can take one of the values from the set $\{\beta_1, \beta_2, \dots, \beta_n\}$.

If the optimum value $K(y_1, \dots, y_n) = \beta_g$, then all jobs j with costs $\beta_j \le \beta_g$ can be compressed down to their smallest processing times $\underline{p}_j$ and in the corresponding EDD-schedule all jobs meet their deadlines. In order to find the smallest K-value, binary search can be used to consider the trial values β_g one by one, checking each time if in the EDD-schedule all jobs meet their deadlines. Since for the given EDD-sequence, the feasibility check can be done in $O(n)$ time, the time complexity of solving the reverse problem $1|\text{adjustable } p_j,\ L^*|L_{\max}$ under the Hamming norm $\ell^{\max}_{H,\alpha,\beta}$ is $O(n \log n)$.

5 Conclusions

In this paper, we have studied the inverse and reverse counterparts of the single-machine scheduling problem $1\|L_{\max}$ in the case of adjustable due dates or processing times under five different types of the norm: ℓ_1, ℓ_2, ℓ_∞, ℓ^{Σ}_{H}, and $\ell^{\max}_{H}$. Two problems appeared to be NP-hard; for the remaining problems, we have produced their mathematical programming formulations and developed efficient solution algorithms.

The results are summarized in Table 1. Interestingly, the inverse problems with adjustable due dates appear to be easier than those with adjustable processing times, while some reverse problems with adjustable due dates appear to be more difficult than those with adjustable processing times.

Observe that the results summarized in Table 1 can be reformulated for inverse and reverse counterparts of problem $1|r_j|C_{\max}$. The forward problem $1|r_j|C_{\max}$ consists in scheduling the jobs without overlapping starting not earlier than their release times r_j, so that the finish time of the last job is minimum. The necessary and sufficient conditions for optimality of a given job sequence for problem $1|r_j|C_{\max}$ are symmetric to those specified in Theorem 1 for problem $1\|L_{\max}$, see Lin and Wang (2007).

It is an interesting research goal to study the inverse and reverse counterparts of problems $1\|L_{\max}$ and $1|r_j|C_{\max}$ with two types of parameters adjustable simultaneously: adjustable processing times and due dates for the first problem or adjustable processing times and release times for the second one.

Table 1 Time complexity of the inverse and reverse counterparts of problem $1||L_{\max}$

	Adjustable d_j	Norm	Section	Adjustable p_j	Norm	Section
Inverse problem	$O(n\log n)$	$\ell_1, \ell_2,$	3.1	$O(n^2\log n)$	$\ell_1, \ell_2,$	4.1.1
with a given π	$O(n\log n)$	ℓ_∞	3.1	$O(n^3)$	ℓ_∞	4.1.2
	$O(n\log n)$	ℓ_H^Σ	3.1	NP-hard	ℓ_H^Σ	4.1.3
	$O(n\log n)$	$\ell_H^{\max}$	3.1	$O(n^2)$	$\ell_H^{\max}$	4.1.4
Reverse problem	$O(n^2)$	ℓ_1, ℓ_2	3.2	$O(n\log n)$	ℓ_1, ℓ_2	Hochbaum and Hong (1995); Katoh and Ibaraki (1998)
with a given L^*	$O(n^2)$	ℓ_∞	3.2	$O(n\log n + cn)$	ℓ_∞	Choi et al. (1998)
	$O(n^2)$	ℓ_H^Σ	3.2	NP-hard	ℓ_H^Σ	4.2.3
	$O(n^2)$	$\ell_H^{\max}$	3.2	$O(n\log n)$	$\ell_H^{\max}$	4.2.4

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Appendix

Proof of Lemma 1 Suppose that job h is critical for the original (non-adjusted) due dates $\mathbf{d}$, but it is not critical for some optimum adjusted due dates $\hat{\mathbf{d}}$. If another job k is critical for $\hat{\mathbf{d}}$, then

$$L_h = C_h - d_h \geq C_k - d_k = L_k$$

(h is critical for due dates $\mathbf{d}$),

$$\hat{L}_h = C_h - \hat{d}_h < C_k - \hat{d}_k = \hat{L}_k \tag{28}$$

(k is critical for due dates $\hat{\mathbf{d}}$).

In addition, we will use the following two conditions which hold for any job $u \in N$:

$$L_u = C_u - d_u \leq C_h - d_h = L_h \tag{29}$$

(h is critical for due dates $\mathbf{d}$),

$$\hat{L}_u = C_u - \hat{d}_u \leq C_k - \hat{d}_k = \hat{L}_k \tag{30}$$

(k is critical for due dates $\hat{\mathbf{d}}$).

To prove the lemma, we demonstrate that there exists another set of optimum due dates $\hat{\hat{\mathbf{d}}}$ such that the following properties hold:

(i) Both jobs k and h are critical for due dates $\hat{\hat{\mathbf{d}}}$.

(ii) Due date boundaries are observed, i.e., $\hat{\hat{d}}_j \in [\underline{d}_j, \overline{d}_j]$ for all jobs $j \in N$.

(iii) Due date deviation of $\hat{\hat{\mathbf{d}}}$ from the initial due dates is no larger than that of $\hat{\mathbf{d}}$:

$$\left\|\hat{\hat{\mathbf{d}}} - \mathbf{d}\right\| \leq \|\hat{\mathbf{d}} - \mathbf{d}\|. \tag{31}$$

(iv) The necessary and sufficient conditions of Theorem 1 of optimality of permutation π are satisfied for the modified due dates $\hat{\hat{\mathbf{d}}}$ and job k.

The latter property implies that π is an optimal permutation for $\hat{\hat{\mathbf{d}}}$, which, together with Property (i), proves the lemma.

We define the new values $\hat{\hat{\mathbf{d}}}$ so that the maximum lateness $L^0 = \max_{j\in N}\{C_j - \hat{\hat{d}}_j\}$ calculated for permutation π and due dates $\hat{\hat{\mathbf{d}}}$ satisfies

$$L^0 = \min\{\hat{L}_k, L_h\}. \tag{32}$$

Consider the two cases.

Case 1

$$L^0 = \hat{L}_k \leq L_h. \tag{33}$$

In this case, condition (30) implies

$$\hat{L}_u \leq \hat{L}_k = L^0,$$

so that any job $u \in N\backslash\{k\}$ has the lateness with respect to due dates $\hat{\mathbf{d}}$ no larger than L^0, and Property (i) holds for job k.

In order to achieve Property (i) for job h, its due date should be modified to the value

$$\hat{\hat{d}}_h = C_h - L^0, \tag{34}$$

so that

$$\hat{\hat{L}}_h = C_h - \hat{\hat{d}}_h = L^0.$$

Observe that the modified due date $\hat{\hat{d}}_h$ is smaller than $\hat{d}_h$:

$$\hat{\hat{d}}_h - \hat{d}_h = (C_h - L^0) - \hat{d}_h = \hat{L}_h - L^0 \overset{(28)}{<} \hat{L}_k - L^0 \overset{(33)}{=} 0.$$

Thus the due dates $\hat{\hat{\mathbf{d}}}$ are obtained from $\hat{\mathbf{d}}$ by decreasing one component $\hat{d}_h$ down to $\hat{\hat{d}}_h$.

We demonstrate that Properties (ii)–(iii) hold for job h:

$$\hat{\hat{d}}_h \overset{(34)}{=} C_h - L^0 \overset{(33)}{\geq} C_h - L_h = d_h \geq \underline{d}_h,$$

so that

$$d_h \leq \hat{\hat{d}}_h \leq \hat{d}_h \leq \overline{d}_h.$$

Thus condition (31) is satisfied.

Consider now Property (iv). Due to Property (i), job k is critical for the due dates $\hat{\mathbf{d}}$ and $\hat{\hat{\mathbf{d}}}$, so that condition (1) holds. We need to demonstrate that condition (2) holds for $\hat{\hat{\mathbf{d}}}$ and any job j that precedes k. Indeed, according to the assumption, due dates $\hat{\mathbf{d}}$ are optimum, so that condition (2) holds for $\hat{\mathbf{d}}$ and any job j that precedes the critical job k. Clearly, after one component of $\hat{\mathbf{d}}$ is decreased leading to $\hat{\hat{\mathbf{d}}}$, condition (2) will still hold for $\hat{\hat{\mathbf{d}}}$ and the same critical job k.

Case 2

$$L^0 = L_h < \hat{L}_k. \tag{35}$$

In order to achieve Property (i) and to decrease the maximum lateness value down to L^0, consider the due dates $\hat{\mathbf{d}}$ and the set of jobs $U = \{u \in N \mid \hat{L}_u > L^0\}$ which lateness is larger than L^0. Clearly, set U includes job k. For each job $u \in U$, its due date $\hat{d}_u$ should be modified to the value

$$\hat{\hat{d}}_u = C_u - L^0, \tag{36}$$

so that its lateness decreases to the value L^0:

$$\hat{\hat{L}}_u = C_u - \hat{\hat{d}}_u = L^0. \tag{37}$$

Observe that each modified due date $\hat{\hat{d}}_u$ is larger than the corresponding due date $\hat{d}_u$:

$$\hat{\hat{d}}_u - \hat{d}_u = (C_u - L^0) - \hat{d}_u = \hat{L}_u - L^0 > 0.$$

Due to condition (37), after the due dates of the jobs from U are increased, all of them (including job k) become critical for $\hat{\hat{\mathbf{d}}}$.

Consider now Property (i) for job h. If $h \in U$, then job h is properly adjusted as described above, and it becomes critical for $\hat{\hat{\mathbf{d}}}$. Otherwise, $h \notin U$, or equivalently

$$\hat{L}_h \leq L^0, \tag{38}$$

and its due date $\hat{d}_h$ should be modified to the value

$$\hat{\hat{d}}_h = C_h - L^0, \tag{39}$$

so that

$$\hat{\hat{L}}_h = C_h - \hat{\hat{d}}_h = L^0.$$

Observe that the modified due date $\hat{\hat{d}}_h$ is smaller than $\hat{d}_h$:

$$\hat{\hat{d}}_h - \hat{d}_h = (C_h - L^0) - \hat{d}_h = \hat{L}_h - L^0 \overset{(38)}{\leq} 0.$$

Now we demonstrate that Properties (ii)–(iii) hold for the jobs from U and for job h. The due date of any job $u \in U$ increases from $\hat{d}_u \geq \underline{d}_u$ to

$$\hat{\hat{d}}_u \overset{(36)}{=} C_u - L^0 \overset{(35)}{=} C_u - L_h \overset{(29)}{\leq} d_u \leq \overline{d}_u.$$

It follows that

$$\underline{d}_u \leq \hat{d}_u \leq \hat{\hat{d}}_u \leq d_u,$$

so that the due date deviation of any job $u \in U$ does not increase:

$$\left|\hat{\hat{d}}_u - d_u\right| \leq \left|\hat{d}_u - d_u\right|.$$

Consider now the adjustment of the due date of job h. If $h \in U$, then the above arguments hold for job h. Otherwise the due date of job h decreases from $\hat{d}_h \leq \overline{d}_h$ to

$$\hat{\hat{d}}_h \overset{(39)}{=} C_h - L^0 \overset{(35)}{=} C_h - L_h = d_h \geq \underline{d}_h,$$

so that

$$d_h \leq \hat{\hat{d}}_h \leq \hat{d}_h \leq \overline{d}_h.$$

Thus condition (31) is satisfied.

Finally, we prove Property (iv). Due to property (i), job k is critical for the due dates $\hat{\mathbf{d}}$ and $\hat{\hat{\mathbf{d}}}$, so that condition (1) holds. We need to demonstrate that condition (2) holds for $\hat{\hat{\mathbf{d}}}$ and any job j that precedes k. Since the necessary and sufficient conditions of optimality of permutation π hold for the critical job k and due dates $\hat{\mathbf{d}}$, we have

$$\hat{d}_j \leq \hat{d}_k.$$

If $j \notin U$, then its due date is not increased, so that

$$\hat{\hat{d}}_j \leq \hat{\hat{d}}_k.$$

If $j = u \in U$, then condition (2) is violated only if

$$C_u < C_k$$

and

$$\hat{\hat{d}}_u > \hat{\hat{d}}_k,$$

which implies

$$C_u - \hat{\hat{d}}_u < C_k - \hat{\hat{d}}_k = L^0,$$

a contradiction to (37).

Lemma 1 is proved. □

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