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Single-machine scheduling to stochastically minimize maximum lateness

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Abstract We study the problem of scheduling a set of jobs on a single machine, to minimize the maximum lateness *ML* or the maximum weighted lateness *MWL* under stochastic order. The processing time P_i, the due date D_i, and the weight W_i of each job i may all be random variables. We obtain the optimal sequences in the following situations: (i) For *ML*, the $\{P_i\}$ can be likelihood-ratio ordered, the $\{D_i\}$ can be hazard-rate ordered, and the orders are *agreeable*; (ii) For *MWL*, $\{D_i\}$ are exponentially distributed, $\{P_i\}$ and $\{W_i\}$ can be likelihood-ratio ordered and the orders are agreeable with the rates of $\{D_i\}$; and (iii) For *ML*, P_i and D_i are exponentially distributed with rates μ_i and ν_i, respectively, and the sequence $\{\nu_i(\nu_i + \mu_i)\}$ has the same order as $\{\nu_i(\nu_i + \mu_i + A_0)\}$ for some sufficiently large A_0. Some related results are also discussed.

Keywords Stochastic scheduling · Stochastic order · Deterministic or stochastic processing times · Random due dates · Maximum weighted lateness

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1 Introduction

In this paper we are concerned with a single-machine stochastic scheduling problem to stochastically minimize the objective function of *maximum lateness* or *maximum weighted lateness*. The performance measure based on maximum lateness has been considered by a number of authors in the literature; see, for example, Jackson (1955), Sarin et al. (1991), Zhou and Cai (1997), Muller and Stoyan (2002), Pinedo (2002) and Cai and Zhou (2005), among others. The focus of our study in this paper is, however, to investigate the problem to optimize the performance measure under *stochastic order*. Under such an order, a random variable X is considered as (stochastically) smaller than another random variable Y, if X is always more likely than Y to take on small values. Therefore, the optimal schedule under stochastic order is always more likely to produce a smaller value of the objective function than any other schedule. Such a desired property is not generally available for the optimal schedule that minimizes the mean value of the objective function.

Scheduling problems involving stochastic order have been studied by some researchers. Brown and Solomon (1973) considered the problem of optimal issuing policies under stochastic order, which is equivalent to a scheduling problem. Shanthikumar and Yao (1991) proposed a bivariate characterization, which is extremely useful for interchange arguments in scheduling. They considered the problem of minimizing the total flowtime and obtained optimization results with likelihood-ratio ordered processing times. Chang and Yao (1993) further demonstrated this theory and applied some stochastic rearrangement inequalities to obtain solutions to the stochastic counterpart of many classical deterministic scheduling problems. Boxma and Forst (1986) showed that, when processing times are stochastically ordered and due dates are independent and identically distributed (i.i.d.), the SEPT (shortest expected processing time) rule minimizes the expected number of tardy jobs. Chang and Yao (1993) permitted the rearrangement of weights and processing times separately. In the case of agreeable due dates, so that the SEPT rule is identical to the SEDD (shortest expected due date) rule—also known as the EEDD (earliest expected due date) rule, SEPT minimizes certain classes of functions of lateness or tardiness.

The main results we present in this paper are the optimal sequences for the problem of minimizing the maximum lateness (or weighted maximum lateness) in a number of situations, including: (A) The likelihood ratios of the processing times and the hazard rates of the due dates meet an agreeability condition; (B) The due dates are exponentially distributed with rates agreeable with the likelihood ratios of the processing times and the weights; and (C) The processing times and the due dates are exponentially distributed.

The rest of the paper is organized as follows. In Sect. 2 we specify the basic problems and assumptions. Results on stochastic minimization of maximum lateness and maximum weighted lateness are presented in Sect. 3. Section 4 derives some more delicate results with exponentially distributed processing times and due dates.

2 Basic problems and assumptions

We study the following problems: A set of n jobs are to be processed on a single machine, which are all available at time zero. The *processing times* P_i of jobs i, $i = 1, 2, \dots, n$, are independent random variables. Each job i has a due date D_i. The due dates $D_1, \dots, D_n$ are independent random variables and independent of $\{P_i\}$. The machine can process at most one job at a time. We consider the *maximum lateness*:

$$ML(\pi) = \max_{1\le i\le n} \bigl(C_i(\pi) - D_i\bigr), \tag{1}$$

and the *maximum weighted lateness*:

$$MWL(\pi) = \max_{1\le i\le n} W_i\bigl(C_i(\pi) - D_i\bigr), \tag{2}$$

where

- $\pi = (i_1, \dots, i_n)$ is a permutation of $\{1, \dots, n\}$, called a *sequence*, that determines the order to process the n jobs, with $i_k = j$ if and only if job j is the k-th to be processed;
- $C_i(\pi)$ is the completion time of job i under sequence π; and
- W_i is the weight assigned to job i, $i = 1, \dots, n$; $W_1, \dots, W_n$ are independent random variables, independent of $\{P_i\}$ and $\{D_i\}$.

In order to formulate the scheduling problems under stochastic order, we give the definitions of three types of stochastic orders below.

Definition 2.1 Let X and Y be two random variables. X is said to be "stochastically less than or equal to Y", written $X \le_{\mathrm{st}} Y$ or $Y \ge_{\mathrm{st}} X$, if

$$\Pr(X \le x) \ge \Pr(Y \le x), \quad \text{for all } x \in (-\infty, \infty),$$

or, equivalently,

$$\Pr(X < x) \ge \Pr(Y < x), \quad \text{for all } x \in (-\infty, \infty).$$

In other words, X is more likely to take small values than Y over all real values. This order is referred to as the "usual stochastic order" (cf. Shaked and Shanthikumar 1994).

Our problem is to find an optimal sequence π^* such that

$$ML\bigl(\pi^*\bigr) \le_{\mathrm{st}} ML(\pi), \quad \text{for all } \pi, \tag{3}$$

or

$$MWL\bigl(\pi^*\bigr) \le_{\mathrm{st}} MWL(\pi), \quad \text{for all } \pi. \tag{4}$$

Such a π^* is said to *stochastically minimize* $ML(\pi)$ (or $MWL(\pi)$).

Remark 2.1 Let $F_X(x) = \Pr(X \le x)$ and $\bar{F}_X(x) = \Pr(X > x) = 1 - F_X(x)$. Then $X \le_{\mathrm{st}} Y$ if and only if $F_X(x) \ge F_Y(x)$ or, equivalently, $\bar{F}_X(x) \le \bar{F}_Y(x)$ for all x. Furthermore, $X \le_{\mathrm{st}} Y$ is also equivalent to $E[f(X)] \le E[f(Y)]$ for all increasing functions $f(x)$. Therefore, $X \le_{\mathrm{st}} Y \Longrightarrow E[X] \le E[Y]$, and so the solution to (3) or (4) also minimizes $E[ML(\pi)]$ or $E[MWL(\pi)]$.

For the next definition, let $h(x)$ denote the hazard rate function of a nonnegative random variable X, which is a nonnegative function such that $\int_0^\infty h(t)\,dt = \infty$. When X is a continuous random variable, $h(x)$ determines the distribution of X by

$$\bar{F}(x) = P(X > x) = \exp\left\{-\int_0^{x\vee 0} h(t)\,dt\right\}. \tag{5}$$

Definition 2.2 Let X and Y be two nonnegative continuous random variables with hazard rate functions h_X and h_Y, respectively. If $h_X(t) \ge h_Y(t)$ for all $t \ge 0$, then X is said to be "less than or equal to Y in hazard-rate order", written $X \le_{\mathrm{hr}} Y$ or $Y \ge_{\mathrm{hr}} X$.

Definition 2.3 Let X and Y be two continuous random variables with density functions f and g, respectively. If $f(u)/g(u) \ge f(v)/g(v)$ for all $u \le v$, then X is said to be "less than or equal to Y in likelihood-ratio order", written $X \le_{\mathrm{lr}} Y$ or $Y \ge_{\mathrm{lr}} X$.

Remark 2.2 There are many commonly considered families of distributions that can be likelihood-ratio and/or hazard-rate ordered. For example, exponential distributions, Weibull distributions with a common shape parameter, Gamma distributions with a common shape parameter, Poisson distributions and geometric distributions can all be hazard-rate ordered as well as likelihood-ratio ordered. In such cases, the likelihood-ratio or hazard-rate order is equivalent to the order of the expectations. In fact, we have the following implications about the orders between random variables:

$$X \le_{\text{lr}} Y \quad \Longrightarrow \quad X \le_{\text{hr}} Y \quad \Longrightarrow \quad X \le_{\text{st}} Y \\ \Longrightarrow E[X] \le E[Y].$$

Our main results include:

(A) If the processing times $\{P_i\}$ can be likelihood-ratio ordered, the due dates $\{D_i\}$ can be hazard-rate ordered, and the orders are *agreeable* in the sense that

$$P_i \le_{\text{lr}} P_j \quad \Longleftrightarrow \quad D_i \le_{\text{hr}} D_j, \quad \text{for all } i, j \in \{1, \dots, n\},$$

then $ML(\pi)$ is stochastically minimized by the sequence in nondecreasing likelihood-ratio order of $\{P_i\}$ or, equivalently, in nondecreasing hazard-rate order of $\{D_i\}$. That is, EEDD, which is here equivalent to SEPT, stochastically minimizes *ML*.

(B) When the due dates $\{D_i\}$ are exponentially distributed with $\{\nu_i\}$, if the processing times $\{P_i\}$ and the weights $\{W_i\}$ can be likelihood-ratio ordered and the orders are *agreeable* with $\{D_i\}$ in the sense that

$$P_i \le_{\text{lr}} P_j \quad \Longleftrightarrow \quad \nu_i \ge \nu_j \quad \Longleftrightarrow \quad W_i \ge_{\text{lr}} W_j, \\ \text{for all } i, j \in \{1, \dots, n\},$$

then $MWL(\pi)$ is stochastically minimized by the sequence in nondecreasing likely-hood-ratio order of $\{P_i\}$, or in nonincreasing order of $\{\nu_i\}$, or in nonincreasing likelihood-ratio order of $\{W_i\}$. Therefore, in this case the optimal sequence is equivalent to SEPT and EEDD as well as the *largest mean weight first* rule.

(C) If P_i and D_i are exponentially distributed with rates μ_i and ν_i, respectively, and the sequence $\{\nu_i(\nu_i + \mu_i)\}$ has the same order as $\{\nu_i(\nu_i + \mu_i + A_0)\}$ for some sufficiently large A_0, then $ML(\pi)$ is stochastically minimized by the sequence in nonincreasing order of $\{\nu_i(\nu_i + \mu_i)\}$.

3 Stochastic minimization of maximum lateness

It has been known that in a deterministic environment, the maximum lateness $ML(\pi)$ is minimized by the EDD (Earliest Due Date) rule. That is, the optimal sequence π^* is in nonincreasing order of the deterministic due dates $\{D_i\}$, regardless of the processing times. As a result, if $\{P_i\}$ are random variables but $\{D_i\}$ remain deterministic, then the EDD rule minimizes $ML(\pi)$ almost surely (with probability 1) for any π, which implies (3) with $\pi^* =$ EEDD. Thus, when the due dates are deterministic, the EDD rule remains optimal under stochastic order. But it was unclear, what would happen if the due dates are also stochastic. In the following we will show that, when the due dates $\{D_i\}$ are random variables, the optimal schedule will depend on the processing times, and give the optimal solution under certain conditions.

First we state a lemma regarding a characterization of the likelihood-ratio order, which will play a crucial rule in the proof of our first theorem. It is a result from Theorem 1.C.14 of Shaked and Shanthikumar (1994).

Lemma 3.1 *Let X and Y be two independent random variables and $\phi_1(u, v)$, $\phi_2(u, v)$ be two bivariate real-valued functions. If $X \le_{\text{lr}} Y$ and $u \le v$ implies $\phi_1(u, v) \le \phi_2(u, v)$ and $\phi_1(u, v) + \phi_1(v, u) \le \phi_2(u, v) + \phi_2(v, u)$, then*

$$E[\phi_1(X, Y)] \le E[\phi_2(X, Y)].$$

Remark 3.1 Brown and Solomon (1973) provided a lemma with regard to a pairwise interchange of likelihood-ratio ordered distributions, which is another characterization for the likelihood-ratio order and is similar to Theorem 1.C.13 of Shaked and Shanthikumar (1994). It is, however, insufficient to prove our results, as our objective functions involve random elements other than the processing times, such as the due dates and weights.

Our first main result is as follows.

Theorem 3.1 *Let $P_1, \dots, P_n$ be independent random processing times and $D_1, \dots, D_n$ be independent random due dates, independent of $\{P_i\}$. If the $\{P_i\}$ can be likelihood-ratio ordered, the $\{D_i\}$ can be hazard-rate ordered, and the orders satisfy the following agreeability condition*:

$$P_i \le_{\text{lr}} P_j \quad \Longleftrightarrow \quad D_i \le_{\text{hr}} D_j, \\ \textit{for all } \ i, j \in \{1, \dots, n\},$$

then the maximum lateness $ML(\pi)$ is stochastically minimized by the sequence in nondecreasing likelihood-ratio order of the processing times $\{P_i\}$ or, equivalently, in nondecreasing hazard-rate order of the processing times $\{D_i\}$ (i.e., SEPT or EEDD).

Proof By the independence of the processing times $P_1, \dots, P_n$ and the due dates $D_1, \dots, D_n$, we have

$$\Pr(ML(\pi) < x) \\ = E[\Pr(ML(\pi) < x \mid P_1, \dots, P_n)]$$

$$= E\Big[\Pr\Big(\max_{1\le i\le n}\big(C_i(\pi) - D_i\big) < x \mid P_1, \ldots, P_n\Big)\Big]$$

$$= E\big[\Pr\big(C_i(\pi) - D_i < x, \imath = 1, \ldots, n \mid P_1, \ldots, P_n\big)\big]$$

$$= E\left[\prod_{i=1}^{n} \Pr\big(D_i > C_i(\pi) - x \mid P_1, \ldots, P_n\big)\right]$$

$$= E\left[\prod_{i=1}^{n} \bar{F}_i\big(C_i(\pi) - x\big)\right]$$

$$= E\left[\prod_{i=1}^{n} \bar{F}_i\big(C_i(\pi) - x\big)\right], \tag{6}$$

where the expectation is with respect to $\{P_i\}$ and

$$\bar{F}_i(x) = P(D_i > x) = \exp\left\{-\int_0^{x\vee 0} h_i(t)\,dt\right\},$$
$$i = 1, \ldots, n, \text{ (cf. (5))} \tag{7}$$

where $h_i(t)$ is the hazard rate function of D_i.

For an arbitrary job sequence $\pi = (\ldots, r, s, \ldots)$, let $\pi' = (\ldots, s, r, \ldots)$ denote the sequence after interchanging two neighboring jobs r and s in π. Let C be the completion time of the job sequenced just before r under π. Then we have

$$C_r(\pi) = C + P_r, \qquad C_s(\pi) = C + P_r + P_s,$$
$$C_s(\pi') = C + P_s, \qquad C_r(\pi') = C + P_r + P_s,$$
$$C_i(\pi') = C_i(\pi), \quad \text{for } i \ne r, s.$$

Hence, by (6),

$$\Pr\big(ML(\pi) < x\big)$$
$$= E\big[\bar{F}_r(C + P_r - x)\bar{F}_s(C + P_r + P_s - x)H(x)\big] \tag{8}$$

and

$$\Pr\big(ML(\pi') < x\big)$$
$$= E\big[\bar{F}_s(C + P_s - x)\bar{F}_r(C + P_s + P_r - x)H(x)\big], \tag{9}$$

where

$$H(x) = H\big(x; P_r, P_s, \{P_i, i \ne r, s\}\big)$$
$$= \prod_{i\ne r,s} \bar{F}_i\big(C_i(\pi) - x\big). \tag{10}$$

Suppose $P_r \le_{\mathrm{lr}} P_s$ and $D_r \le_{\mathrm{hr}} D_s$. Then $h_r(t) \ge h_s(t)$ for all t, which together with (7) leads to

$$\frac{\bar{F}_r(C+u-x)}{\bar{F}_r(C+u+v-x)}$$
$$= \exp\left\{-\int_0^{(C+u-x)\vee 0} h_r(t)\,dt + \int_0^{(C+u+v-x)\vee 0} h_r(t)\,dt\right\}$$
$$= \exp\left\{\int_{(C+u-x)\vee 0}^{(C+u+v-x)\vee 0} h_r(t)\,dt\right\}$$
$$\ge \exp\left\{\int_{(C+u-x)\vee 0}^{(C+u+v-x)\vee 0} h_s(t)\,dt\right\}$$
$$= \frac{\bar{F}_s(C+u-x)}{\bar{F}_s(C+u+v-x)}.$$

Hence,

$$\bar{F}_r(C+u-x)\bar{F}_s(C+u+v-x)$$
$$\ge \bar{F}_s(C+u-x)\bar{F}_r(C+u+v-x) \tag{11}$$

and, by the same arguments,

$$\bar{F}_r(C+v-x)\bar{F}_s(C+u+v-x)$$
$$\ge \bar{F}_s(C+v-x)\bar{F}_r(C+u+v-x). \tag{12}$$

As $\bar{F}_r(x)$ is a nonincreasing function, (12) also implies that

$$\bar{F}_r(C+u-x)\bar{F}_s(C+u+v-x)$$
$$\ge \bar{F}_s(C+v-x)\bar{F}_r(C+u+v-x), \quad \text{if } u \le v. \tag{13}$$

Now given $P_i = p_i$ for $i \ne r, s$, and for fixed x, define

$$\phi_1(u, v) = \bar{F}_s(C+v-x)\bar{F}_r(C+u+v-x)$$
$$\times H\big(x; u, v, \{p_i, i \ne r, s\}\big)$$

and

$$\phi_2(u, v) = \bar{F}_r(C+u-x)\bar{F}_s(C+u+v-x)$$
$$\times H\big(x; u, v, \{p_i, i \ne r, s\}\big).$$

Note $H(x; u, v, \{p_i, i \ne r, s\}) = H(x; v, u, \{p_i, i \ne r, s\})$, since P_r and P_s are interchangeable in (10). Hence, (11) implies $\phi_2(u, v) \ge \phi_1(v, u)$ and (12) implies $\phi_2(v, u) \ge \phi_1(u, v)$, consequently, $\phi_2(u, v) + \phi_2(v, u) \ge \phi_1(u, v) + \phi_1(v, u)$. Furthermore, (13) shows that $\phi_2(u, v) \ge \phi_1(u, v)$ for $u \le v$. It then follows from Lemma 3.1 that, conditional on $P_i, i \ne r, s$, $P_r \le_{\mathrm{lr}} P_s \Longrightarrow E[\phi_1(P_r, P_s)] \le E[\phi_2(P_r, P_s)]$. Thus, by (8–9), $P_r \le_{\mathrm{lr}} P_s$ and $D_r \le_{\mathrm{hr}} D_s$ imply

$$\Pr\big(ML(\pi) < x\big)$$
$$= E\big[E\big[\bar{F}_r(C + P_r - x)\bar{F}_s(C + P_r + P_s - x)H(x) \mid \{P_i, i \ne r, s\}\big]\big]$$
$$= E\big[E\big[\phi_2(P_r, P_s) \mid \{P_i, i \ne r, s\}\big]\big]$$
$$\ge E\big[E\big[\phi_1(P_r, P_s) \mid \{P_i, i \ne r, s\}\big]\big]$$

$$= E\big[E\big[\bar{F}_s(C+P_s-x)\bar{F}_r(C+P_r+P_s-x)H(x) \mid \{P_i, i\neq r,s\}\big]\big]$$
$$= \Pr\big(ML(\pi')<x\big), \quad \text{for all } x\in(-\infty,\infty)$$
$$\Longrightarrow \quad ML(\pi)\leq_{\rm st} ML(\pi').$$

This shows that an optimal solution to minimize $ML(\pi)$ stochastically is given by SEPT (EEDD). □

Remark 3.2 When $P_i = p_i$ are deterministic and $\{D_i\}$ follow a common distribution, equations (8–9) reduce to $\Pr(ML(\pi) < x) = \bar{F}(C + p_r - x)\bar{F}(C + p_r + p_s - x)H(x)$ and $\Pr(ML(\pi') < x) = \bar{F}(C + p_s - x)\bar{F}_r(C + p_r + p_s - x)H(x)$. Hence, it is easy to see that $ML(\pi) \leq_{\rm st} ML(\pi') \Longleftrightarrow p_r \leq p_s$. Thus, unlike in the case of deterministic due dates, the optimal solution does depend on the processing times, when D_i's are random, even in this very special case. Therefore, the optimal solution can no longer be given by any rule independent of $\{P_i\}$, such as the EEDD rule, without an agreeability condition.

When the due dates $\{D_i\}$ are exponentially distributed, we have the following result on the maximum weighted lateness.

Theorem 3.2 *Suppose that the due dates $D_1,\dots,D_n$ are exponentially distributed with rates $\nu_1,\dots,\nu_n$, respectively. If $\{P_i\}$ and $\{W_i\}$ can be likelihood-ratio ordered and satisfy the following agreeability condition with $\{D_i\}$:*

$$P_i \leq_{\rm lr} P_j \quad\Longleftrightarrow\quad \nu_i\geq\nu_j \quad\Longleftrightarrow\quad W_i\geq_{\rm lr} W_j \quad \text{for all } i,j\in\{1,\dots,n\},$$

then the maximum weighted lateness $MWL(\pi)$ is stochastically minimized by EEDD or, equivalently, by SEPT, or the largest expected weight first rule.

Proof Similar to (6) and (8–10), we get

$$\Pr\big(MWL(\pi)<x\big) = E\left[\prod_{i=1}^{n}\bar{F}_i\big(C_i(\pi)-X_i\big)\right], \tag{14}$$

where $X_i = x/W_i$ and the expectation is with respect to $\{P_i\}$ and $\{W_i\}$,

$$\Pr\big(MWL(\pi)<x\big) = E\big[\bar{F}_r(C+P_r-X_r)\bar{F}_s(C+P_r+P_s-X_s)H(x)\big] \tag{15}$$

and

$$\Pr\big(MWL(\pi')<x\big) = E\big[\bar{F}_s(C+P_s-X_s)\bar{F}_r(C+P_s+P_r-X_r)H(x)\big], \tag{16}$$

where

$$H(x) = H\big(x; P_r, P_s, \{P_i, W_i, i\neq r,s\}\big) = \prod_{i\neq r,s}\bar{F}_i\big(C_i(\pi)-X_i\big).$$

Suppose that $P_r \leq_{\rm lr} P_s$, $\nu_r\geq\nu_s$ and $W_r\geq_{\rm lr} W_s$. Let $\{P_i = p_i,\ i\neq r,s\}$ and $\{W_i = w_i,\ i=1,\dots,n\}$ be given, and for now suppose $w_r\geq w_s$. Then $X_i = x_i = x/w_i$, $i = 1,\dots,n$. Define

$$\phi_1(u,v) = \bar{F}_s(C+v-x_s)\bar{F}_r(C+u+v-x_r) \times H\big(x; u, v, \{p_i, w_i, i\neq r,s\}\big)$$

and

$$\phi_2(u,v) = \bar{F}_r(C+u-x_r)\bar{F}_s(C+u+v-x_s) \times H\big(x; u, v, \{p_i, w_i, i\neq r,s\}\big).$$

Let $u\leq v$. Since D_i is exponential with rate ν_i, $\bar{F}_i(x) = \exp\{-\nu_i(x\vee 0)\}$, $i=1,\dots,n$. Thus $\phi_1(u,v)\leq\phi_2(u,v)$ if

$$\nu_r\big[(C+u+v-x_r)\vee 0-(C+u-x_r)\vee 0\big] \geq \nu_s\big[(C+u+v-x_s)\vee 0-(C+v-x_s)\vee 0\big]. \tag{17}$$

First consider the case $x\geq 0$. Then $0\leq x_r = x/w_r\leq x/w_s\leq x_s$. If $C+u\geq x_r$ and $C+v\geq x_s$, then (17) becomes $\nu_r v\geq\nu_s u$, which holds, since $u\leq v$ and $\nu_r\geq\nu_s$. If $C+u\geq x_r$ and $C+v<x_s$, then (17) becomes $\nu_r v\geq\nu_s(C+u+v-x_s)\vee 0$, which also holds because $C+u+v-x_s<u$ in this case. If $C+u<x_r$ and $C+v\geq x_s$, then (17) becomes $\nu_r(C+u+v-x_r)\vee 0\geq\nu_s u$, which again holds because $C+u+v-x_r\geq C+u+v-x_s>u$. Finally, as $x_r\leq x_s$ and $\nu_r\geq\nu_s$ imply, $\nu_r(C+u+v-x_r)\vee 0\geq\nu_s(C+u+v-x_s)\vee 0$, (17) still holds when $C+u<x_r$ and $C+v<x_s$. In summary, (17) holds for $x\geq 0$. When $x<0$, (17) reduces to $\nu_r v\geq\nu_s u$ and, hence, holds as well. Therefore, we have shown that $u\leq v\Longrightarrow\phi_1(u,v)\leq\phi_2(u,v)$. So given $\{P_i=p_i, i\neq r,s\}$ and $\{W_i=w_i\}$ with $w_r\geq w_s$, it follows from Lemma 3.1 that $E[\phi_1(P_r,P_s)]\leq E[\phi_2(P_r,P_s)]$, when $P_r\leq_{\rm lr}P_s$.

By similar (and, in fact, simpler) arguments, we can show that $\phi_1(u,v)\leq\phi_2(v,u)$ and $\phi_1(v,u)\leq\phi_2(u,v)$, so that $\phi_1(u,v)+\phi_1(v,u)\leq\phi_2(u,v)+\phi_2(v,u)$. Thus, according to Lemma 3.1, conditional on $\{P_i=p_i, i\neq r,s\}$, $E[\phi_1(P_r,P_s)]\leq E[\phi_2(P_r,P_s)]$, when $P_r\leq_{\rm lr}P_s$. Consequently, by (15–16), conditional on $W_r = w_r\geq w_s = W_s$,

$$\Pr\big(MWL(\pi)<x\big) = E\big[E\big[\bar{F}_r(C+P_r-x_r)\bar{F}_s(C+P_r+P_s-x_s)H(x) \mid \{P_i, W_i\}_{i\neq r,s}\big]\big]$$
$$= E\big[E\big[\phi_2(P_r,P_s) \mid \{P_i,W_i\}_{i\neq r,s}\big]\big]$$

$$\geq E\big[E\big[\phi_1(P_r,P_s)\mid\{P_i,W_i\}_{i\neq r,s}\big]\big]$$
$$= E\big[E\big[\bar{F}_s(C+P_s-x_s)\bar{F}_r(C+P_r+P_s-x_r)H(x)\mid \{P_i,W_i\}_{i\neq r,s}\big]\big]$$
$$= \Pr\big(ML(\pi')<x\big),\quad \text{for all } x\in(-\infty,\infty). \tag{18}$$

Next define

$$\psi_1(u,v)=\Pr\big(MWL(\pi')<x\mid W_s=u,W_r=v\big)$$

and

$$\psi_2(u,v)=\Pr\big(MWL(\pi)<x\mid W_s=u,W_r=v\big).$$

Then (18) shows that $\psi_1(u,v)\leq\psi_2(u,v)$, whenever $u\leq v$. It can also be checked that $\psi_1(u,v)+\psi_1(v,u)\leq\psi_2(u,v)+\psi_2(v,u)$ for $u\leq v$. Thus, applying Lemma 3.1 again gives

$$W_s\leq_{\mathrm{lr}}W_r\quad\Longrightarrow\quad E\big[\psi_1(W_s,W_r)\big]\leq E\big[\psi_2(W_s,W_r)\big].$$

To summarize, we have shown that $P_r\leq_{\mathrm{lr}}P_s$, $\nu_r\geq\nu_s$ and $W_r\geq_{\mathrm{lr}}W_s$ imply

$$\Pr\big(MWL(\pi)<x\big)$$
$$=E\big[\Pr\big(MWL(\pi)<x\mid W_s,W_r\big)\big]=E\big[\psi_2(W_s,W_r)\big]$$
$$\geq E\big[\psi_1(W_s,W_r)\big]=E\big[\Pr\big(MWL(\pi')<x\mid W_s,W_r\big)\big]$$
$$=\Pr\big(MWL(\pi')<x\big),\quad\text{for all } x\in(-\infty,\infty).$$

It follows that an optimal solution to minimize $MWL(\pi)$ stochastically is given by the sequence in nondecreasing likelihood-ratio order of $\{P_i\}$ or, equivalently, by SEPT, EEDD, or the largest mean weight first rule. □

4 Optimal solutions with exponential processing times and due dates

In this section we show that, when both $\{P_i\}$ and $\{D_i\}$ are exponentially distributed, the agreeability condition in Theorem 3.1 can be substantially relaxed in order to minimize the maximum lateness $ML(\pi)$ stochastically.

To begin, by (8–9), we can write

$$\Pr\big(ML(\pi)<x\big)$$
$$=E\big[\Pr\big(ML(\pi)<x\mid P_1,\dots,P_n\big)\big]$$
$$=E\big[\bar{F}_r(C+P_r-x)\bar{F}_s(C+P_r+P_s-x)H(x)\big]$$
$$=E\big[E\big[\bar{F}_r(P_r-a)\bar{F}_s(P_r+P_s-a)H(x)\mid P_i,i\neq r,s\big]\big], \tag{19}$$

where $H(x)$ is given by (10), which depends on $P_1,\dots,P_n$. Similarly,

$$\Pr\big(ML(\pi')<x\big)$$
$$=E\big[E\big[\bar{F}_s(C+P_r-x)\bar{F}_r(C+P_r+P_s-x)H(x)\mid P_i,i\neq r,s\big]\big]. \tag{20}$$

Let $\mathcal{B}=\mathcal{B}(\pi)$ and $\mathcal{A}=\mathcal{A}(\pi)$ denote the sets of jobs scheduled, respectively, before and after jobs r,s under π. Write $H(x)=H_1(x)H_2(x)$, where

$$H_1(x)=\prod_{j\in\mathcal{B}}\bar{F}_j\big(C_j(\pi)-x\big),$$

$$H_2(x)=\prod_{j\in\mathcal{A}}\bar{F}_j\big(C_j(\pi)-x\big).$$

Note that $H_1(x)$ is independent of P_r and P_s, as $C_j(\pi)$ only involves those jobs sequenced before jobs r and s. But $H_2(x)$ still depends on P_r and P_s. Thus, by (19) and (20), $\Pr(ML(\pi)<x)\geq\Pr(ML(\pi')<x)$ holds if

$$E\big[\bar{F}_r(C+P_r-x)\bar{F}_s(C+P_r+P_s-x)H_2(x)\mid P_i,i\neq r,s\big]$$
$$\geq E\big[\bar{F}_s(C+P_r-x)\bar{F}_r(C+P_r+P_s-x)H_2(x)\mid P_i,i\neq r,s\big], \tag{21}$$

for every instance of $\{P_i,i\neq r,s\}$.

Define $A_j=x-C_j(\pi)+P_r+P_s$ for $j\in\mathcal{A}$. Then, as $C_j(\pi)-P_r-P_s$ represents the sum of the processing times over jobs up to job j under π, excluding r and s, A_j is independent of P_r, P_s and $A_j<x-C$. Given $\{P_i=p_i,i\neq r,s\}$, $A_j=a_j$ and $a=x-C$ are fixed, with $a_j<a$. We can now write

$$H_2(x)=\prod_{j\in\mathcal{A}}\bar{F}_j(C_j-x)=\prod_{j\in\mathcal{A}}\bar{F}_j(P_r+P_s-a_j)$$
$$(a_j<a=x-C). \tag{22}$$

Thus (21) holds if

$$E\bigg[\bar{F}_r(P_r-a)\bar{F}_s(P_r+P_s-a)\prod_{j\in\mathcal{A}}\bar{F}_j(P_r+P_s-a_j)\bigg]$$
$$\geq E\bigg[\bar{F}_s(P_r-a)\bar{F}_r(P_r+P_s-a)\times\prod_{j\in\mathcal{A}}\bar{F}_j(P_r+P_s-a_j)\bigg],\quad\text{when } a_j<a. \tag{23}$$

When $\{P_i\}$ and $\{D_i\}$ are exponentially distributed, we have the following result.

Theorem 4.1 *Suppose that $P_1,\dots,P_n$ are independent and exponentially distributed with rates $\mu_1,\dots,\mu_n$, respectively, $D_1,\dots,D_n$ are independent and exponentially distributed with rates $\nu_1,\dots,\nu_n$ respectively, and $\{P_i\}$ are independent of $\{D_i\}$. Let $\nu_{(1)}\le\cdots\le\nu_{(n)}$ denote the ordered values of $\nu_1,\dots,\nu_n$. If $\{\nu_i\}$ and $\{\mu_i\}$ satisfy the following condition*:

$$\text{the sequence } \{\nu_i(\nu_i+\mu_i), i=1,\dots,n\}$$
$$\text{has the same order as}$$
$$\text{the sequence } \{\nu_i(\nu_i+\mu_i+A_0), i=1,\dots,n\}$$
$$\text{for some } A_0\ge\sum_{i=3}^n \nu_{(i)}, \tag{24}$$

then $ML(\pi)$ is stochastically minimized by the sequence in the nonincreasing order of $\{\nu_i(\nu_i+\mu_i)\}$.

Proof As D_i is exponential with rate ν_i, we can write $\bar F_i(x)=\Pr(D_i>x)$ as

$$\bar F_i(x)=1_{\{x<0\}}+e^{-\nu_i x}1_{\{x\ge 0\}},\quad i=1,\dots,n, \tag{25}$$

where 1_E is the indicator of an event E which takes value 1 if E occurs and 0, otherwise.

Let $\pi=(\dots,r,s,\dots)$ and $\pi'=(\dots,s,r,\dots)$. By (25) we have

$$\begin{aligned}&\bar F_r(P_r-a)\bar F_s(P_r+P_s-a)\\&\quad=\big[1_{\{P_r<a\}}+e^{-\nu_r(P_r-a)}1_{\{P_r\ge a\}}\big]\\&\qquad\times\big[1_{\{P_r+P_s<a\}}+e^{-\nu_s(P_r+P_s-a)}1_{\{P_r+P_s\ge a\}}\big]\\&\quad=1_{\{P_r+P_s<a\}}+e^{-\nu_s(P_r+P_s-a)}1_{\{P_r<a\le P_r+P_s\}}\\&\qquad+e^{-(\nu_r+\nu_s)(P_r-a)-\nu_s P_s}1_{\{P_r\ge a\}}\end{aligned}$$

and

$$\begin{aligned}&\bar F_j(P_r+P_s-a_j)\\&\quad=1_{\{P_r+P_s<a_j\}}+e^{-\nu_j(P_r+P_s-a_j)}1_{\{P_r+P_s\ge a_j\}},\quad j\in\mathcal A.\end{aligned}$$

It follows that

$$\begin{aligned}&\bar F_r(P_r-a)\bar F_s(P_r+P_s-a)\prod_{j\in\mathcal A}\bar F_j(P_r+P_s-a_j)\\&\quad=\prod_{j\in\mathcal A}\bar F_j(P_r+P_s-a_j)1_{\{P_r+P_s<a\}}\\&\qquad+e^{-(\nu_s+\nu_{\mathcal A})(P_r+P_s)+\nu_s a+(\nu a)_{\mathcal A}}1_{\{P_r<a\le P_r+P_s\}}\\&\qquad+e^{-(\nu_r+\nu_s+\nu_{\mathcal A})P_r-(\nu_s+\nu_{\mathcal A})P_s}e^{(\nu_r+\nu_s)a+(\nu a)_{\mathcal A}}1_{\{P_r\ge a\}},\end{aligned} \tag{26}$$

where $\nu_{\mathcal A}=\sum_{j\in\mathcal A}\nu_j$ and $(\nu a)_{\mathcal A}=\sum_{j\in\mathcal A}\nu_j a_j$. Similarly,

$$\begin{aligned}&\bar F_s(P_r-a)\bar F_r(P_r+P_s-a)\prod_{j\in\mathcal A}\bar F_j(P_r+P_s-a_j)\\&\quad=\prod_{j\in\mathcal A}\bar F_j(P_r+P_s-a_j)1_{\{P_r+P_s<a\}}\\&\qquad+e^{-(\nu_r+\nu_{\mathcal A})(P_r+P_s)+\nu_r a+(\nu a)_{\mathcal A}}1_{\{P_s<a\le P_r+P_s\}}\\&\qquad+e^{-(\nu_r+\nu_s+\nu_{\mathcal A})P_s-(\nu_r+\nu_{\mathcal A})P_r}e^{(\nu_r+\nu_s)a+(\nu a)_{\mathcal A}}1_{\{P_s\ge a\}}.\end{aligned} \tag{27}$$

Let

$$E_1=E\big[e^{-(\nu_s+\nu_{\mathcal A})(P_r+P_s)+\nu_s a}1_{\{P_r<a\le P_r+P_s\}}\big],$$
$$E_2=E\big[e^{-(\nu_r+\nu_s+\nu_{\mathcal A})P_r-(\nu_s+\nu_{\mathcal A})P_s+(\nu_r+\nu_s)a}1_{\{P_r\ge a\}}\big],$$
$$E_1'=E\big[e^{-(\nu_r+\nu_{\mathcal A})(P_r+P_s)+\nu_r a}1_{\{P_s<a\le P_r+P_s\}}\big],$$
$$E_2'=E\big[e^{-(\nu_r+\nu_s+\nu_{\mathcal A})P_s-(\nu_r+\nu_{\mathcal A})P_r+(\nu_r+\nu_s)a}1_{\{P_s\ge a\}}\big],$$

where the expectations are with respect to P_r and P_s only, and a is a fixed real value. Then, by (26–27), we can see that (23) holds if

$$E_1+E_2-E_1'-E_2'\ge 0,\quad\text{for all } a\in(-\infty,\infty). \tag{28}$$

First consider the case of $a\ge 0$. Define

$$I(a;\mu_r,\mu_s)=\begin{cases}\frac{e^{-\mu_s a}-e^{-\mu_r a}}{\mu_r-\mu_s}, & \text{if } \mu_r\ne\mu_s,\\ ae^{-\mu_s a}, & \text{if } \mu_r=\mu_s,\end{cases}$$

(note that $I(a;\mu_r,\mu_s)=I(a;\mu_s,\mu_r)\ge 0$ for any $a\ge 0$).

Then

$$\begin{aligned}E_1&=E\big[e^{-(\nu_s+\nu_{\mathcal A})(P_r+P_s)+\nu_s a}1_{\{P_r<a\le P_r+P_s\}}\big]\\&=\int\int_{u<a\le u+v}e^{-(\nu_s+\nu_{\mathcal A})(u+v)+\nu_s a}\\&\qquad\times\mu_r e^{-\mu_r u}\mu_s e^{-\mu_s v}\,du\,dv\\&=\mu_r\mu_s e^{\nu_s a}\\&\qquad\times\int_0^a\bigg(\int_{a-u}^\infty e^{-(\nu_s+\mu_s+\nu_{\mathcal A})v}dv\bigg)e^{-(\nu_s+\mu_r+\nu_{\mathcal A})u}\,du\\&=\frac{\mu_r\mu_s e^{\nu_s a}}{\nu_s+\mu_s+\nu_{\mathcal A}}\\&\qquad\times\int_0^a e^{-(\nu_s+\mu_s+\nu_{\mathcal A})(a-u)}e^{-(\nu_s+\mu_r+\nu_{\mathcal A})u}du\\&=\frac{\mu_r\mu_s e^{-(\mu_s+\nu_{\mathcal A})a}}{\nu_s+\mu_s+\nu_{\mathcal A}}\int_0^a e^{-(\mu_r-\mu_s)u}\,du\\&=\frac{\mu_r\mu_s e^{-\nu_{\mathcal A}a}}{\nu_s+\mu_s+\nu_{\mathcal A}}I(a;\mu_r,\mu_s).\end{aligned} \tag{29}$$

By the same argument we obtain

$$E_1' = \frac{\mu_r \mu_s e^{-\nu_{\mathcal{A}} a}}{\nu_s + \mu_s + \nu_{\mathcal{A}}} I(a; \mu_r, \mu_s). \quad (30)$$

Furthermore,

$$E_2 = \mu_r \mu_s e^{(\nu_r+\nu_s)a} \int_a^\infty e^{-(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})u}\, du \times \int_0^\infty e^{-(\nu_s+\mu_s+\nu_{\mathcal{A}})v}\, du = \frac{\mu_r \mu_s e^{-(\mu_r+\nu_{\mathcal{A}})a}}{(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})}$$

and, similarly,

$$E_2' = \frac{\mu_r \mu_s e^{-(\mu_s+\nu_{\mathcal{A}})a}}{(\nu_r+\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\mu_r+\nu_{\mathcal{A}})}.$$

Therefore, by writing $e^{-(\mu_r+\nu_{\mathcal{A}})a} = e^{-\nu_{\mathcal{A}} a}(e^{-\mu_r a} - e^{-\mu_s a}) + e^{-(\mu_s+\nu_{\mathcal{A}})a}$, we get

$$\begin{aligned} E_2 - E_2' &= \frac{\mu_r \mu_s e^{-\nu_{\mathcal{A}} a}(e^{-\mu_r a} - e^{-\mu_s a})}{(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})} + \frac{\mu_r \mu_s e^{-(\mu_s+\nu_{\mathcal{A}})a}}{(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})} - \frac{\mu_r \mu_s e^{-(\mu_s+\nu_{\mathcal{A}})a}}{(\nu_r+\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\mu_r+\nu_{\mathcal{A}})} \\ &= \frac{\mu_r \mu_s (\mu_s - \mu_r) e^{-\nu_{\mathcal{A}} a}}{(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})} I(a; \mu_r, \mu_s) \\ &\quad + \mu_r \mu_s \big[\nu_r(\nu_r+\mu_r+\nu_{\mathcal{A}}) - \nu_s(\nu_s+\mu_s+\nu_{\mathcal{A}})\big] \times e^{-(\mu_s+\nu_{\mathcal{A}})a} \\ &\quad \times \big[(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}}) \times (\nu_r+\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\mu_r+\nu_{\mathcal{A}})\big]^{-1}. \end{aligned} \quad (31)$$

Since

$$\frac{1}{(\nu_s+\mu_s+\nu_{\mathcal{A}})} - \frac{1}{(\nu_r+\mu_r+\nu_{\mathcal{A}})} + \frac{1}{(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})} = \frac{\nu_r(\nu_r+\mu_r+\nu_{\mathcal{A}}) - \nu_s(\nu_s+\mu_s+\nu_{\mathcal{A}})}{(\nu_r+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})},$$

by combining (29–31), we get

$$\begin{aligned} E_1 + E_2 - E_1' - E_2' &= \frac{[\nu_r(\nu_r+\mu_r+\nu_{\mathcal{A}}) - \nu_s(\nu_s+\mu_s+\nu_{\mathcal{A}})] e^{-\nu_{\mathcal{A}} a}}{(\nu_r+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})} \times I(a; \mu_r, \mu_s) \\ &\quad + \mu_r \mu_s \big[\nu_r(\nu_r+\mu_r+\nu_{\mathcal{A}}) - \nu_s(\nu_s+\mu_s+\nu_{\mathcal{A}})\big] \times e^{-(\mu_s+\nu_{\mathcal{A}})a} \\ &\quad \times \big[(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}}) \times (\nu_r+\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\mu_r+\nu_{\mathcal{A}})\big]^{-1}. \end{aligned}$$

Thus, (28) holds for $a \ge 0$, and so does (23), if

$$\nu_r(\nu_r+\mu_r+A) \ge \nu_s(\nu_s+\mu_s+A), \quad \text{for all } 0 \le A \le \sum_{i \ne r,s} \nu_i. \quad (32)$$

If $a < 0$, then

$$\begin{aligned} &\bar{F}_r(P_r - a)\bar{F}_s(P_r + P_s - a)H_2(x) \\ &\quad = e^{-(\nu_r+\nu_s+\nu_{\mathcal{A}})P_r - (\nu_s+\nu_{\mathcal{A}})P_s} e^{(\nu_r+\nu_s)a + (\nu a)_{\mathcal{A}}}, \\ &\bar{F}_s(P_s - a)\bar{F}_r(P_r + P_s - a)H_2(x) \\ &\quad = e^{-(\nu_r+\nu_s+\nu_{\mathcal{A}})P_s - (\nu_r+\nu_{\mathcal{A}})P_r} e^{(\nu_r+\nu_s)a + (\nu a)_{\mathcal{A}}}. \end{aligned}$$

Hence, (23) holds if

$$E\big[e^{-(\nu_r+\nu_s+\nu_{\mathcal{A}})P_r - (\nu_s+\nu_{\mathcal{A}})P_s}\big] \ge E\big[e^{-(\nu_r+\nu_s+\nu_{\mathcal{A}})P_s - (\nu_r+\nu_{\mathcal{A}})P_r}\big].$$

Since

$$\begin{aligned} &E\big[e^{-(\nu_r+\nu_s+\nu_{\mathcal{A}})P_r - (\nu_s+\nu_{\mathcal{A}})P_s}\big] - E\big[e^{-(\nu_r+\nu_s+\nu_{\mathcal{A}})P_s - (\nu_r+\nu_{\mathcal{A}})P_r}\big] \\ &\quad = \frac{\mu_r \mu_s}{(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}})} - \frac{\mu_r \mu_s}{(\nu_r+\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\mu_r+\nu_{\mathcal{A}})} \\ &\quad = \mu_r \mu_s \big[\nu_r(\nu_r+\mu_r+\nu_{\mathcal{A}}) - \nu_s(\nu_s+\mu_s+\nu_{\mathcal{A}})\big] \\ &\qquad \times \big[(\nu_r+\nu_s+\mu_r+\nu_{\mathcal{A}})(\nu_s+\mu_s+\nu_{\mathcal{A}}) \times (\nu_r+\nu_s+\mu_s+\nu_{\mathcal{A}})(\nu_r+\mu_r+\nu_{\mathcal{A}})\big]^{-1}, \end{aligned}$$

(32) again implies (23). Now, under condition (24), if $\nu_r(\nu_r+\mu_r) \ge \nu_s(\nu_s+\mu_s)$, then $\nu_r(\nu_r+\mu_r+A_0) \ge \nu_s(\nu_s+\mu_s+A_0)$ for some $A_0 \ge \sum_{i\ge 3} \nu_{(i)}$. If $\nu_r \ge \nu_s$, then

$$\begin{aligned} &\nu_r(\nu_r+\mu_r+A) \\ &\quad = \nu_r(\nu_r+\mu_r) + \nu_r A \ge \nu_s(\nu_s+\mu_s) + \nu_s A \\ &\quad = \nu_s(\nu_s+\mu_s+A), \quad \forall A \ge 0. \end{aligned}$$

If $\nu_r < \nu_s$, on the other hand, then for any $0 \le A \le \sum_{i\ne r,s} \nu_i \le \sum_{i\ge 3} \nu_{(i)} \le A_0$,

$$\nu_r(\nu_r+\mu_r+A) = \nu_r(\nu_r+\mu_r+A_0) - \nu_r(A_0 - A)$$

$$\geq \nu_s(\nu_s + \mu_s + A_0) - \nu_s(A_0 - A)$$
$$= \nu_s(\nu_s + \mu_s + A).$$

In either case we see that (32) holds, when $\nu_r(\nu_r + \mu_r) \geq \nu_s(\nu_s + \mu_s)$. Consequently, under condition (24), $\nu_r(\nu_r + \mu_r) \geq \nu_s(\nu_s + \mu_s)$ implies $\Pr(ML(\pi) < x) \geq \Pr(ML(\pi') < x)$ for all x. It follows that $ML(\pi)$ is stochastically minimized by the sequence in nonincreasing order of $\{\nu_i(\nu_i + \mu_i)\}$. □

The following two corollaries are straightforward consequences of Theorem 4.1.

Corollary 4.1 *If* $\{\nu_i\}$ *and* $\{\mu_i\}$ *satisfy the condition*:

$$\nu_i(\nu_i + \mu_i) \geq \nu_j(\nu_j + \mu_j) \iff \nu_i \geq \nu_j, \quad \text{for all } i, j \in \{1, \ldots, n\},$$

then $ML(\pi)$ *is stochastically minimized by the sequence in nonincreasing order of* $\{\nu_i\}$. *In other words, the EEDD rule is optimal.*

Corollary 4.2 *If* $\{\nu_i\}$ *and* $\{\mu_i\}$ *satisfy the condition*:

$$\nu_i \geq \nu_j \iff \nu_i + \mu_i \geq \nu_j + \mu_j, \quad \text{for all } i, j \in \{1, \ldots, n\},$$

then $ML(\pi)$ *is stochastically minimized by the sequence in nonincreasing order of* $\{\nu_i\}$.

The next corollary shows that the EEDD rule is optimal under a different type of sufficient conditions: the variations among the rates of processing times are dominated by those among the due dates.

Corollary 4.3 *If* $\{\nu_i\}$ *and* $\{\mu_i\}$ *satisfy following condition*:

$$|\mu_i - \mu_j| \leq |\nu_i - \nu_j|, \quad \text{for all } i, j \in \{1, \ldots, n\}, \tag{33}$$

then $ML(\pi)$ *is stochastically minimized by the sequence in nonincreasing order of* $\{\nu_i\}$.

Proof Let $\nu_i \geq \nu_j$. Then, by condition (33), for any $A \geq 0$,

$$\nu_j(\mu_j - \mu_i) \leq \nu_j|\mu_i - \mu_j|$$
$$\leq (\nu_i + \nu_j + A)(\nu_i - \nu_j) + \mu_i(\nu_i - \nu_j)$$
$$\implies (\nu_i + \nu_j + A)(\nu_i - \nu_j) + \mu_i(\nu_i - \nu_j) - \nu_j(\mu_j - \mu_i) \geq 0$$
$$\implies (\nu_i + \nu_j + A)(\nu_i - \nu_j) + \nu_i\mu_i - \nu_j\mu_j \geq 0$$
$$\implies \nu_i^2 + \nu_i\nu_j + \nu_i A - \nu_i\nu_j - \nu_j^2 - \nu_j A + \nu_i\mu_i - \nu_j\mu_j \geq 0$$
$$\implies \nu_i(\nu_i + \mu_i + A) - \nu_j(\nu_j + \mu_j + A) \geq 0.$$

Thus the sequence $\{\nu_i(\nu_i + \mu_i + A)\}$ has the same order as $\{\nu_i\}$ for any $A \geq 0$. The conclusion of Corollary 4.3 then follows immediately from Theorem 4.1. □

Remark 4.1 If $\{\mu_i\}$ and $\{\nu_i\}$ have the same order, then the result of Theorem 4.1 becomes a special case of Theorem 3.1 or Corollary 3.1. The condition of Theorem 4.1, however, is substantially weaker than the same order between $\{\mu_i\}$ and $\{\nu_i\}$. As a simple example, suppose that $\mu_1 = 1, \mu_2 = 2, \ldots, \mu_n = n$ and $\nu_1 = n, \nu_2 = n-1, \ldots, \nu_n = 1$, which are in totally opposite order. Then $\{\nu_i(\nu_i + \mu_i + A)\} = \{\nu_i(n + A)\}$ has the same order as $\{\nu_i\}$ for any $A \geq 0$. Hence, according to Theorem 4.1, the optimal sequence is in nonincreasing order of $\{\nu_i\}$, despite the opposite orders between $\{\mu_i\}$ and $\{\nu_i\}$.

Moreover, in condition (24), the rates $\{\nu_i\}$ of the due dates tend to have much greater influence in the determination of optimal sequence than the rates $\{\mu_i\}$ of the processing times, as demonstrated by the above example and Corollary 4.3. Intuitively, this may be viewed as close to the case of deterministic due dates, where the due dates determine the optimal sequence, while the processing times play no role.

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