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ON DYNAMICS OF QUANTUM STATES GENERATED BY THE CAUCHY PROBLEM FOR THE SCHRÖDINGER EQUATION WITH DEGENERATION ON THE HALF-LINE

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Abstract. The paper considers the Cauchy problem for the Schrödinger equation with operator degenerate on the semiaxis and the family of regularized Cauchy problems with uniformly elliptic operators whose solutions approximate the solution of the degenerate problem. The author studies the strong and weak convergences of the regularized problems and the convergence of values of quadratic forms of bounded operators on solutions of the regularized problems when the regularization parameter tends to zero.

Introduction

In this paper, we study the Cauchy problem for the Schrödinger equation on the line with variable-type operator that is a second-order differential operator on the semiaxis that degenerates to a first-order operator on the complement to the semiaxis.

The conditions for the well-posedness of the statement of initial-value problems for second-order mixed-type equations were studied in [3]; in particular, the domains of boundary conditions depending on the form of the differential operator were indicated there. In [8], the well-posedness of the initial-value problem with second-order operator having a nonnegative characteristic form was studied by using the vanishing-viscosity method. This method was effectively applied in studying degenerate variable-type equations and first-order partial differential equations (see [1]). For example, to study the well-posedness of the boundary-value problem with degenerate second-order operator in divergence form, Zhikov [12] considers a family of regularized problems with uniformly elliptic operators approximating the initial problem with degenerate operator when the regularization parameter ε tends to 0. The existence was proved and a description of the set of partial limits of solutions of the regularized problems as $\varepsilon \to 0$ was given; moreover, for every partial limit, the variational formulation of the limit degenerate problem for which this partial limit is its unique solution was found.

In this work, we consider the model problem

$$i\frac{\partial u(t,x)}{\partial t} = \mathbf{L}u(t,x), \quad t > 0, \quad x \in \mathbb{R}, \tag{1}$$

$$u(+0,x) = u_0(x), \quad x \in \mathbb{R}, \tag{2}$$

$$\mathbf{L}v = \frac{\partial}{\partial x}\left[g(x)\frac{\partial v}{\partial x}\right] + \frac{i}{2}\left(a(x)\frac{\partial v}{\partial x} + \frac{\partial a(x)v}{\partial x}\right), \tag{3}$$

where $u_0(x)$ is a given function, $v(x)$ is a test function, the functions $g(x)$ and $a(x)$ are real-valued, and $g(x) \geq 0$. In the work, we consider the model problem for the function $g(x) = \theta(-x)$ and the function $a(x) = a\theta(x)$, $a \in \mathbb{R}$; here, $\theta(x)$ is the Heaviside function. The operator $\mathbf{L}$ is a second-order operator on the set $\mathbb{R}_-$, and it is a first-order operator on the set $\mathbb{R}_+$.

A direct study of the Cauchy problem (1), (2) meets certain difficulties (see, e.g., [3,10]). To find the solution of the problem (1), (2), we use the ideas of the vanishing-viscosity method. Let us consider the

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family of operators $\mathbf{L}_\varepsilon$ depending on the regularization parameter $\varepsilon \in (0,1)$ and given by the expressions

$$\mathbf{L}_\varepsilon v = \frac{\partial}{\partial x}\left[g_\epsilon(x)\frac{\partial v}{\partial x}\right] + \frac{i}{2}\left(a(x)\frac{\partial v}{\partial x} + \frac{\partial a(x)v}{\partial x}\right)$$

in which the function $a(x) = a\theta(x)$, $a \in \mathbb{R}$, and the functions $g_\varepsilon(x) = 1 - (1-\varepsilon)\theta(x)$ depend on the real parameter $\varepsilon \in (0,1)$.

For every $\varepsilon \in (0,1)$, the operator $\mathbf{L}_\varepsilon$ is a second-order elliptic-type operator on the axis $\mathbb{R}$. The characteristic forms of the regularized operators $\mathbf{L}_\varepsilon$ uniformly converge to the characteristic form of the degenerate operator $\mathbf{L}$ on each compact set. Along with the problem (1), (2) in which the operator $\mathbf{L}$ is a variable-type operator degenerating on the semiaxis, we consider the family of regularized Cauchy problems with initial condition (2) for the family of equations

$$i\frac{\partial u(t,x)}{\partial t} = \mathbf{L}_\varepsilon u(t,x), \quad t > 0, \quad x \in \mathbb{R}, \quad \varepsilon \in (0,1). \tag{4}$$

The regularized Cauchy problems (2), (4) have been well studied (see [9, Chap. 8]), and it is known that for any initial condition $u_0(x) \in L_2(\mathbb{R})$, there exists a unique solution $u_\varepsilon(t,x)$ of the regularized problem, and, moreover, the correspondence $\mathbf{U}_{\mathbf{L}_\varepsilon}(t)\colon u_0(x) \to u_\varepsilon(t,x)$, $t \in \mathbb{R}$, is a unitary group.

In [11], the convergence of the family of solutions of the regularized problems (2), (4) as $\varepsilon \to 0$ was studied. It was proved that the weak convergence of the family as $\varepsilon \to 0$ holds for any choice of the initial conditions $u_0 \in L_2(\mathbb{R})$ and the parameter of the operator $a \in \mathbb{R}$. The conditions on the parameters of the problem that are necessary and sufficient for the norm-convergence of the family as $\varepsilon \to 0$ were found.

In quantum-mechanics problems, it is required to find the dynamics of values of observables that are bounded operators in H. Denote by $\mathcal{B}(H)$ the Banach space of bounded operators on a Hilbert space H with the operator norm. The Cauchy problem for the Schrödinger equation having a unique solution determines the dynamics of values of bounded operators, i.e., the mapping $\mathbb{R}_+ \times \mathcal{B}(H) \to \mathbb{C}$ acting according to the rule $(t,\mathbf{A}) \to \big(u(t), \mathbf{A}u(t)\big)$. One asks if the solution of the Cauchy problem (1), (2) is approximated by solutions of the regularized problems (2), (4) whether or not the sequence of regularized dynamics of values of bounded operators determines the dynamics of values of bounded operators for the limit problem (1), (2) when the regularization parameter tends to zero?

Undoubtedly, the strong convergence of the sequence of regularized solutions $u_\varepsilon(t) \to u(t)$ as $\varepsilon \to 0$ implies the convergence of the regularized dynamics $\big(u_\varepsilon(t), \mathbf{A}u_\varepsilon(t)\big)$ to the limit dynamics $\big(u(t), \mathbf{A}u(t)\big)$. However, the weak convergence of the sequence of regularized solutions guarantees only the convergence of values of all linear continuous functionals on the space $L_2(\mathbb{R}) = H$ on solutions, but it cannot guarantee the convergence of values on the sequence of solutions of quadratic forms of bounded operators.

In the present work, we show that in the case of only weak convergence of the family of regularized problems, the convergence of quadratic forms of all bounded operators is impossible. However, for a certain narrower class of bounded operators $B_1 \subset \mathcal{B}(H)$, it is possible to find an infinitely small sequence of regularization parameters such that the sequence of dynamics $\big(u_{\varepsilon_n}(t), \mathbf{A}u_{\varepsilon_n}(t)\big)$ corresponding to it converges for any operator $\mathbf{A}$ from the class B_1.

In [4], Gerard studied the convergence of probability measures on the coordinate space $\mathbb{R}$ whose density is defined by a weakly convergent sequence $\{u_n(x)\}$ of elements of the space $L_2(\mathbb{R})$. The convergence of values on this sequence of elements of quadratic forms for a certain class of pseudodifferential operators was considered. In the present paper, we study the dynamics of values of quadratic forms of all operators from the algebra of multiplication operators by a continuous function and unitary equivalent Abelian operator algebras (see [5]). For this purpose, we consider the convergence as $\varepsilon \to 0$ of a family of probability measures $P_\varepsilon(t)$ with distribution functions $F_\varepsilon(t,x)$ defined by solutions $u_\varepsilon(t)$ and the orthogonal decomposition $\mathbf{E}(\lambda)$ of the identity operator on $L_2(\mathbb{R})$ according to the rule

$$F_\varepsilon(t,\lambda) = \int\limits_{-\infty}^{\lambda} \big(u_\varepsilon(t), d\mathbf{E}(\mu)u_\varepsilon(t)\big).$$

In terms of the behavior of the family of indicated probability measures, we found necessary and sufficient conditions for the convergence of values of quadratic forms of all multiplication operators by a function on the family of solutions of the regularized problems (2), (4) as $\varepsilon \to 0$.

The Cauchy problem (1), (2) with degenerate variable-type operator studied in the paper can arise in studying the linearization of the Cauchy problem for a nonlinear equation. The problems considered in the paper arise in describing the motion of mechanical systems with variable effective mass whose examples occur in rigid-body physics. The example of a family of quantum systems whose dynamics is described by the family of problems (2), (4) is presented in [10].

Definition of a Solution of the Cauchy Problem

The maximal domain of the operator $\mathbf{L}$ generated by the differential expression (3) is the linear variety $D(\mathbf{L})$ consisting of those elements $u(x) \in L_2(\mathbb{R})$ for which the application of the differential expression (3) to $u(x)$ has the meaning as an element $\mathbf{L}u(x) \in L_2(\mathbb{R})$:

$$D(\mathbf{L}) = \{u(x) \in L_2(\mathbb{R})\colon \mathbf{L}u(x) \in L_2(\mathbb{R})\}. \tag{5}$$

The maximal domain of the operator $\mathbf{L}$ is uniquely defined by this condition for any $a \neq 0$ (see [10]).

The domain of the operator $D(\mathbf{L})$ is dense in the space $L_2(\mathbb{R})$, and the operator $\mathbf{L}$ is symmetric and closed. It is easy to prove that the adjoint operator $\mathbf{L}^*$ acts according to the same formula (3) and its domain is wider. Then one can directly verify that if $a \neq 0$, then $\mathrm{Ker}(\mathbf{L}^* - ia\mathbf{I}) = \{0\}$, and $\mathrm{Ker}(\mathbf{L}^* + ia\mathbf{I})$ is a nontrivial, one-dimensional linear subspace and the deficiency indices of the operator $\mathbf{L}$ are different. Hence the spectrum of the operator $\mathbf{L}$ fills in the whole real axis.

Transform the domain $D(\mathbf{L})$ into a Hilbert space by equipping it with the norm of the graph of the operator $\mathbf{L}$. Furthermore, denote by $C^m\big((a,b), X\big)$ the space of m-times continuously differentiable (with respect to the norm of the space X) mappings $x(t)$ of the interval (a,b) into a linear normed space X with the norm

$$\|x(t)\|_{C^m((a,b),X)} = \max_{j=0,1,\dots,m} \Big\{ \sup_{t\in(a,b)} \|x^{(j)}(t)\|_X \Big\}.$$

Denote by $X \cap Y$ the intersection of the spaces X and Y equipped with the norm $\|f\|_{X\cap Y} = \max\{\|f\|_X, \|f\|_Y\}$.

Let us consider the family of regularized problems (2), (4) for $\varepsilon \in (0,1)$ approximating the problem (1), (2) as $\varepsilon \to 0$.

The maximal domain $D(\mathbf{L}_\varepsilon)$ of the operator $\mathbf{L}_\varepsilon$ was described in [8], where it was proved that for any $\varepsilon > 0$, the differential operator $\mathbf{L}_\varepsilon$ is a self-adjoint operator on the space $L_2(\mathbb{R})$. Therefore (see [9]), for any $\varepsilon > 0$, the Cauchy problem (2), (4) defines a group of unitary transformations $U_{L_\varepsilon}(t) = \exp(iL_\varepsilon t)$, $t \in \mathbb{R}$, on the space $L_2(\mathbb{R})$.

Definition 1. A function $u(t,x) \in C\big(\mathbb{R}_+, L_2(\mathbb{R})\big)$ is called a strong approximate solution of the problem (1), (2) if there exists a sequence $\{\varepsilon_k\}$, $\lim\limits_{k\to\infty} \varepsilon_k = 0$, such that the following condition holds for any $T > 0$:

$$\lim_{\varepsilon\to 0} \|U_{L_\varepsilon}(t)u_0(x) - u(t,x)\|_{C([0,T],L_2(\mathbb{R}))} = 0.$$

Definition 2. A weakly continuous mapping $u(t,x)$ of the semiaxis $\mathbb{R}_+$ into the space $L_2(\mathbb{R})$ is called a weak approximate solution of the problem (1), (2) if there exists a sequence $\{\varepsilon_k\}$, $\lim\limits_{k\to\infty} \varepsilon_k = 0$, such that the following condition holds for any function $v(x) \in L_2(\mathbb{R})$ and any $T > 0$:

$$\lim_{k\to\infty} \sup_{t\in[0,T]} \big|\big(U_{L_{\varepsilon_k}}(t)u_0(x) - u(t,x), v(x)\big)\big| = 0,$$

where $(\cdot,\cdot)$ denotes the inner product of the space $L_2(\mathbb{R})$.

The strong and weak approximate solutions satisfy Eq. (1) in the sense of the integral identity (see [11]) and condition (2) in the sense of the strong and weak convergences in L_2, respectively.

On the Convergence of Solutions

The following result on the convergence of the family of regularized problems was proved in [10].

Theorem 1. *Let $a \leq 0$. Then for any $u_0 \in L_2(\mathbb{R})$, there exists a unique strong approximative solution $u(t,x)$, and, moreover, the following relation holds for any $T > 0$:*

$$\lim_{\varepsilon\to+0} \|u_\varepsilon(t,x) - u(t,x)\|_{C([0,T],L_2(\mathbb{R}))} = 0.$$

The following theorem is an insignificant reformulation and generalization of the assertions of Theorem 5 in [11].

Theorem 2. *For any $a > 0$, there exist subspaces H_0 and H_1 of the space $H = L_2(\mathbb{R})$ such that*

(1) *for any $u_0(x) \in H_0$, there exists a unique strong approximate solution $u(t,x)$ of the Cauchy problem* (1), (2), *and, moreover, for any $t > 0$, the conditions $u(t,x) \in H_0$ and $\|u(t,x)\|_{L_2(\mathbb{R})} = \|u_0(x)\|_{L_2(\mathbb{R})}$ hold;*
(2) *for any $u_0(x) \in H_1$, there exists a unique weak approximate solution $u^*(t,x)$ of the Cauchy problem* (1), (2), *and, moreover, for any $t > 0$, the conditions $u^*(t,x) \in H_1$ and $\lim\limits_{t\to+\infty} \|u^*(t,x)\|_{L_2(\mathbb{R})} = 0$ hold;*
(3) *the Cauchy problem* (1), (2) *has a strong approximate solution if and only if $u_0(x) \in H_0$.*

The proof of the theorem is published in [11]; also, the representation of the subspaces H_0 and H_1 through the parameters of the operators of the problem (1), (2) is found there.

On the Convergence of Probability Measures

In [11], the author performed a study of the convergence of probability measures on the coordinate space that are defined by the sequence of solutions of regularized problems and the orthogonal partition of unity of the multiplication operator by the coordinate when the regularization parameter tends to zero. The goal of this paper is to generalize the result of [11] to the case of an arbitrary partition of unity $\mathbf{E}(\lambda)$. However, in this case, we cannot prove the differentiability of densities of the measures in the parameter λ and cannot use the Nikol'skii embedding theorems as in [11]. In the present paper, we overcome this gap using the Helly choice principle, which allows us to prove Theorem 3. In the case of smooth initial conditions, we prove the equicontinuity in t of the family of distribution functions of probability measures on the interval $(0, +\infty)$, which allows us to prove Theorem 4. Theorem 5 is proved by using the continuous dependence of the distribution function on the initial conditions proved in Lemma 2; it allows us to continuously extend the result of Theorem 6 to the whole space $L_2(\mathbb{R})$ of initial conditions.

Everywhere in what follows, let $\mathbf{E}(\lambda)$, $\lambda \in \mathbb{R}$, be an orthogonal partition of unity in the Hilbert space $L_2(\mathbb{R}) \equiv H$. Then the following assertion holds.

Theorem 3. *Let $u_0(x) \in L_2(\mathbb{R})$ and let $\{\varepsilon_n\}$ be a certain infinitely small sequence. Then for any $t > 0$, there exist a subsequence ε_{n_k} of the sequence $\{\varepsilon_n\}$ and a function $G(t,\xi) \in L_{1,\mathrm{loc}}(\mathbb{R})$ defined on the whole axis $\mathbb{R}$ such that it monotonically increases and satisfies the inequalities $0 \leq G(t,\xi) \leq 1$ for which the sequence $\{G_{n_k}(t,\xi)\}$, where*

$$G_{n_k}(t,\xi) = \int\limits_{-\infty}^{\xi} \big(u_{\varepsilon_{n_k}}(t), d\mathbf{E}(\lambda) u_{e_{n_k}}(t)\big),$$

converges to the function $G(t,\xi)$ in the space $L_{1,\mathrm{loc}}(\mathbb{R})$.

Proof. Let ε_n be an arbitrary infinitely small sequence. For a fixed $t > 0$, consider the sequence of functions

$$G_n(t,\lambda) = \int\limits_{-\infty}^{\lambda} \big(u_{\varepsilon_n}(t), d\mathbf{E}(\mu) u_{\varepsilon_n}(t)\big), \quad \lambda \in \mathbb{R}, \quad n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N}$, the function $G_n(t,\lambda)$ monotonically increases, $0 \le G_n(t,\lambda) \le 1$, and $G_n(0,\lambda) = G_0(\lambda)$, $n \in \mathbb{N}$.

For a fixed $t > 0$, consider the sequence of functions $G_n(t,\lambda)$, $n \in \mathbb{N}$. Since all elements of the above sequence are uniformly bounded in the uniform norm and are monotone functions with respect to the variation, it follows that by the Helly theorem (see [6]), we can extract a subsequence $G_{n_k}(t,\lambda)$ from the sequence $G_n(t,\lambda)$ that pointwise converges to the function $G(t,\lambda)$, which is monotone (and hence is of finite variation) and such that $0 \le G(t,\lambda) \le 1$ on the axis $\mathbb{R}$.

Fix a certain $l > 0$. Then according to the Egorov theorem (see [9]), for any $\delta > 0$, on the closed interval $[-l,l]$, there exists a set Ω, $\operatorname{mes}\Omega \le \delta$, such that the sequence $G_n(t,x)$ uniformly converges on the set $[-l,l]\setminus\Omega$. Then it is easy to see that by the uniform boundedness of the functions $G_n(t,x)$ and $G(t,x)$ and the arbitrariness of $\delta > 0$, the sequence $G_n(t,x)$ converges in the space $L_1([-l,l])$, which implies the assertion of Theorem 3. □

Denote by $K_{T,l}$, $T > 0$, $l > 0$, the set $[0,T]\times(-l,l]$ and by $C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big) \equiv C_{\mathrm{loc}}$ (see [7]) the linear topological space of functions $u(t,x)$ such that $u(t,x)|_{K(T,l)} \in C\big([0,T], L_1([-l,l])\big)$ for any $T > 0$ and $l > 0$ in which the convergence is defined by the family of seminorms $p_{T,l}(u) = \|u(t,x)|_{K(T,l)}\|_{C([0,T],L_1(-l,l))}$.

Lemma 1. *If the sequence $f_n(t,x)$ is a Cauchy sequence in the space C_{loc} (i.e., for any $T, l > 0$, the sequence $f_n(t,x)|_{K(T,l)}$ is a Cauchy sequence in $C\big([0,T], L_1(-l,l)\big)$), then it has a limit in the space C_{loc}.*

Proof. The space $C\big([0,T], L_1(-l,l)\big)$ is complete; therefore, for any $T, l > 0$, the sequence $f_n(t,x)|_{K(T,l)}$ has the limit $f^{T,l}(t,x) \in C\big([0,T],(-l,l)\big)$. Moreover, if $T_1 > T$ and $l_1 > l$, then by the uniqueness of the limit, we have $F^{T_1,l_1}(t,x)|_{K(T,l)} = f^{T,l}(t,x)$. Therefore, the function $f(t,x) \in C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$ such that $f(t,x)|_{K(T,l)} = f^{T,l}(t,x)$ for any $K(T,l)$ is uniquely defined. Then $f(t,x)$ is the limit of the sequence $f_n(t,x)$ in the space C_{loc}. Lemma 1 is proved. □

Theorem 4. *For any function $u_0 \in \bigcap_{\varepsilon\in[0,1]} D(\mathbf{L}_\varepsilon)$ and any infinitely small sequence $\{\varepsilon_n\}$, there exist its subsequence ε_k and a function $F(t,\lambda) \in C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$ such that the sequence $F_{\varepsilon_k}(t,\lambda)$ converges to $F(t,\lambda)$ in $C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$. Moreover, $0 \le F(t,\lambda) \le 1$, and for each $t > 0$, the function $F(t,\lambda)$ is monotone on $\mathbb{R}$.*

Proof. Let t_j, $j \in \mathbb{N}$, be a certain sequence defining the enumeration of all rational numbers from the half-open interval $[0,+\infty)$. Then from the sequence ε_n, we can extract a subsequence $\varepsilon^1_{n_k}$ and find a function $F(t_1,\lambda) \in L_{1,\mathrm{loc}}(\mathbb{R})$ for which $F_{e^{(1)}_{n_k}}(t_1,\lambda)$ converges to $F(t_1,\lambda)$ in $L_{1,\mathrm{loc}}(\mathbb{R})$ as $k \to \infty$; moreover, according to Theorem 3, $0 \le F(t_1,\lambda) \le 1$ and the function $F(t_1,\lambda)$ monotonically increases on the axis $\mathbb{R}$.

Analogously, from the sequence $\varepsilon^{(1)}_{n_k}$, we can extract a subsequence $\varepsilon^{(2)}_{n_k}$ and find a function $F(t_2,\lambda) \in L_{1,\mathrm{loc}}(\mathbb{R})$ for which $F_{e^{(2)}_{n_k}}(t_1,\lambda)$ converges to $F(t_1,\lambda)$ in $L_{1,\mathrm{loc}}(\mathbb{R})$ as $k \to \infty$, and the function $F(t_2,\lambda)$ satisfies the same boundedness and monotonicity conditions.

Therefore, for any $p \in \mathbb{N}$, there exists a subsequence $\varepsilon^{(p)}_{n_k}$ that is a subsequence of the sequence $\varepsilon^{(p-1)}_{n_k}$ and there exists a function $F(t_p,\lambda) \in L_{1,\mathrm{loc}}(\mathbb{R})$ for which $F_{e^p_{n_k}}(t_p,\lambda)$ converges to $F(t_p,\lambda)$ in $L_{1,\mathrm{loc}}(\mathbb{R})$ as $k \to \infty$, and the function $F(t_p,\lambda)$ satisfies the boundedness and monotonicity conditions.

Then the sequence $\varepsilon^{(p)}_{n_p}$ that is a subsequence of the sequence ε_n is such that for any $q \in \mathbb{N}$, the sequence $F_{\varepsilon^{(p)}_{n_p}}(t_q,\lambda)$ converges to $F(t_q,\lambda)$ in the space $L_{1,\mathrm{loc}}(\mathbb{R})$.

Hence for any $t_q \in \mathbb{Q}^+$, any $l > 0$, and any $\sigma > 0$, there exists a number $p_0 \in \mathbb{N}$ such that for any $p \ge p_0$,

$$\big\|F_{\varepsilon^{(p)}_{n_p}}(t_q,\lambda) - F(t_q,\lambda)\big\|_{L_1([-l,l])} \le \sigma. \tag{6}$$

Let us show that there exists a function $F(t,\lambda) \in C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$ such that the sequence $F_{\varepsilon^{(p)}_{n_p}}(t,\lambda)$ converges to $F(t,\lambda)$ in the space $C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$ as $p \to \infty$, i.e., for any $T > 0$, $l > 0$, and $\sigma > 0$, there exists p_0 such that the following inequality holds for all $p > p_0$:

$$\big\|F_{\varepsilon^{(p)}_{n_p}}(t,\lambda)|_{K(T,l)} - F(t,\lambda)|_{K(T,l)}\big\|_{C([0,T],L_1([-l,l]))} < \sigma.$$

According to Eq. (1), the following relation holds for any $n \in \mathbb{N}$:

$$\frac{\partial}{\partial t}F_{\varepsilon_n}(t,\lambda) = j_n(t,\lambda),$$

where

$$j_n(t,\lambda) = i\int\limits_{-\infty}^{\lambda}\big[\big(\mathbf{L}_{\varepsilon_n}u_{\varepsilon_n}(t), d\mathbf{E}(\mu)u_{\varepsilon_n}(t)\big) - \big(u_{\varepsilon_n(t)}, d\mathbf{E}(\mu)\mathbf{L}_{\varepsilon_n}u_{\varepsilon_n}(t)\big)\big].$$

Hence, for any $n \in \mathbb{N}$, the following estimate holds in accordance with the Cauchy–Bunyakovskii inequality:

$$\sup_{t>0,\,\lambda\in\mathbb{R}}|j_n(t,\lambda)| \le 2\sup_{t>0,\,\varepsilon\in(0,1)}\Bigg[\int\limits_{-\infty}^{\lambda}\big(\mathbf{L}_{\varepsilon_n}u_{\varepsilon_n}(t), d\mathbf{E}(\mu)\mathbf{L}_{\varepsilon_n}u_{\varepsilon_n}(t)\big)\Bigg]^{1/2}\Bigg[\int\limits_{-\infty}^{\lambda}\big(u_{\varepsilon_n}(t), d\mathbf{E}(\mu)u_{\varepsilon_n}(t)\big)\Bigg]^{1/2}$$
$$\le \|\mathbf{L}_\varepsilon u_\varepsilon(t)\|_{L_2}\|u_\varepsilon(t)\|_{L_2} = 2\|\mathbf{L}_1 u_0\|_{L_2},$$

since the transformations $\mathbf{U}_{\mathbf{L}_\varepsilon}(t)$ are unitary and the relations $\|\mathbf{L}_\varepsilon u_0\| \le \|\mathbf{L}_1 u_0\|$, $\varepsilon \in (0,1)$, hold for any $u_0 \in \bigcap\limits_{\varepsilon\in[0,1]} D(\mathbf{L}_\varepsilon)$.

Therefore, for any $l > 0$, there exists a constant $c(l) > 0$ such that for all $n \in \mathbb{N}$,

$$\sup_{t\ge 0}\bigg\|\frac{\partial}{\partial t}F_{\varepsilon_n}(t,\lambda)|_{\mathbb{R}_+\times[-l,l]}\bigg\|_{L_1([-l,l])} \le c(l).$$

Hence, for any $t_1, t_2 \in \mathbb{R}_+$ such that $|t_2 - t_1| \le \sigma$, the following inequality holds for any $n \in \mathbb{N}$:

$$\|F_{e_n}(t_2,\lambda) - F_{\varepsilon_n}(t_1,\lambda)\|_{L_1([-l,l])} \le c(l)\sigma. \tag{7}$$

For each $m \in \mathbb{N}$, on the half-open interval $[0,+\infty)$, let us choose points $t_j^{(m)} = 2^{-m}j$, $j,m \in \mathbb{N}$; then $t_j^{(m)} \in \mathbb{Q}$ and $t_j^{(m)} - t_{j-1}^{(m)} = 2^{-m}$, $j \in \mathbb{N}$.

Let $p^{(m)}(t,\lambda) \in C(\mathbb{R}_+, L_{1,\mathrm{loc}})$ be a mapping of the semiaxis $\mathbb{R}_+$ into the linear topological space $L_{1,\mathrm{loc}}(\mathbb{R})$ that is piecewise-linear on $\mathbb{R}_+$, linear on the intervals $\big(t_{j-1}^{(m)}, t_j^{(m)}\big)$, $j = 1,\dots,N$, and such that

$$p^{(m)}\big(t_j^{(m)},\lambda\big) = F\big(t_j^{(m)},\lambda\big), \quad j \in \mathbb{N}. \tag{8}$$

Proposition 1. *The sequence $\{p^{(m)}(t,\lambda)\}$ is a Cauchy sequence in the space C_{loc}.*

Proof. Fix certain $T, l > 0$. Then the sequence $p^{(m)}(t,\lambda)|_{K(T,l)}$ is a Cauchy sequence in $C\big([0,T], L_1([-l,l])\big)$, since the following inequality holds according to (7) and (8):

$$\sup_{t\in[0,T]}\|p^{(m+q)}(t,\lambda) - p^{(m)}(t,\lambda)\|_{L_1([-l,l])} \le 2^{-m}c(l). \tag{9}$$

The proposition is proved. □

It follows from Lemma 1 and Proposition 1 that there exists $F(t,\lambda) \in C_{\mathrm{loc}}$ such that the sequence $p^{(m)}(t,\lambda)$ converges to $F(t,\lambda)$ in the space C_{loc} as $m \to \infty$.

We stress that the sequence of functions $p^{(m)}(t,\lambda)$ and hence its limit are independent of the parameters T and l. Only the constant in the estimate (9) of the Cauchy sequence depends on the parameter l.

Let us show that the sequence $F_{\varepsilon^{(p)}_{n_p}}(t,\lambda)$ converges to the function $F(t,\lambda)$ in the space $C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$ as $p \to \infty$.

Choose certain $T, l > 0$ and estimate the norm of the difference

$$\begin{aligned}&\big\|p^{(m)}(t,\lambda) - F_{\varepsilon_{n_p}^{(p)}}(t,\lambda)\big\| \\ &\quad\le \big\|p^{(m)}(t,\lambda) - p^{(m)}\big(t_j^{(m)},\lambda\big)\big\| + \big\|p^{(m)}\big(t_j^{(m)},\lambda\big) - F_{\varepsilon_{n_p}^{(p)}}\big(t_j^{(m)},\lambda\big)\big\| + \big\|F_{\varepsilon_{n_p}^{(p)}}\big(t_j^{(m)},\lambda\big) - F_{\varepsilon_{n_p}^{(p)}}(t,\lambda)\big\|,\end{aligned}$$

where t_j^m is the nearest number to t among the numbers $t_i^{(m)}$, $i \in \mathbb{N}$. Then by inequality (7), the first and third summands do not exceed $2^{-m}c(l)$, and, according to (8) and (6), for the second summand, we have

$$\big\|p^{(m)}\big(t_j^{(m)},\lambda\big) - F_{\varepsilon_{n_p}^{(p)}}\big(t_j^{(m)},\lambda\big)\big\| = \big\|F\big(t_j^{(m)},\lambda\big) - F_{\varepsilon_{n_p}^{(p)}}\big(t_j^{(m)},\lambda\big)\big\| \le \sigma$$

for any $p \ge p_0$.

Therefore,

$$\big\|p^{(m)}(t,\lambda) - F_{\varepsilon_{n_p}^{(p)}}(t,\lambda)\big\| \le 2^{-m+1}c(l) + \sigma$$

for any $p \ge p_0$. Therefore,

$$\begin{aligned}&\big\|F(t,\lambda) - F_{\varepsilon_{n_p}^{(p)}}(t,\lambda)\big\|_{C([0,T],L_1([-l,l]))} \\ &\quad\le \|F(t,\lambda) - p^{(m)}(t,\lambda)\|_{C([0,T],L_1([-l,l]))} + \big\|p^{(m)}(t,\lambda) - F_{\varepsilon_{n_p}^{(p)}}(t,\lambda)\big\|_{C([0,T],L_1([-l,l]))} \\ &\quad\le \|F(t,\lambda) - p^{(m)}(t,\lambda)\|_{C([0,T],L_1([-l,l]))} + 2^{-m+1}c(l) + \sigma.\end{aligned}$$

Passing to the limit as $m \to \infty$ in the latter estimate, we conclude that for any $\sigma > 0$ and any $T, l > 0$, there exists $p_0 \in \mathbb{N}$ such that the following inequality holds for any $p > p_0$:

$$\big\|F(t,\lambda) - F_{\varepsilon_{n_p}^{(p)}}(t,\lambda)\big\|_{C([0,T],L_1([-l,l]))} \le \sigma.$$

Theorem 4 is proved. □

Lemma 2. *Let $u, v \in H$, and let $\mathbf{E}(\lambda)$, $\lambda \in \mathbb{R}$, be an orthogonal partition of unity in the Hilbert space H. Then the following inequality holds for any $\lambda \in \mathbb{R}$:*

$$3|(u, \mathbf{E}(\lambda)u) - (v, \mathbf{E}(\lambda)v)| \le \|u - v\|_H(\|u\|_H + \|v\|_H).$$

Proof. The following chain of estimates follows from direct calculations and the Cauchy–Bunyakovskii inequality:

$$\begin{aligned}|(u, \mathbf{E}u) - (v, \mathbf{E}v)| &= \big|\mathrm{Re}\big\{\big(u - v, \mathbf{E}(u+v)\big)\big\}\big| \\ &\le \big|\big(u - v, \mathbf{E}(u+v)\big)\big| \le \|u - v\|_H\|\mathbf{E}(u+v)\|_H \le \|u - v\|_H(\|u\|_H + \|v\|_H).\end{aligned}$$

Lemma 2 is proved. □

Corollary 1. *Let $u_0, v_0 \in \bigcap_{\varepsilon\in[0,1]} D(\mathbf{L}_\varepsilon)$, and let an infinitely small sequence ε_m be such that there exist nonnegative functions $F(t,\lambda)$ and $G(t,\lambda)$ monotone for each $t > 0$, bounded from above by unity, having the properties formulated in Theorem 4, and such that the sequences of functions $\{F_{\varepsilon_k}(t,\lambda)\}$ and $\{G_{\varepsilon_k}(t,\lambda)\}$ converge to the functions $F(t,\lambda)$ and $G(t,\lambda)$, respectively, in the space C_{loc}. Then the following inequality holds for any $T, l > 0$:*

$$\|F(t,\lambda) - G(t,\lambda)\|_{C([0,T],L_1(K_{T,l}))} \le C\|u_0 - v_0\|_{L_2(\mathbb{R})}.$$

Proof. Fix a certain $\sigma > 0$. Then the following condition holds for any $T > 0$, $l > 0$ and any $k \in \mathbb{N}$:

$$\begin{aligned}\|F(t,\lambda) - G(t,\lambda)\|_{C([0,T],L_1([-l,l]))} &\le \|F(t,\lambda) - F_{\varepsilon_k}(t,\lambda)\|_{C([0,T],L_1([-l,l]))} \\ &\quad+ \|G(t,\lambda) - G_{\varepsilon_k}(t,\lambda)\|_{C([0,T],L_1([-l,l]))} + \|F_{\varepsilon_k}(t,\lambda) - G_{\varepsilon_k}(t,\lambda)\|_{C([0,T],L_1([-l,l]))},\end{aligned}$$

and according to Theorem 4, there exists $k_0 = k_0(T,l,\sigma) \in \mathbb{N}$ such that

$$\|F(t,\lambda) - F_{\varepsilon_k}(t,\lambda)\|_{C([0,T],L_1([-l,l]))} + \|G(t,\lambda) - G_{\varepsilon_k}(t,\lambda)\|_{C([0,T],L_1(-l,l))} \le \sigma$$

for all $k \geq k_0$. According to Lemma 2, the following estimate holds for the third summand:

$$\begin{aligned}
&\|F_{\varepsilon_k}(t,\lambda) - G_{\varepsilon_k}(t,\lambda)\|_{C([0,T],L_1([-l,l]))} \\
&\quad = \sup_{[0,T]}\big(u_{\varepsilon_k}(t), \big(\mathbf{E}(l) - \mathbf{E}(-l)\big)u_{\varepsilon_k}(t)\big) - \big(v_{\varepsilon_k}(t), \big(\mathbf{E}(l) - \mathbf{E}(-l)\big)v_{\varepsilon_k}(t)\big) \\
&\quad \leq \|\mathbf{U}_{\mathbf{L}_{\varepsilon_k}}(u_0 - v_0)\|_{L_2}\|\mathbf{U}_{\mathbf{L}_{\varepsilon_k}}(u_0 + v_0)\|_{L_2} \leq 2\|u_0 - v_0\|_{L_2}.
\end{aligned}$$

Hence, by the arbitrariness of $\sigma > 0$, the following inequality holds for any compact set $K_{T,l}$, $T, l > 0$:

$$\|F(t,\lambda) - G(t,\lambda)\|_{C([0,T],L_1([-l,l])} \leq 2\|u_0 - v_0\|_{L_2(\mathbb{R})}.$$

Corollary 1 is proved. □

Theorem 5. *Let $\{\varepsilon_n\}$ be a certain infinitely small sequence. Then for any partition of unity $\mathbf{E}(\lambda)$, $\lambda \in \mathbb{R}$, in the Hilbert space H, there exists a subsequence ε_{n_k} such that for any initial function $u_0 \in L_2(\mathbb{R})$, there exists a nonnegative function $G(t,\lambda) \in C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$ monotonically increasing for each $t > 0$ and such that the sequence $G_{\varepsilon_n}(t,\lambda)$ converges to the function $G(t,\lambda)$ in $C\big(\mathbb{R}_+, L_{1,\mathrm{loc}}(\mathbb{R})\big)$.*

Proof. Let $\mathbf{E}(\lambda)$, $\lambda \in \mathbb{R}$, be a certain orthogonal partition of unity in the Hilbert space $H = L_2(\mathbb{R})$, and let u_{0l}, $l \in \mathbb{N}$, be a certain orthonormal basis in the space $L_2(\mathbb{R})$. Consider the following countable family of functions in the space C_{loc}:

$$G_{e_n}^{ji}(t,x) = \int_{-\infty}^{x} (\mathbf{U}_{\mathbf{L}_{\varepsilon_n}}(t)u_{0j}, d\mathbf{E}(\xi)\mathbf{U}_{\mathbf{L}_\varepsilon}(t)u_{0i}), \quad i,j \in \mathbb{N}.$$

As in the proof of Theorem 4, using the *procedure of extracting a diagonal subsequence*, we can show that there exist a subsequence ε_{n_k} of the sequence ε_n and a family of functions $G^{ji}(t,x) \in C_{\mathrm{loc}}$, $i,j \in \mathbb{N}$, such that for any $j,i \in \mathbb{N}$, the sequence of functions $G_{\varepsilon_n}^{ji}(t,x)$ converges to the function $G^{ji}(t,x)$ in the space C_{loc} as $n \to \infty$.

Therefore, for the partition of unity $\mathbf{E}(\lambda)$ and the orthonormal basis u_{0l}, we choose the above sequence ε_{n_k}. Then for any element $u_0(x) \in L_2(\mathbb{R})$ and any $\sigma > 0$, there exists a tuple of numbers $\alpha_1, \ldots, \alpha_N$ such that $\|u_0 - v_0\|_{L_2(\mathbb{R})} \leq \sigma/4$, where

$$v_0(x) = \sum_{l=1}^{N} \alpha_l u_{0l}; \quad \sum_{l=1}^{N} |\alpha_j|^2 \leq \|u_0\|.$$

We set $u_k(t) = \mathbf{U}_{\mathbf{L}_{e_{n_k}}}(t)u_0$ and $v_k(t) = \mathbf{U}_{\mathbf{L}_{e_{n_k}}}(t)v_0$.

Let

$$G_{\varepsilon_m}^{w}(t,x) = \int_{-\infty}^{x} (\mathbf{U}_{\mathbf{L}_{\varepsilon_m}}(t)w, d\mathbf{E}(\xi)\mathbf{U}_{\mathbf{L}_{\varepsilon_m}}(t)w),$$

where $w = u, v$. Then according to Lemma 2, the following estimate holds for any $m \in \mathbb{N}$:

$$\|G_{\varepsilon_m}^{u}(t,x) - G_{\varepsilon_m}^{v}\|_{C([0,T],[-l,l])} \leq \|u_{\varepsilon_m}(t) - v_{\varepsilon_m}(t)\|_{L_2(\mathbb{R})}\|u_{\varepsilon_m} + v_{\varepsilon_m}\|_{L_2(\mathbb{R})} \leq \frac{\sigma}{2}. \tag{10}$$

Therefore, for any $T, l > 0$ and any $\sigma > 0$, there exist a number N and a tuple of numbers $\alpha_1, \ldots, \alpha_N$ such that inequality (10) holds.

According to the choice of the sequence ε_n, for the above $\sigma > 0$ and N, there exists a number n_0 such that the following inequalities hold for all $n \geq n_0$ and all $i,j \in \{1, \ldots, N\}$:

$$\|G_{\varepsilon_n}^{ji}(t,x) - G^{ji}(t,x)\|_{C([0,T],L_1([-l,l]))} \leq \sigma 2^{-(j+i+2)}.$$

Then

$$\|G^v_{m+q}(t,x) - G^v_m(t,x)\|_{C([0,T],L_1(-l,l))}$$
$$\le \sum_{i,j=1}^{N} |\alpha_i|\,|a_j|\,\big\|\big(G^{ji}_{\varepsilon_{m+q}}(t,x) - G^{ji}_{\varepsilon_m}(t,x)\big)\big|_{K(T,l)}\big\|_{C([0,T],L_1(-l,l))} \le \sum_{i,j=1}^{N} \alpha_i \bar{a}_j \sigma 2^{-(j+i+1)} \le \frac{\sigma}{4}.$$

Therefore, the following estimate holds for any $m \ge n_0$ and $q \in \mathbb{N}$:

$$\big\|\big(G^u_{\varepsilon_{m+q}}(t,x) - G^u_{\varepsilon_m}(t,x)\big)\big|_{K(T,l)}\big\|_{C([0,T],L_1(-l,l))} \le \|(G^u_{\varepsilon_{m+q}}(t,x) - G^v_{\varepsilon_{m+q}}(t,x)\|_{C([0,T],L_1(-l,l))}$$
$$+ \|G^v_{\varepsilon_{m+q}}(t,x) - G^v_{\varepsilon_m}(t,x)\|_{C([0,T],L_1(-l,l))} + \|(G^v_{\varepsilon_m}(t,x) - G^u_{\varepsilon_m}(t,x)|_{K(T,l)}\|_{C([0,T],L_1(-l,l))} \le \frac{5}{4}\sigma.$$

Therefore, the sequence $G^u_{\varepsilon_m}(t,x)|_{K(T,l)}$ is a Cauchy sequence in $C\big([0,T],L_1(-l,l)\big)$ for any $T,l > 0$, and according to Lemma 1, in the space C_{loc}, the sequence $G^u_{\varepsilon_m}(t,x)$ converges to the limit function $G(t,x) \in C_{\mathrm{loc}}$, which is a monotonically increasing function for each $t > 0$ on the axis $x \in \mathbb{R}$ and assumes values in the closed interval $[0,1]$. Theorem 5 is proved. □

Application to the Dynamics of Observables

Let $\mathbf{E}(\lambda)$ be an orthogonal partition of unity in H, and let $\mathcal{B}^b_{\mathbf{E}}(H)$ be the subalgebra of the algebra $\mathcal{B}(H)$ of all bounded operators acting on the space H according to the rule

$$\mathbf{A}u = \int_{\mathbb{R}} a(\lambda)\, d\mathbf{E}(\lambda)u,$$

where the function $a(\lambda)$ belongs to the class $C(\mathbb{R})$ and has the limits $\lim_{\lambda\to\pm\infty} a(\lambda) = a_\pm$. Let $\mathcal{B}^c_{\mathbf{E}}(H)$ be the subalgebra of the algebra $\mathcal{B}(H)$ of bounded self-adjoint operators commuting with $\mathbf{E}(\lambda)$, $\lambda \in \mathbb{R}$, whose action on an arbitrary function $u \in L_2(\mathbb{R})$ is given by the formula

$$\mathbf{A}u = \int_{\mathbb{R}} a(\lambda)\, d\mathbf{E}(\lambda)u,$$

where $a(\lambda) \in C(\mathbb{R})$ (see [5]).

Theorem 6. *Let $u_0(x) \in L_2(\mathbb{R})$, let $\mathbf{E}(\lambda)$ be an orthogonal partition of unity, and let $\{\varepsilon_k\}$ be the sequence whose existence is asserted by Theorem 7. Then for any $A \in \mathcal{B}^b_{\mathbf{E}}$ and any $T > 0$, the sequence $\big(u_{\varepsilon_k}(t), Au_{\varepsilon_k}(t)\big)$ uniformly converges to*

$$\bar{A}(t) = a_+\big(1 - F(t,+\infty)\big) + a_- F(t,\infty) + \int_{\mathbb{R}} a(\xi)dF(t,\xi),$$

on $[0,T]$, where $\int_{\mathbb{R}} a(\lambda)\,dF(t,\lambda)$ is the Stieltjes integral of the continuous function $a(\lambda)$ with respect to the monotone bounded function $F(t,\lambda)$ (see [6, Chap. 8, Sec. 6]*).*

For an arbitrary partition of unity $\mathbf{E}(\lambda)$, the proof of Theorem 6 literally repeats the proof of the corresponding assertion for the partition of unity $\mathbf{E}(x)$ in [11].

There arises the following question: does there exist an infinitely small sequence $\{\varepsilon_n\}$ such that the sequence of means $(u_{\varepsilon_n}, Au_{\varepsilon_n})$ converges for any operator $A \in \mathcal{B}(H)$? The answer is negative.

Theorem 7. *If $\mathbf{P}_{H_1}u_0(x) \ne 0$ and $t > t^*$ (see Corollary 1), then for any infinitely small sequence ε_n, we can find a bounded self-adjoint operator $\mathbf{A} \in \mathcal{B}(H)$ such that the sequence $\big(u_{\varepsilon_n}(t), \mathbf{A}u_{\varepsilon_n}(t)\big)$ diverges.*

Remark. The assertion of Theorem 7 can be deduced as a consequence of Theorem 1 in [2]. However, for completeness of the presentation and for obtaining consequences of Theorem 7, in what follows we present the proof based on an approach different from that in [2].

Proof. Let $\mathbf{P}_{H_1} u_0 \neq 0$ and let $t > t^*$. Then according to Corollary 1, for any sequence $\{\varepsilon_n\}$, $\varepsilon_n \to 0$ as $n \to \infty$, the sequence of solutions $\{u_{\varepsilon_n}(t,x)\}$ of the corresponding regularized problems weakly converges to the function $u_*(t,x)$ in $L_2(\mathbb{R})$ and diverges in the norm of the space $L_2(\mathbb{R})$. Moreover, any subsequence of the sequence $\{u_{\varepsilon_n}(t,x)\}$ also diverges in $L_2(\mathbb{R})$. Therefore, the sequence $\{u_{\varepsilon_n}(t,x)\}$ is not compact in $L_2(\mathbb{R})$, and hence there exists $\sigma > 0$ such that for any finite tuple of elements $x_1, \ldots, x_N$, the σ-net $\bigcup\limits_{j=1}^{N} O_\sigma(x_j)$ does not cover all elements of the sequence $\{u_{\varepsilon_n}(t,x)\}$, i.e., there exists $m \in \mathbb{N}$ such that $u_{\varepsilon_m}(t,x) \notin \bigcup\limits_{j=1}^{N} O_\sigma(x_j)$. Here,

$$O_\sigma(M) = \{x \in L_2(\mathbb{R})\colon \rho_{L_2}(x, M) < \sigma\},$$

where ρ_{L_2} is the metric of the space $L_2(\mathbb{R})$ and M is a certain set of the space $L_2(\mathbb{R})$.

Then the following assertion holds: there exists $\sigma > 0$ such that for any $n \in \mathbb{N}$, there exists $m > n$ such that $u_{\varepsilon_m}(t,x) \notin T(n,\sigma)$, where

$$T(n,\sigma) = \big\{x \in L_2(\mathbb{R})\colon x \in O_\sigma\big(\operatorname{lin}(u_*, u_1, \ldots, u_n)\big);\ |\|x\| - 1| < \sigma\big\}.$$

Indeed, if for any $\sigma > 0$, there exists $n \in \mathbb{N}$ such that $u_{\varepsilon_m}(t,x) \notin T(n,\sigma)$ for any $m > n$, then the sequence $\{u_n\}$ can be covered by a finite σ-net, which contradicts its noncompactness.

Let us extract a linearly independent systems of vectors f_k from the sequence $\{u_n\}$ according to the following rule.

Let $f_1 = u_*(t)$; then there exists $m_1 \geq 1$ such that $u_{\varepsilon_{m_1}}(t,x) \notin T(1,\sigma)$. We set $f_1 = u_{m_1}$. Then there exists $m_2 > m_1$ such that $u_{\varepsilon_{m_2}}(t,x) \notin T(m_1,\sigma)$. We set $f_2 = u_{m_2}$, and so on. Then, by induction, there exists a sequence $f_k \in L_2$ that is a subsequence of $\{u_n\}$ such that for any $k \in \mathbb{N}$,

$$f_{k+1} \notin V(k,\sigma) \equiv \big\{x \in L_2(\mathbb{R})\colon x \in O_\sigma\big(\operatorname{lin}(u_*, f_1, \ldots, f_n)\big);\ |\,\|x\| - 1| < \sigma\big\}. \tag{11}$$

Let us subject the family of elements $\{f_k\}$ to the standard orthogonalization procedure. We set

$$g_1 = f_1, \quad g_2 = (I - P_1) f_2, \quad \ldots, \quad g_{k+1} = (I - P_k) f_{k+1}, \quad \ldots,$$

where P_k is the orthogonal projection on the linear span $\operatorname{lin}\{f_1, f_2, \ldots, f_k\}$. Then by (11), the estimate $\|g_k\|_{L_2(\mathbb{R})} \geq \sigma$ holds for any $k \in \mathbb{N}$.

We set $h_k = (\|g_k\|)^{-1} g_k$, $k \in \mathbb{N}$. Then $\{h_k\}$ is an orthonormal system of vectors. Note that according to the construction of the orthogonal system $\{g_k\}$,

$$(f_j, h_k) = 0 \quad \forall j < k.$$

The following inequality holds according to estimate (11):

$$(f_k, h_k) \geq \sigma. \tag{12}$$

According to Theorem 2, the sequence $\{f_j\}$ weakly converges to u^* in $L_2(\mathbb{R})$; therefore,

$$\lim_{j \to \infty} (f_j, h_k) = 0 \quad \forall k \in \mathbb{N}. \tag{13}$$

Consider the bounded self-adjoint operator $\mathbf{Q} \in L_2(\mathbb{R})$ on the space L_2 whose action on any vector $\psi \in L_2$ is given by the relation

$$\mathbf{Q}\psi = \sum_{j=1}^{\infty} a_j (h_j, \psi) h_j,$$

where $\{a_j\}$ is a certain bounded sequence of real numbers. Let us show that there exists a sequence of numbers $\{a_j\}$ assuming values from $\{-1, 0, 1\}$ such that the sequence $(f_k, \mathbf{Q} f_k)$ and hence the sequence $\big(u_{\varepsilon_n}(t), \mathbf{Q} u_{\varepsilon_n}(t)\big)$ diverge.

For arbitrary natural numbers m and p, let us consider the quantity

$$\|(f_k, \mathbf{Q} f_k) - (f_{k+p}, \mathbf{Q} f_{k+p})\| = \left\| \sum_{j=1}^{\infty} a_j |(h_j, f_k)|^2 - \sum_{j=1}^{\infty} a_j |(h_j, f_{k+p})|^2 \right\|,$$

which is equal to

$$\left\| \sum_{j=1}^{k} a_j |(h_j, f_k)|^2 - \sum_{j=1}^{k+p} a_j |(h_j, f_{k+p})|^2 \right\|$$

according to the remarks.

Let $a_1 = 1$; then according to (13), there exists $p_2 > 1$ such that $(h_1, f_{p_2}) \le \sigma/2^2$. We set $a_j = 0$, $j = 2, 3, \dots, p_2 - 1$; $a_{p_2} = -1$. Then there exists $p_3 > p_2$ such that $(h_i, f_{p_3}) \le \sigma/2^3$, $i = 1, 2, \dots, p_2$. We set $a_j = 0$, $j = p_2 + 1, \dots, p_3 - 1$; $a_{p_3} = 1$. By induction, there exists a strictly monotone sequence of natural numbers $\{p_l\}$ such that $(h_i, f_{p_l}) \le \sigma/2^l$, $i = 1, 2, \dots, p_{l-1}$, for any $l = 4, 5, \dots$. We set $a_j = 0$, $j \ne p_l$, and $a_{p_l} = (-1)^{l-1}$.

Then for any $n \in \mathbb{N}$, we can find $p_n > n$ such that

$$(f_{p_n}, \mathbf{Q} f_{p_n}) = \sum_{j=1}^{p_n} a_j |(h_j, f_{p_n})|^2 = (-1)^{n-1} |(f_{p_n}, h_{p_n})|^2 + \alpha_n,$$

where

$$|\alpha_n| \le \sum_{i=1}^{n-1} \delta^2 2^{-n} \to 0$$

as $n \to \infty$. By (12), the sequence $\{(f_{n_p}, \mathbf{Q} f_{n_p})\}$ diverges.

Hence, for any infinitely small sequence $\{\varepsilon_n\}$, we can find a bounded self-adjoint operator $\mathbf{Q}$ such that the sequence $\big(u_{\varepsilon_n}(t), \mathbf{Q} u_{\varepsilon_n}(t)\big)$ diverges. Theorem 7 is proved. □

Corollary 2. *If the sequence $u_n(t)$ of regularized solutions weakly converges in H as $n \to \infty$, then there exists a partition of unity $\mathbf{E}(\lambda)$ such that the inequality $\lim\limits_{\lambda \to \infty} \big(F(\lambda) - F(-\lambda)\big) < 1$ holds for the limit $F(\lambda)$ of any subsequence $F_n(\lambda)$ convergent in the space $L_{1,\mathrm{loc}}(\mathbb{R})$.*

Proof. In the space H, we choose an orthonormal basis $\{e_n\}$ including the orthonormal system $\{h_k\}$ constructed in Theorem 7 as a subsystem. Consider the partition of unity

$$\mathbf{E}(\lambda) = \sum_{j=1}^{[\lambda]} \mathbf{P}_{e_j}.$$

We set $F_i(\lambda) = \big(u_i(t), \mathbf{E}(\lambda) u_i(t)\big)$. Let n_k be a sequence of serial numbers such that $e_{n_k} = h_k$. Then according to inequality (11), for any $\lambda_0 > 0$, there exists a serial number k_0 such that $F_{n_k}(\lambda_0) < 1 - \sigma$ holds for all $k > k_0$. Then the following assertion holds for the upper limit $\bar{F}(\lambda)$ of the sequence $F_{n_k}(\lambda)$: the inequality $\bar{F}(\lambda) \le 1 - \sigma$ holds for any $\lambda > 0$ (since the assumption that there exists $\lambda_0 > 0$ such that $\bar{F}(\lambda_0) > 1 - \sigma$ leads to a contradiction). Corollary 2 is proved. □

Corollary 3. *Let $u_0(x) \in L_2(\mathbb{R})$, let $\mathbf{E}(\lambda)$ be an orthogonal partition of unity, and let $\{\varepsilon_k\}$ be an infinitely small sequence such that the sequence of functions $F_k(t, \lambda)$ converges to a function $F(t, \lambda)$ in C_{loc}. Then the sequence $\big(u_{\varepsilon_k}(t), \mathbf{A} u_{\varepsilon_k}(t)\big)$ converges for any $\mathbf{A} \in \mathcal{B}^{\mathrm{c}}_{\mathbf{E}}$ if and only if $F_+(t) - F_-(t) = 1$.*

Proof. Let us prove the sufficiency. Fix a certain $\varepsilon > 0$. It follows from the convergence of the sequence $F_k(t, \lambda)$ in the space C_{loc} and the condition $F_+(t) - F_-(t) = 1$ for the limit function that there exists $L \ge 0$ such that the following estimates hold for all $k \in \mathbb{N}$:

$$\left| \int\limits_{|\lambda| > L} dF_k(T, \lambda) \right| < \varepsilon,$$

and also the following estimate holds:

$$\left| \int\limits_{|\lambda| > L} dF(T, \lambda) \right| < \varepsilon.$$

Then the following inequality holds for any function $a(\lambda) \in C(\mathbb{R})$ with a finite norm A:

$$\left| \int\limits_{|\lambda|>L} a(\lambda)\, dF(t,\lambda) \right| < A\varepsilon.$$

Hence there exists $\alpha(\lambda) \in C^b(\mathbb{R})$ such that

$$\left| \int\limits_{\mathbb{R}} \big(a(\lambda) - \alpha(\lambda)\big)\, dF(t,\lambda) \right| < 2A\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the sufficiency follows from Theorem 6.

Let us prove the necessity. Let there exist $\tau \geq 0$ and $\delta > 0$ such that $F_+(\tau) - F_-(\tau) = 1 - \delta$. Then there exists a closed interval $[-L, L]$ such that

$$F(\tau, L) - F(\tau, -L) \geq 1 - \frac{9}{8}\delta,$$

and, moreover, the sequence of restrictions $\{F_n(\tau,\lambda)|_{[-L,L]}\}$ converges to $F(\tau,\lambda)|_{[-L,L]}$ in the space $L_1([-L,L])$. Then we can find monotonically increasing sequences of numbers $n_k \in \mathbb{N}$ and $L_k \in (L, +\infty)$ such that

$$\int\limits_{|\lambda|\in[L_k+1, L_{k+1}]} dF_{n_k}(\tau,\lambda) > \frac{7\delta}{8}$$

and

$$\int\limits_{|\lambda|\leq L} dF_{n_k}(\tau,\lambda) > 1 - \frac{5\delta}{4}.$$

Let $a(\lambda)$ be a continuous piecewise-linear function monotone on the intervals $(L_k, L_k + 1)$ such that $a(\lambda) = (-1)^k$ for $\lambda \in [L_k + 1, L_{k+1}]$. Then the sequence $\left\{ \int\limits_{\mathbb{R}} a(l) dF_{n_k}(\lambda) \right\}$ is not a Cauchy sequence. Corollary 3 is proved. □

Example. Let $\{u_n(x)\}$ be an orthogonal sequence of functions from the space $L_2(\mathbb{R})$ whose supports belong to the closed interval $[0,1]$ and the restrictions $\{u_n(x)|_{[0,1]}\}$ compose a basis in $L_2([0,1])$. Then $u_n(x)$ weakly converges to zero in $L_2(\mathbb{R})$. Therefore, there exist a partition of unity $E(\mu)$ and a subsequence $\{u_{n_m}\}$ such that the inequality $\Delta F < 1$ holds for any partial limit $F(\mu)$ of the sequence $\{F_{n_m}(\mu)\}$. But since the supports of all measures with distribution functions $F_n(x) = \int\limits_{-\infty}^{x} |u_n(s)|^2\, ds$ corresponding to the orthogonal partition of unity $E(x)$ have a common compact support $[0,1]$, it follows that the relation $\Delta F = F(1) - F(0) = 1$ holds for any partial limit $F(x)$ of the sequence $F_n(x)$. The example shows that for a sequence of elements of the space H that is only weakly convergent, there can exist partitions of unity of the space H of the following two classes: those for which there exist limit measures on the whole coordinate space $\mathbb{R}$ whose variations are less than 1 and those for which any limit measure on $\mathbb{R}$ is of variation 1.

Remark. Since for any bounded operator $\mathbf{Q} \in \mathcal{B}(H)$ and any infinitely small sequence $\{\varepsilon_n\}$, the sequence of mean values $\big(u_{\varepsilon_n}(t), \mathbf{Q}u_{\varepsilon_n}(t)\big)$ of the operator $\mathbf{Q}$ on solutions of the family of solutions $u_{\varepsilon_n}(t)$ of regularized problems is bounded, the set of its partial limits is bounded, and it follows from Theorem 7 that it consists of more than one point.

There arises the following question: does the Cauchy problem (1), (2) allow us to give the dynamics of values of the observables $\mathbf{A} \in \mathcal{B}(H)$? Is this dynamics uniquely defined?

The intention of applying the vanishing viscosity method to finding the dynamics of values of observables for the Cauchy problem (1), (2) leads to the consideration of multivalued mappings. Define the mapping such that to each infinitely small sequence $\{\varepsilon_n\}$, the initial condition $u_0(x)$, a number $t > 0$, and an operator $\mathbf{A} \in \mathcal{B}(H)$, it puts in correspondence the set of all possible partial limits of the sequence of mean values $\big(u_{\varepsilon_n}(t), \mathbf{Q}u_{\varepsilon_n}(t)\big)$ as $n \to \infty$:

$$T\colon E \times H \times \mathbb{R}_+ \times \mathcal{B}(H) \to 2^{\mathbb{C}}\colon \quad (\{\varepsilon_n\}, u_0, t, \mathbf{Q}) \to T(\{\varepsilon_n\}, u_0, t, \mathbf{Q}),$$

where E is the set of all infinitely small sequences of nonnegative numbers, $T(\{\varepsilon_n\}, u_0, t, \mathbf{Q})$ is the set of all partial limits of the sequence $\{(\mathbf{U}_{\mathbf{L}_{\varepsilon_n}}(t)u_0, \mathbf{Q}\mathbf{U}_{\mathbf{L}_{\varepsilon_n}}(t)u_0)\}$, and $2^{\mathbb{C}}$ is the metric space of all subsets of the complex plane $\mathbb{C}$ equipped with the Hausdorff metric. In this case, the dynamics of mean values loses the single-valued property and is given by a multivalued mapping.

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