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QUANTIFYING NONCLASSICALITY OF $su(1,1)$ SQUEEZED STATES BY QUANTUM FISHER INFORMATION

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Abstract

Evaluating quantum Fisher information is an essential task in the parameter estimation and quantum metrology. It quantifies the sensitivity of a quantum state to probe and capture variations in an unknown parameter, which is aimed to be estimated. In this context, the amount of quantum Fisher information measures the operational nonclassicality of a given state, regarded as a quantifiable resource for quantum metrology. We construct $su(1,1)$ coherent states, using the Perelomov formalism, and present their various optical realizations forming a general class of $su(1,1)$ algebraic squeezed states. We analyze the nonclassicality of these states and evaluate the corresponding Fisher information. Also, we find that $su(1,1)$ algebraic squeezed states surpass the standard quantum limit, thereby exhibiting a quantum metrological advantage.

Keywords: quantum metrology, quantum Fisher information, Cramer–Rao bound, phase estimation, operational resource theory, nonclassical states, $su(1,1)$ Lie algebra, $su(1,1)$ squeezed states, Perelomov coherent states.

1. Introduction

The classical–quantum correspondence of dynamical systems has been a subject of great interest since the very beginning of quantum mechanics [1]. In his seminal work on the quantum theory of optical coherence [2, 3], Glauber introduced the coherent states of the electromagnetic field expressed as $|\alpha\rangle = \sum_0^\infty c_n(\alpha)|n\rangle$, where $|c_n(\alpha)|^2$ represents the probability of detecting n photons in the field. As a quantum superposition of number states $|n\rangle$, Glauber coherent states are inherently quantum-mechanical states but establish a one-to-one correspondence with the properties of classical coherent electromagnetic fields.

The benchmark for the classical behavior of a quantum optical state is defined by the celebrated notion of the Glauber–Sudarshan P-function [2, 4]. Within this framework, a state is deemed classical, if its underlying P-function is positive definite; otherwise, the state is considered nonclassical. States exhibiting nonclassical properties are found to be of great importance, particularly, in quantum interferometry [5], quantum information processing [6–8], and quantum computation [9]. Nonclassicality is recognized as a key resource for numerous quantum technologies [10–12], including quantum teleportation [13, 14], quantum metrology [15], and the implementation of various quantum gates [9, 16].

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The coherent states, along with their generalizations in various contexts [17–29], have played an instrumental role in broad areas of research. A major focus has been on exploring their nonclassical properties, such as through discrete excitation or photon addition to coherent states associated with various general systems [30–33]. In this context, a seminal theoretical proposal by Agarwal and Tara [34, 35] introduced the coherent addition of a discrete number of photons to continuous-variable coherent states. These photon-added coherent states (PACSs) exhibit strong nonclassical properties [34, 35]. Laboratory experiments have realized such PACSs, demonstrating a smooth classical-to-quantum transition through a single-photon addition to the coherent state of light [36] and experimentally probing quantum commutation rules [37].

Originating from the formulation of Glauber coherent states, using the Heisenberg–Weyl algebra of the harmonic oscillator [2], the generalization of this concept has been extended, using algebras related to various Lie groups. Typical examples include Perelomov coherent states [18] and Barut–Girardello coherent states [17]. The Lie group $SU(1,1)$ and its associated algebra find vast applications in quantum optics [38–45].

In our previous papers [46–48], we introduced a general class of coherent states based on the $su(1,1)$ Lie algebra and discussed various optical realizations of the $su(1,1)$ Lie algebra and its associated coherent states. We observed that, under particular optical realizations, $su(1,1)$ Perelomov coherent states map onto various types of squeezed states, referred to as $su(1,1)$ squeezed states. Furthermore, we analyzed the nonclassical properties of our constructed coherent states and found that nonclassicality was enhanced by multiphoton excitation [46–48]. Being highly nonclassical in nature, $su(1,1)$ squeezed states hold a special place in quantum optics [49, 50] and various related areas [51], especially in the theory of quantum metrology [52, 53].

However, quantifying the nonclassicality of optical states for quantum metrology is a fundamental and crucial task [54–57]. In this context, quantum Fisher information provides an operational definition of nonclassicality, quantifying the sensitivity of a quantum state to probe and capture variations of an unknown parameter. In this work, we present a class of $su(1,1)$ squeezed coherent states and analyze their nonclassical behavior: (i) first, using photon counting statistics and the Mandel Q-parameter; (ii) then, by computing quantum Fisher information. We find that our constructed $su(1,1)$ squeezed states have the potential to better perform in quantum metrology and surpass the standard quantum limit.

This paper is organized as follows.

In Sec. 2, first we review the $su(1,1)$ Lie algebra and associated Perelomov coherent states. Subsequently, using the single-mode bosonic realization of the $su(1,1)$ algebra and the relevant unitary irreducible representations, we present the $su(1,1)$ squeezed states and analyze the nonclassicality of $su(1,1)$ squeezed states, employing the photon detection probability. In Sec. 3, we present the Mandel Q-parameter and then discuss how quantum Fisher information quantifies the underlying nonclassicality of a quantum state, within the framework of quantum metrology. Finally, we conclude our work in Sec. 4.

2. The $su(1,1)$ Squeezed States

We define $su(1,1)$ squeezed states as the optical realization of $su(1,1)$ Perelomov coherent states [18]. Before digging into a detailed discussion of the $su(1,1)$ squeezed states, first we provide a quick review of the $SU(1,1)$ group, its relevant unitary irreducible representations, and the associated Perelomov coherent states. By expressing the elements of the $su(1,1)$ Lie algebra in terms of single-mode bosonic ladder operators, we explicitly demonstrate how $su(1,1)$ Perelomov coherent states map onto single-mode squeezed states.

2.1. The $su(1,1)$ Algebraic Model

The $su(1,1)$ Lie algebra comprises three generators $\hat{L}_+$, $\hat{L}_-$, and $\hat{L}_0$ satisfying the commutation relations,

$$[\hat{L}_0, \hat{L}_\pm] = \pm\hat{L}_\pm, \qquad [\hat{L}_-, \hat{L}_+] = 2\hat{L}_0, \tag{1}$$

where $\hat{L}_\pm$ collectively denotes both $\hat{L}_+$ and $\hat{L}_-$. These operators adhere to the Hermiticity conditions, $(\hat{L}_-)^\dagger = \hat{L}_+$, $(\hat{L}_+)^\dagger = \hat{L}_-$, and $(\hat{L}_0)^\dagger = \hat{L}_0$. Additionally, we define combinations $\hat{L}_1 = (\hat{L}_+ + \hat{L}_-)/2$ and $\hat{L}_2 = (\hat{L}_+ - \hat{L}_-)/2$, which, along with $\hat{L}_0$, constitute the set of generators belonging to the $SU(1,1)$ group. The generator $\hat{L}_0$ generates compact $SU(1,1)$ transforms of the elliptic class, while $\hat{L}_1$ and $\hat{L}_2$ generate noncompact $SU(1,1)$ transforms of the hyperbolic class [58]. Expressed in terms of $\hat{L}_1$, $\hat{L}_2$, and $\hat{L}_0$, the $su(1,1)$ Lie algebra is given by

$$[\hat{L}_1, \hat{L}_2] = -i\hat{L}_0, \qquad [\hat{L}_2, \hat{L}_0] = i\hat{L}_1, \qquad [\hat{L}_0, \hat{L}_1] = i\hat{L}_2. \tag{2}$$

The corresponding Casimir operator is expressed as

$$\hat{\mathcal{C}} = \hat{L}_0^2 - \hat{L}_1^2 - \hat{L}_2^2 = \hat{L}_0^2 - \frac{1}{2}(\hat{L}_+\hat{L}_- + \hat{L}_+\hat{L}_-). \tag{3}$$

Having defined the structure of associated algebra, the relevant irreducible representations of $SU(1,1)$ are given by positive discrete series $\mathcal{D}^\varkappa : \{|\varkappa, m\rangle, \varkappa > 0; m = 0, 1, 2, \ldots\}$ satisfying the eigenvalue equations,

$$\hat{\mathcal{C}}|\varkappa, m\rangle = \varkappa(\varkappa - 1)|\varkappa, m\rangle, \tag{4}$$

$$\hat{L}_0|\varkappa, m\rangle = (m + \varkappa)|\varkappa, m\rangle, \tag{5}$$

along with the following relations:

$$\hat{L}_-|\varkappa, m\rangle = \sqrt{m(2\varkappa + m - 1)}\ |\varkappa, m-1\rangle, \qquad \hat{L}_+|\varkappa, m\rangle = \sqrt{(m+1)(2\varkappa + m)}\ |\varkappa, m+1\rangle, \tag{6}$$

where $\varkappa$ is the so-called Bargmann index.

In Eq. (6), we regard the operators $\hat{L}_-$ and $\hat{L}_+$ as $SU(1,1)$ annihilation and creation operators, respectively, and the $SU(1,1)$ ground state can be defined as

$$\hat{L}_-|\varkappa, 0\rangle = 0. \tag{7}$$

In this sense, the state $|\varkappa, m\rangle$ can be generated by repeated action of $\hat{L}_+$ on the ground state $|\varkappa, 0\rangle$, according to

$$|\varkappa, m\rangle = \left[\frac{\Gamma(2\varkappa)}{m!\Gamma(2\varkappa + m)}\right]^{1/2} (\hat{L}_+)^m|\varkappa, 0\rangle. \tag{8}$$

2.2. The $su(1,1)$ Perelomov Coherent States

In his seminal work on the quantum theory of optical coherence, Glauber defined *coherent states* [2] as displaced vacuum states; they read

$$|\alpha\rangle = D(\alpha)|0\rangle, \qquad D(\alpha) = \exp\,[\alpha a^\dagger - \alpha^* a], \tag{9}$$

where α is a complex parameter representing the amplitude of a coherent electromagnetic field. Equivalently, the coherent states are defined as eigenstates of bosonic annihilation operator, i.e., $a|\alpha\rangle = \alpha|\alpha\rangle$. Expanding the coherent state $|\alpha\rangle$ as superposition of photon-number states $|n\rangle$ yields

$$|\alpha\rangle = e^{-(|\alpha|^2)/2}\sum_0^\infty \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{10}$$

In analogy, by generalizing the above-mentioned Glauber's formalism, the $su(1,1)$ displacement operator [18] can be defined as

$$D(\beta) = \exp(\beta\hat{L}_+ - \beta^*\hat{L}_-); \qquad \beta = -\frac{r}{2}e^{-i\varphi}, \tag{11}$$

where $0 < r < \infty$ and $0 \le \varphi \le 2\pi$ are group parameters. The Perelomov coherent state [18] associated to $su(1,1)$ algebra is defined by applying displacement operator, given in Eq. (11), to the ground state

$$|\beta,\varkappa\rangle = D(\beta)|\varkappa,0\rangle. \tag{12}$$

In view of the $SU(1,1)$ disentangling theorem [39], the displacement operator can be written as

$$D(\beta) = \exp\,(\beta\hat{L}_+)\exp\,(\Lambda\hat{L}_0)\exp\,(-\beta^*\hat{L}_-), \tag{13}$$

where $\beta = -e^{-i\varphi}\tanh(r/2)$ and $\Lambda = \ln\,(1-|\beta|^2)$. The parameter $|\beta|$ is thus limited within unit circle, $0 \le |\beta| < 1$, on the complex plane. Now it is easy to express the Perelomov coherent states in terms of basis $|\varkappa, m\rangle$ as follows:

$$|\beta,\varkappa\rangle = (1-|\beta|^2)^\varkappa \sum_{n=0}^\infty \sqrt{\frac{\Gamma(2\varkappa+m)}{m!\Gamma(2\varkappa)}}\;\beta^m|\varkappa,m\rangle. \tag{14}$$

Below, we connect the Perelomov coherent states to single-mode squeezed states under a specific bosonic realization of $su(1,1)$ algebra.

2.3. Single-Mode $su(1,1)$ Algebraic Squeezed States

The elements of the $su(1,1)$ Lie algebra, as defined in Eq. (3), can be realized through various combinations of standard bosonic annihilation and creation operators a and $a^\dagger$. In a single-mode realization, the operators L_+, L_-, and L_0 are expressed as

$$\hat{L}_+ = \frac{1}{2}a^{\dagger 2}, \qquad \hat{L}_- = \frac{1}{2}a^2, \qquad \hat{L}_0 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right). \tag{15}$$

In this case, the Casimir operator becomes $C = -3/16$, and the possible values of Bargmann indices are $\varkappa = 1/4$ and $\varkappa = 3/4$. Moreover, the usual boson number states $|n\rangle; n = 0, 1, 2, \ldots$ map onto the unitary irreducible representations of $SU(1,1)$ [59] according to

$$|n\rangle \leftrightarrow |\varkappa, m\rangle \qquad \text{for} \qquad n = 2(m + \varkappa) - 1/2. \tag{16}$$

It is important to note from Eq. (16) that, for $\varkappa = 1/4$, we get $n = 2m$. This implies that, for $\varkappa = 1/4$, only the even number of photons are mapped onto the unitary irreducible representation of $SU(1,1)$. On the other hand, using $\varkappa = 3/4$ in Eq. (16) reveals that the odd photon-number states with $n = 2m+1$ are mapped onto the other unitary representation of $SU(1,1)$. It is worth mentioning that the ground states, corresponding to these representations, link up to the Fock states as $|1/4, 0\rangle = |0\rangle$ and $|3/4, 0\rangle = |1\rangle$.

Using the single-mode bosonic realization of $SU(1,1)$ operators, given in Eq. (15), the $SU(1,1)$ displacement operator can be written as

$$D(\beta) = \exp\left(\frac{1}{2}\beta a^{\dagger 2} - \frac{1}{2}\beta^* a^2\right), \tag{17}$$

which is exactly the same as a single-mode squeezing operator. For $\varkappa = 1/4$, the corresponding Perelomov $SU(1,1)$ coherent states, in terms of photon-number states, are given by

$$|\beta, 1/4\rangle = (1 - |\beta|^2)^{1/4} \sum_{m=0}^{\infty} \left[\frac{\Gamma(m + 1/2)}{m!\Gamma(1/2)}\right]^{1/2} \beta^m |2m\rangle. \tag{18}$$

It is important to note that the state in Eq. (18) is the single-mode squeezed vacuum state, in which only the even photon-number states are populated. On the other hand, for $\varkappa = 3/4$, the corresponding Perelomov coherent state reads

$$|\beta, 3/4\rangle = (1 - |\beta|^2)^{3/4} \sum_{m=0}^{\infty} \left[\frac{\Gamma(m + 3/2)}{m!\Gamma(3/2)}\right]^{1/2} \beta^m |2m + 1\rangle, \tag{19}$$

which is a single-mode squeezed one-photon state, containing only odd photon-number states being populated.

3. Quantification of Nonclassicality and Quantum Fisher Information

A quantum optical state $\hat{\rho}$ can be expressed by a diagonal representation of coherent states; it reads

$$\hat{\rho} = \int P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha| d^2\alpha. \tag{20}$$

Here, $P(\alpha, \alpha^*)$ is known as the Glauber–Sudarshan P-function. The state $\hat{\rho}$ is considered as classical state, if its corresponding Glauber–Sudarshan P-function is positive definite; otherwise, if the probability distribution $P(\alpha, \alpha^*)$ is negative or narrower than a delta-function [2,4], the state is defined as nonclassical state. However, directly characterizing $P(\alpha, \alpha^*)$ is challenging in many practical situations; alternative suitable indicators are employed in various scenarios [5].

Within the framework of quantum metrology, the operational definition of nonclassicality refers to the exploitation of uniquely quantum features to achieve precision beyond what is classically achievable. This operational interpretation of nonclassicality relies on evaluating the amount of Fisher information related to some appropriate generator of the unitary evolution of an unknown parameter aimed to be estimated. In the following, we analyze the nonclassicality of $su(1,1)$ squeezed coherent states using photon-counting statistics and quantum Fisher information, within the framework of quantum metrology.

3.1. Sub-Poissonian Statistics

The most convenient way to explore the nonclassical nature is to analyze the underlying photon counting probability distribution

$$P_n(|\beta|) = |\langle n,k|\beta,k\rangle|^2. \tag{21}$$

In Eq. (10), one can see that the photon detection probability $|\langle n|\alpha\rangle|^2$ of Glauber coherent states exhibits Poissonian statistics, which is a benchmark of classical behavior. In the case of $su(1,1)$ squeezed states, the probability distribution is given as

$$P_n(|\beta|) = (1-|\beta|^2)^{2\varkappa}\frac{\Gamma(2\varkappa+m)}{m!\Gamma(2\varkappa)}\beta^m. \tag{22}$$

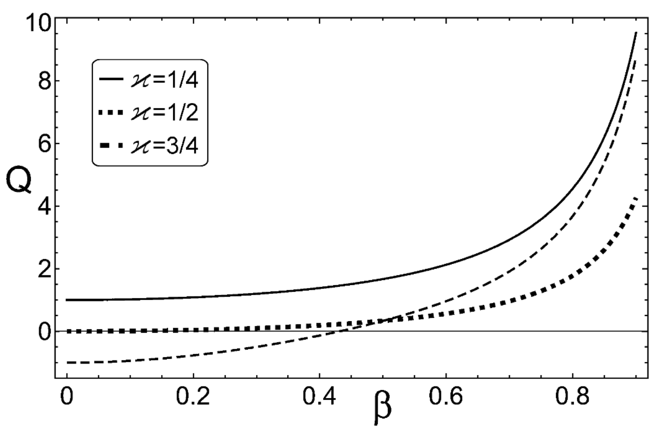

Fig. 1. Mandel Q-parameter versus $|\beta|$ for $su(1,1)$ Perelomov coherent states.

However, it is difficult to ascertain the exact nature of the underlying statistics for these coherent states from Eq. (22). To identify the underlying probability distribution, we can characterize the Mandel Q-parameter [60] defined as

$$Q = \frac{\left[\langle(\hat{a}^\dagger\hat{a})^2\rangle - \langle\hat{a}^\dagger\hat{a}\rangle^2\right]}{\langle\hat{a}^\dagger\hat{a}\rangle} - 1. \tag{23}$$

For $Q = 0$, the distribution is Poissonian, and for $Q < 0$ ($Q > 0$), the distribution is sub-Poissonian (super-Poissonian). The detection of sub-Poissonian statistics is an indicator of nonclassicality. We numerically compute the Mandel Q-parameter for $su(1,1)$ squeezed states; the results are displayed in Fig. 1. These plots show that $Q > 0$ for all chosen values of β, indicating that $su(1,1)$ squeezed states exhibit super-Poissonian statistics. Hence, in this case, the analysis of the Mandel Q-parameter alone is not enough to detect and quantify the correct amount of nonclassicality.

3.2. Quantum Fisher Information as a Measure of Nonclassicality

Quantum Fisher information is an essential element in quantum metrology and parameter estimation. While estimating an unknown parameter, say λ, the Fisher information quantifies the sensitivity of a quantum state $\hat{\rho}$ to variations in λ during some unitary evolution introduced by some generator $\hat{G}$. For a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ evolving under a unitary transform $\hat{\rho}(\lambda) = U(\lambda)\hat{\rho}U^\dagger(\lambda)$, where $U(\lambda) = e^{-i\lambda\hat{G}}$, with $\hat{G}$ being a Hermitian operator, the quantum Fisher information [61,62] is given by

$$F_Q(\rho,\lambda) = 4(\Delta G)^2_{|\psi\rangle}, \tag{24}$$

where $(\Delta G)^2_{|\psi\rangle} = \langle\psi|\hat{G}^2|\psi\rangle - \langle\psi|\hat{G}|\psi\rangle^2$.

For example, in the case of a single-mode phase estimation, the generator $\hat{G} = \hat{n}/2$, where $\hat{n} = \hat{a}^\dagger\hat{a}$, is the bosonic number operator. In view of Eq. (24), for classical pure states, for instance, coherent states $\hat{\rho} = |\alpha\rangle\langle\alpha|$, the quantum Fisher information is

$$F_Q(\rho,\lambda) = 4(\Delta n/2)^2_{|\alpha\rangle} = (\Delta n)^2 = \langle n\rangle_\alpha. \tag{25}$$

This is due to the fact that the number distribution of the coherent state is Poissonian; therefore, the variance and mean of the number distribution are the same. For a classical mixed state defined in terms of coherent states $\rho_{\rm cl} = \int P_{\rm cl}(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha$, where $P_{\rm cl}(\alpha,\alpha^*)$ is a positive probability distribution function, the quantum Fisher information reads

$$F_Q(\rho_{\rm cl}, n/2) \leq \int d^2\alpha\ P_{\rm cl}(\alpha,\alpha^*)\langle n\rangle_\alpha = {\rm Tr}\left[\rho_{cl} n\right] = \langle n\rangle_{\rho_{\rm cl}}. \tag{26}$$

The measurement precision in the parameter estimation is related with quantum Fisher information by the quantum Cramer–Rao bound [61, 62],

$$\Delta\phi \geq \frac{1}{\sqrt{\langle n\rangle_{\rho_{\rm cl}}}}. \tag{27}$$

The above expression is called the standard quantum limit or the shot noise limit, which states that, for classical light sources, the measurement precision scales with the inverse square root of the mean photon number at best. It is important to note that the quantum Fisher information, for classical states $F_Q(\rho_{\rm cl}, n/2)$, scales with the average photon number $\langle n\rangle_{\rho_{\rm cl}}$ at best. Hence, for any quantum state $\hat{\rho}$, if

$$F_Q(\rho, n/2) > F_Q(\rho_{\rm cl}, n/2) = \langle n\rangle_{\rho_{\rm cl}}, \tag{28}$$

it implies that $\hat{\rho}$ must be nonclassical and may provide useful quantum metrological advantage.

As a typical example, we consider the constructed $su(1,1)$ coherent states given in Eq. (14). For the value of the Bargmann index $\varkappa = 1/4$, the $su(1,1)$ basis states $|m,\varkappa\rangle$ map onto even number states $|2n\rangle$, i.e., $|m, 1/4\rangle = |2n\rangle$. In this case, the $su(1,1)$ squeezed states are connected with squeezed vacuum states; see Eq. (18). The mean photon number and variance for squeezed vacuum states can be calculated as $\langle n\rangle_{\rho_{\rm sq}} = \sinh^2(|\beta|)$ and $(\Delta n/2)^2_{\rho_{\rm sq}} = 2\cosh^2(|\beta|)\sinh^2(|\beta|)$. Using Eq. (24), we can calculate the quantum Fisher information as follows:

$$F_Q(\rho_{\rm sq}, n/2) = 2(\langle n\rangle^2_{\rho_{\rm sq}} + \langle n\rangle_{\rho_{\rm sq}}). \tag{29}$$

Equation (29) clearly shows that $F_Q(\rho_{\rm sq}, n/2) > F_Q(\rho_{\rm cl}, n/2)$, indicating the operational nonclassicality of squeezed vacuum states available to surpass the standard quantum limit, i.e., $F_Q(\rho_{\rm sq}, n/2) > \langle n\rangle_{\rho_{\rm sq}}$.

We numerically plot quantum Fisher information for $su(1,1)$ squeezed states; the results are presented in Fig. 2. Moreover, we numerically compute the phase uncertainty for various values of Bargmann indices $\varkappa$, belonging to various classes of $su(1,1)$ squeezed states. The corresponding results are displayed in Fig. 3.

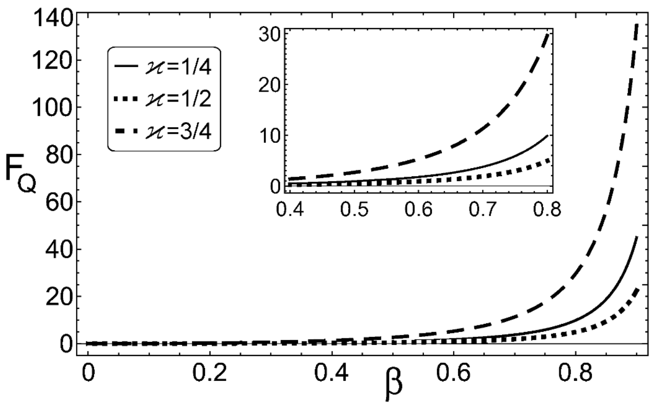

Fig. 2. The quantum Fisher information for $su(1,1)$ Perelomov coherent states as a function of $|\beta|$.

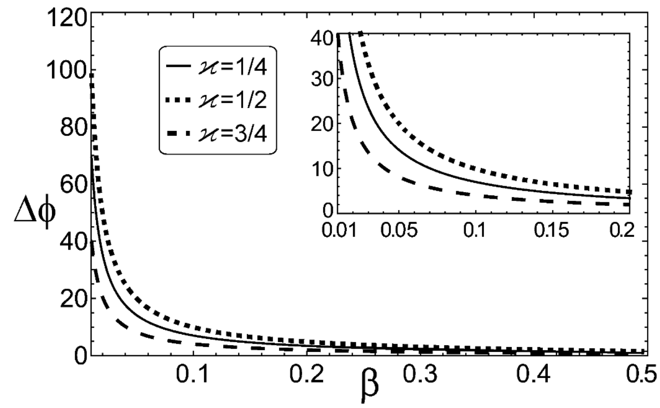

Fig. 3. The phase uncertainty $\Delta\phi$ for $su(1,1)$ Perelomov coherent states as a function of $|\beta|$.

4. Summary and Conclusions

Detecting and quantifying the nonclassicality of quantum optical states is a crucial task due to its profound impact on advancing quantum technologies. The nonclassicality of a quantum optical state is defined by the Glauber–Sudarshan P-function [2, 4]. A state is considered nonclassical, if its underlying P-function is negative or narrower than a delta-function. However, in many practical situations, direct analysis of the P-function may become challenging, leading to the use of various alternative measures to detect and analyze the inherent nonclassical properties of given quantum states. Nonetheless, harnessing nonclassicality to perform a specific quantum task beyond the classical limit requires a resource-theoretic approach. In this article, we demonstrated that the amount of quantum Fisher information measured operational nonclassicality, serving as a quantifiable resource for quantum metrology.

In this article, employing a specific realization of the $su(1,1)$ Lie algebra and associated Perelomov coherent states, we introduced a large class of squeezed states. The $su(1,1)$ algebra finds extensive applications in describing various special situations in the quantum optical field. Specifically, we utilized the single-mode realization, expressing the generators of the $su(1,1)$ algebra in terms of standard bosonic ladder operators. Correspondingly, we mapped Perelomov coherent states onto a class of $su(1,1)$ squeezed states. Then, we computed quantum Fisher information for the constructed $su(1,1)$ squeezed states, quantifying operational nonclassicality for performing quantum metrological tasks better than classical states. The results indicate that $su(1,1)$ algebraic squeezed states have the potential to surpass the standard quantum limit, thereby exhibiting a quantum metrological advantage.

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