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COMMENTS ON 100 YEARS OF QUANTUM MECHANICS: NEW RESULTS IN ITS UNDERSTANDING AND APPLICATIONS IN MODERN QUANTUM TECHNOLOGIES

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Due to the century of development of quantum mechanics, when basic foundations for creations of new technologies like artificial intellect, quantum computing, and new aspects of quantum information were suggested, we open a special rubric reflecting some aspects of the progress in better understanding still unusual for human intuitive ideas based on every day practice, which can be associated, for example, with understanding the notion of state of usual particle in classical and quantum mechanics. In every day life, we associate such a state with the position q of the particle and its velocity or momentum $p = m\dot{q}$, where m is the particle's mass. If we assume $m = 1$, then $\dot{q} = p$; thus, two parameters describe the particle state. The time evolution of this classical particle state is associated with the trajectory $q(t)$ and $p(t)$ in the particle's phase space. This picture means that we understand the meaning of time and how to measure the time t by a device like watch. Also, we can measure both characteristics $q(t)$ and $p(t)$ of the particle state at the same time.

Several centuries ago, Isaac Newton discovered that the state of classical particle could be described by the equation $f = m\ddot{q}(t)$ known as Newton's law, with $\ddot{q}(t)$ being the particle acceleration, where the force was defined by the potential energy $U(q)$, namely, $f = -\dfrac{\partial U}{\partial q}$; the energy E contained both contributions of the kinetic energy and the potential energy, i.e., $E = \dfrac{p^2}{2m} + U(q)$. During many centuries, all the results of technical achievements like planes, cars, factories, etc. were based on these formulas.

In the beginning of the last century, it became clear that Newton's Picture of the World was not complete, and the particles like atoms and molecules demonstrated that this picture was even incorrect. For example, it turned out that it was impossible, in principle, to simultaneously measure the particle's position and velocity (momentum) due to the existence of uncertainty relations [1–3] in the nature. Also, the position and momentum could not be described by numbers, but should be associated with operators $\hat{q}$ and $\hat{p}$ acting on the states. The operators acted on the vectors in Hilbert spaces, and these vectors were associated with complex wave functions $\psi(x,t) = |\psi(x,t)| \exp[i\phi(x,t)]$. The particle's state was described by the wave function obeying to the Schrödinger equation [4, 5].

New era of the science development started, and important steps in the quantum revolution in science were better understandings of the notion introduced. To understand means to associate the quantum picture with our every day intuition and connection with our experience in the classical picture of the state, position, and momentum; for example, to describe the usual particle states like the states of atoms, molecules, and electrons, which we use for radio, TV, etc. These attempts fast started to find

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the understanding based on our knowledge of classical states associated with temperature T of the gas in the room, where one particle like molecule was moving and kicked by the other atoms and molecules. This means that both the position and momentum of this particle fluctuate, and these fluctuations do not permit the particle to have exact position and momentum; they have dispersions δq and δp, but at temperature $T \to 0$, they are permitted to be equal to zero; so, for $T = 0$, we have the picture, where the state can be determined by exact position q and momentum p.

The idea arrived to employ the probability theory, where the particle states, being random fluctuating variables, were described by the probability distribution $f(q,p)$, which was the basic notion in classical thermodynamics. After discovery of quantum mechanics and the Schrödinger equation for the wave function, the pioneers of quantum mechanics started their attempts to design an analogous picture to find the probability representation of quantum states. Wigner constructed the function [6] similar to the probability density $f(q,p)$ used in classical thermodynamics; there were other similar functions like Husimi function [7] and Glauber–Sudarshan function [8,9]. In fact, these functions were not probability distributions, and there existed the common opinion that there was no way to describe quantum states by probability distribution functions.

Nevertheless, during several decades in the 20th century, there were continuing attempts to find the probability representation of quantum states but without success. Fortunately, the probability representation of quantum particle states was constructed [10–15], and the development of the suggested approach for obtaining the foundation of quantum mechanics similar to intuitively clear picture of the probability description of classical mechanics was continuing. Shortly, the result can be formulated as follows.

The wave function $\psi(X)$ introduced in quantum mechanics can be mapped onto the probability distribution of random variable X, which is the position measured in an ensemble of reference frames in the particle's phase space, where μ and ν determine the axes in the phase space [16]; below, we consider Planck's constant $\hbar = 1$,

$$G\,(X|\mu,\nu) = \frac{1}{2\,\pi\,|\nu|}\left|\int \psi(Y)\exp\left(\frac{i\mu}{2\,\nu}Y^2 - \frac{i}{\nu}XY\right)\,dY\right|^2. \tag{1}$$

This function is the conditional probability distribution of random position X measured in the set of reference frames in the phase space, where axes of the position and momentum were first re-scaled, in view of the formulas $q' = sq$ and $p' = s^{-1}p$, and then they were rotated as $q'' = \cos\Theta\, q' + \sin\Theta\, p'$. It turned out that the conditional probability distribution function $G\,(X|\mu,\nu)$ determined the density matrix [17] $\rho(x,x') = \psi(x)\psi^*(x')$ of the pure state with the wave function $\psi(x)$. The conditional probability distribution function contains the same information on the particle state as the wave function does. The map is constructed, in view of the general approach based on the existence of dequantizer $\hat{U}(\vec{X})$ and quantizer $\hat{D}(\vec{X})$ operators, which provide the equations for the state density operator $\hat{\rho}$ and its symbol – function $f_\rho(\vec{X})$ [18],

$$f_\rho(\vec{X}) = \mathrm{Tr}\left[\hat{\rho}\hat{U}(\vec{X})\right]; \qquad \hat{\rho} = \int f_\rho(\vec{X})\hat{D}(\vec{X})\,d\vec{X}. \tag{2}$$

If $\hat{U}(\vec{X})$ has the properties of density operator, i.e., $\hat{U}^\dagger(\vec{X}) = \hat{U}(\vec{X})$, $\mathrm{Tr}\left[\hat{U}(\vec{X})\right] = 1$, and its eigenvalues are nonnegative, the symbol of operator $\hat{\rho}$ is the probability distribution function. In the case of spin discrete variables, the integral in Eq. (2) is replaced by the corresponding sum.

This approach provided the possibility to construct the probability representation of qubit state [19–21]. Also, such conventional probability distributions [22] were constructed for new systems like inverted harmonic oscillator [23], and its probability representation was found in [12]. Such representation of quantum system states was used in applications to study the Universe properties [24]. New entangled probability distributions were introduced [25] to describe Bell states [26, 27] and the superposition of even and odd coherent states [20] known as the Schrödinger cat states [28]. New aspects of quantum mechanics are in press in J. Russ. Laser Res., where the known solution of Schrödinger equation provides the possibility to study the properties of potential energy [29] and the other aspects [30]. For qubit states, the probability description of density matrix and unitary group transformation of the probabilities were discussed in [31].

Thus, it turned out that the classical-like description of quantum system states by the probability distribution functions was found, and human intuition based on the properties of probability theory begun to work for studying the both classical and quantum systems. Then quantum mechanics started a new century of its life connected with elaborating new results in its understanding and applications in modern quantum technologies.

Below, we consider a well-known example of the ground state of one-dimensional harmonic oscillator. We present recently constructed normal probability distribution given by Eq. (1) and the wave function $\psi(y) = \pi^{-1/4}e^{-y^2/2}$; its explicit form reads

$$G_0\left(X|\mu,\nu\right) = \frac{1}{\sqrt{\pi\left(\mu^2+\nu^2\right)}}\exp\left(-\frac{X^2}{\mu^2+\nu^2}\right). \tag{3}$$

The density operator $\hat{\rho}$ with the density matrix of ground oscillator state $\rho_0\left(x,x'\right) = \dfrac{1}{\sqrt{\pi}}\exp\left(-\dfrac{x^2+x'^{\,2}}{2}\right)$ is expressed in terms of this probability distribution as follows:

$$\hat{\rho} = \frac{1}{4\pi^2}\int G_0\left(X|\mu,\nu\right)\exp\left[i\left(X-\mu\hat{q}-\nu\hat{p}\right)\right]\,dX\,d\mu\,d\nu. \tag{4}$$

Density operators of all other excited states of harmonic oscillators are easily obtained, in view of analogous formulas.

To demonstrate the meaning of probability representations of quantum states, we consider the other basic example of qubit states (spin-1/2 states), with 2×2 density matrix ρ of the form

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \qquad \rho = \rho^\dagger, \qquad \rho_{11}+\rho_{22} = 1. \tag{5}$$

One can check that the conditional probability distribution $w\left(X \mid j\right)$ describes the density matrix

$$\rho = \begin{pmatrix} p_3 & \left(p_1-1/2\right)-i\left(p_2-1/2\right) \\ \left(p_1-1/2\right)+i\left(p_2-1/2\right) & 1-p_3 \end{pmatrix}, \tag{6}$$

where spin projections $X = +1/2$ and $X = -1/2$ onto three axes x, y, and z are random variables, while the conditions $j = 1$, $j = 2$, and $j = 3$ determine these axes.

Here, we have three probability distributions of dichotomic random variables X, namely, $\begin{pmatrix} p_1 \\ 1-p_1 \end{pmatrix}$, $\begin{pmatrix} p_2 \\ 1-p_2 \end{pmatrix}$, and $\begin{pmatrix} p_3 \\ 1-p_3 \end{pmatrix}$. Thus, the conditional probability distribution $w\left(X \mid j\right)$ is expressed in terms of the probabilities $0 \leq p_1, p_2, p_3 \leq 1$ as follows:

$$\begin{aligned} &w\,(+1/2 \mid 1) = p_1; \qquad &w\,(-1/2 \mid 1) = 1 - p_1;\\ &w\,(+1/2 \mid 1) = p_2; \qquad &w\,(-1/2 \mid 1) = 1 - p_2;\\ &w\,(+1/2 \mid 1) = p_3; \qquad &w\,(-1/2 \mid 1) = 1 - p_3. \end{aligned} \tag{7}$$

The constructed conditional probability distribution $w\,(X \mid j)$ determines both pure and mixed qubit states.

Thus, basic quantum systems – oscillators and qubits – are described by probability distribution functions. The evolution equation for qubit states can be written as the equation for the conditional probability distribution $w\,(X \mid j)$ expressed in terms of the density matrix elements for the Hamiltonian $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$. We have the von Neumann evolution equation; assuming that $\hbar = 1$, it reads

$$\begin{aligned} &\frac{\partial}{\partial t}\begin{pmatrix} p_3 & (p_1 - 1/2) - i\,(p_2 - 1/2) \\ (p_1 - 1/2) + i\,(p_2 - 1/2) & 1 - p_3 \end{pmatrix} \\ &+ i\left\{\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} p_3 & (p_1 - 1/2) - i\,(p_2 - 1/2) \\ (p_1 - 1/2) + i\,(p_2 - 1/2) & 1 - p_3 \end{pmatrix}\right. \\ &\left. - \begin{pmatrix} p_3 & (p_1 - 1/2) - i\,(p_2 - 1/2) \\ (p_1 - 1/2) + i\,(p_2 - 1/2) & 1 - p_3 \end{pmatrix}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\right\} = 0. \end{aligned} \tag{8}$$

This equation describes the evolution of the probability distribution $w\,(X \mid j, t)$ of qubit state.

The other important development of the probability representation of quantum states was connected with introducing new kinds of distributions, which were not discussed in classical probability theory. In quantum mechanics, there exists the superposition principle of wave functions ψ_1 and ψ_2. It describes the interference phenomena for the wave function $\psi = C_1\psi_1 + C_2\psi_2$ corresponding to quantum states, which are related to the other two wave functions ψ_1 and ψ_2. In the language of the probabilities corresponding to functions ψ_1, ψ_2, and ψ, for the probability distribution determined by the function ψ of the system with two subsystems, there mandatory exist probability distributions called entangled probability distributions. The entangled probability distributions were not known in classical probability theory; they were discussed only in quantum mechanics for the description of entangled quantum states [10–15].

The entangled probability distributions have specific entropic properties. Formally, in classical probability theory, the normalized sum of all probability distributions reads $W(\vec{X}) = \sum_k p_k W_k(\vec{X})$, where p_k are the probabilities and $W_k(\vec{X})$ are the probability distributions. In quantum mechanics, the sum of terms can exist, where the terms are not probabilities, but the result of summation provides the probability distribution. For example, the entangled probability distributions describe the Schrödinger cat states [32], which are superpositions of even and odd coherent states of two-mode oscillators [33].

Now we expect new revolutionary results due to recent understanding new quantum laws related to the probability distributions and the mathematical theory of symmetry groups like found in [34] dynamical $O(4,2)$ noncompact group description of the Hydrogen atom. An analogous dynamical symmetry was used to study the properties of squeezed states of light [35]. Also, dynamical $SU(1,1)$ noncompact Lie group was applied to study the properties of photons and quantum oscillators in [36]. We expect that the probability distribution functions for the Hydrogen atom states, which realize the irreducible representations of this symmetry group, are completely new and unknown in classical group theory; they provide the possibility to explain the behavior of the water solutions and to influence the process of medical product preparation like in [37].

The other possibility to obtain new results, in view of the quantum picture of the classical processes, is to employ the quantum Schrödinger equation, similar in mathematical aspects with classical equations, to obtain the solutions providing the classical potential energy function, which is difficult to get without solving the quantum equation [29]. Thus, the clarification of the influence of the acceleration of quantum systems in the probability representation of quantum mechanics discussed in [38] is essential; it is connected with Newton's law of the motion of classical objects. It provides extra understanding of intrinsic relations of the classical and quantum pictures of processes in the nature. Extra clarification can be also achieved while considering quantum entanglement of composite systems consisted of several subsystems having quite different properties [39].

The other perspective to obtain completely new results in the probability theory is to generalize the Kolmogorov classical approach [22] and introduce new probability distributions like entangled probability distributions describing entangled states of quantum systems [40–48]. Introducing of irreducible group representations for compact $SU(2)$ group describing the qubit state will be discussed in future publication in the new rubric "100 Years of Quantum Mechanics" using the approach [32]. Also, dynamics of quantum systems traditionally described by the evolution equation for wave function can be described by a new kinetic equation for the probability distribution [45], while the properties of its solution have entropic characteristics associated with entropies determined by these distributions [49].

The quantum-mechanical revolution in science was accompanied by creation of lasers, which were used as optical sources of coherent light for elaborating new quantum technologies.[†]

N. G. Basov was the founder of a new branch of science and technology, known as quantum electronics or laser physics. In 1952, Nikolai Gennadievich Basov, along with Aleksander Mikhailovich Prokhorov, were the first to demonstrate, on the basis of a theoretical analysis, the feasibility of constructing generators and amplifiers of electromagnetic waves, based on stimulated emission of quantum systems employing energy levels with population inversion. As early as 1955, they proposed a highly effective method of achieving population inversion by pumping a three-level system, which is currently widely used in various lasers and spectral ranges. That time saw the construction of fundamentally new devices such as low-noise microwave amplifiers and generators (masers). In 1959, N. G. Basov and A. M. Prokhorov won the Lenin Prize (the Government's highest award in science in the Soviet Union) for the discovery of a new principle of generation and amplification of electromagnetic radiation by quantum systems. In 1964, N. G. Basov and A. M. Prokhorov, together with the American scientist Charles H. Townes, won the Nobel Prize in physics for fundamental investigations in the field of quantum electronics, which led to the discovery of masers and lasers.[‡]

As a result, the problem of construction of the probability representation for quantum systems was solved, and new ways to develop quantum mechanics were opened, along with new opportunities to improve already existed technologies and elaborate new quantum technologies [50].

In addition to discoveries in physics, the probability representation of quantum mechanics opened new aspects of mathematical methods, which are crucial and should be studied in the next century of science. In fact, in classical science, the key role is played by Fourier transform describing the notion of frequency and wavelength in all physical phenomena of optics, electricity, acoustics, etc. It turned out

[†]Ad Memoriam of Nikolai Gennadievich Basov. *J. Russ. Laser Res.*, **43**, 725 (2022); DOI: 10.1007/s10946-022-10100-y

[‡]Obituary of Nikolai Gennadievich Basov is available on `link.springer.com/article/10.1023/A:1017984804522`

that the generalization of Fourier transform called Radon integral transform [51, 52] is fundamental in all constructions of the probability representation of quantum mechanics. Such constructions are important in obtaining new connections of the probability theory with irreducible representations of $su(2)$ Lie algebra [53] as well as irreducible representations of other groups [54]. The introduced connections of group irreducible representations with the probability theory will be studied and applied in the future publications.

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