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DYNAMICAL SYMMETRY AND GENERATION OF SQUEEZED STATES OF LIGHT

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Abstract

Using the Lie-algebraic approach, we develop the theory of generation of squeezed states of light in nonstationary parametric processes of the light interaction with a medium with the quadratic and quartic nonlinearities. The exact solution for the variance of the quadrature component of the field strength is obtained in the case of the quadratic parametric process with the $SU(1,1)$ dynamical symmetry. We show that decay of the field mode in this processes may have strong impact on squeezing. The solution for the standard deviation of the field strength in the case of the quartic parametric process with the approximated $\mathcal{L}_5$ dynamical symmetry is obtained in the first order of smallness with respect to the nonlinearity parameter.

Keywords: dynamical symmetry, Lie algebra, squeezed states of light.

1. Introduction

A squeezed state is a special class of coherent states of quantum systems, for which the variance of one of the canonically conjugate components is smaller than the other. In quantum optics, it is such a coherent field state, in which the variance of one of the quadrature field strength component is smaller than the other; see, e.g., [1–3]. In other words, it is smaller than the standard quantum limit, but the Heisenberg uncertainty principle is not violated. The squeezed states of light can be generated in a variety of nonlinear optical processes including parametric amplification, parametric up and down conversions, generation of the second harmonic in crystals, four-wave mixing in atomic vapors, resonance fluorescence of atoms and other processes; for review, see [4].

One of the main purposes of experiments in quantum physics is the controlled transfer of quantum system from a given initial state to the desired final state for a certain time interval. This is a typical control problem provided that the controlled quantity satisfies one of the equations of motion. The process of control can be considered on the Lie groups of all unitary transformations of a domain of admissible states. Dynamic symmetry concept, based on underlying dynamical Lie algebras, has been successfully applied to solve a number of problems that can be considered in the context of control of different quantum systems, from atoms and cavity field modes to neutrino oscillations; see, e.g., [5–10], and even for partial controlling chaotic quantum systems; see, e.g., [11–15].

Dynamic symmetry is a property of the evolution of dynamical systems that can be strictly formalized in the language of group theory; see, e.g., [16]. It is rooted deep in the nature of things, being more fundamental than differential equations used for description of the evolution. The famous "Eightfold Way"

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in theory of elementary particles has been discovered based solely on considerations of symmetry, while the corresponding differential equations are still unknown. Here, we mean that a quantum system, with a Hamiltonian $\mathcal{H}(t)$ having the dynamic symmetry, if $\mathcal{H}(t)$ generates a Lie algebra, a finite-dimensional one either infinite-dimensional one. This property allows us not only to classify quantum systems, using the structure of underlying Lie algebras, but also to solve some evolution problems.

2. Lie Algebra Approach for Solving the Evolution Problems

The evolution operator in quantum mechanics satisfies the equation

$$i\hbar\frac{d}{dt}U = \mathcal{H}(\mathbf{u},t)\; U,\; U(0) = I. \tag{1}$$

Let $\mathcal{H}$ generates a finite-dimensional Lie algebra

$$\mathcal{H}(t) = \sum_{i=1}^{n} l_i(t)\; L_i. \tag{2}$$

The set of operators $\{L_i\}$ forms the basis of n-dimensional Lie algebra $\mathcal{L}$, and $l_i(t)$ are scalar complex-valued functions of time. It follows from the Frobenius theorem [17], that the solution of (1), at least locally, can be represented in the form

$$U = \prod_{i=1}^{n} \exp\; (g_i(\mathbf{l}, t)\; L_i). \tag{3}$$

Substituting (3) into (1), one gets a system of nonlinear differential equations of the first order for the parameters g_i of the dynamical group G,

$$l_i(t) = N_{ij}\; \dot{g}_l\; (t), \qquad i, j = 1, \ldots, n, \tag{4}$$

where $N_{ij}(g)$ is a $n{\times}n$ matrix with the elements, which are analytic functions of g. The multiplicative parametrization (3) facilitates calculating of the probability amplitudes, observables, and various average values in quantum theory. Equation (4) is invariant with respect to the set of representations and implementations of the associated dynamic algebra $\mathcal{L}$.

The following conclusions can be drawn based only on structural features of dynamical algebras. If $\mathcal{L}$ belongs to a class of solvable algebras, then its basis can obviously be organized to represent the matrix $N(g)$ in a triangular form to reduce the solution (4) to n successive integrations. For solvable algebras, the solution in the form (4) is valid for all t, i.e., globally [18]. In view of the Levi–Maltsev theorem, an arbitrary Lie algebra can be decomposed as follows; see, for example, [19]:

$$\mathcal{L} = \Re \oplus R, \tag{5}$$

where $\Re$ is a semisimple Lie algebra, and R is its radical. Owing to (5), the following further decomposition is possible [18]:

$$U = U_{\Re}\; U_R, \tag{6}$$

where

$$i\hbar\frac{d}{dt}U_{\Re} = \mathcal{H}_{\Re}(t)\; U_{\Re}, \quad U_{\Re}(0) = I, \tag{7}$$

$$i\hbar\frac{d}{dt}U_R = U_{\Re}^{+}\; \mathcal{H}_R(t)\; U_{\Re}\; U_R, \quad U_R(0) = I, \tag{8}$$

$$\mathcal{H}(t) = \mathcal{H}_{\Re}(t) \oplus \mathcal{H}_R(t). \tag{9}$$

The operators $\mathcal{H}_{\Re}$ and $\mathcal{H}_R$ generate the algebras $\Re$ and R, respectively.

Since every semisimple algebra $\Re$ can be uniquely decomposed into a direct sum of simple subalgebras [19]

$$\Re = \Re_1 \oplus \ldots \oplus \Re_k, \tag{10}$$

the further decomposition is possible,

$$S_{\Re} = \prod_{i=1}^{n} S_i, \tag{11}$$

where each of the factors satisfies the equation like (7), with the Hamiltonian $\mathcal{H}_i$ generating the corresponding ideal $\Re_i$ with

$$\mathcal{H}_{\Re}(t) = \mathcal{H}_1 + \cdots + \mathcal{H}_k(t). \tag{12}$$

3. Model Nonlinear Hamiltonian

Let us consider a Hamiltonian of the fourth degree by the boson operators describing a field mode

$$\mathcal{H} = \frac{1}{2}\hbar\left[\omega_0(a^\dagger a + 1/2) + (\alpha a^{\dagger 2} + \alpha^* a^2) + \beta(a^\dagger + a)^4\right], \tag{13}$$

where the first term describes a free field, the second and third terms describe quadratic and quartic quantum processes; the parameters α and β can be time dependent. In the approximation with slowly varying amplitudes, the Hamiltonian can be rewritten in the terms of generators $\mathcal{J}_0 \equiv \frac{1}{2}(a^\dagger a + 1/2)$, $\mathcal{J}_- \equiv \frac{1}{2}a^2$, and $\mathcal{J}_+ \equiv \frac{1}{2}a^{\dagger 2}$ of a $SU(1,1)$ group and their bilinear combinations

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1, \quad \mathcal{H}_0 = \hbar\omega_0\mathcal{J}_0 + \hbar(\alpha\mathcal{J}_+ + \alpha^*\mathcal{J}_-), \quad \mathcal{H}_1 = 2\hbar\beta(\mathcal{C} + 3\mathcal{J}_0^2), \tag{14}$$

where

$$\mathcal{C} = \mathcal{J}_0^2 - \frac{1}{2}(\mathcal{J}_+\mathcal{J}_- + \mathcal{J}_-\mathcal{J}_+) \tag{15}$$

is the Casimir operator of $SU(1,1)$ algebra.

In the first order in the nonlinearity parameter β, the infinite dimensional Lie algebra of the Hamiltonian (14) can be approximated by a five-dimensional Lie algebra $\mathcal{L}_5$, which, in turn, can be decomposed in the direct sum of $SU(1,1)$ and a two-dimensional commutative subalgebra. That allows us to factorize the evolution operator as follows:

$$U(t) = \exp(-2i\beta t\mathcal{C})\exp(-6i\beta t\mathcal{J}_0^2)U_{SU(1,1)}(t), \tag{16}$$

where the operator of the representation of the $SU(1,1)$ group can be factorized into a product of exponentials of the group generators following to (3), with three parameters g_i satisfying the set of nonlinear differential equations (4). The multiplicative parametrization (16) is convenient for calculating the evolution of nonstationary quantum systems of the fourth order.

4. Generation of Squeezed States in the Processes with $SU(1,1)$ Dynamical Symmetry

The squeezed state of light is a kind of generalized coherent states for which the uncertainty relation becomes the equality

$$\langle(\Delta\mathcal{G}_i)^2\rangle\langle(\Delta\mathcal{G}_j)^2\rangle = \frac{1}{4}|c_{ij}^k\langle\mathcal{G}_k\rangle|^2, \quad \langle(\Delta\mathcal{G}_i)^2\rangle = \langle\mathcal{G}_i^2\rangle - \langle\mathcal{G}_i\rangle^2, \tag{17}$$

where $[\mathcal{G}_i,\mathcal{G}_j] = c_{ij}^k\mathcal{G}_k$ and $\mathcal{G}$ are generators of a Lie group, and c_{ij}^k are structural constants.

Among the class of states (17) with equal values of the variances of $\mathcal{G}_i$ and $\mathcal{G}_j$, there exist the squeezed states for which one of the variance is smaller than the other. If $\mathcal{G}_i$ and $\mathcal{G}_j$ are linear combinations of creation and annihilation operators, then one deals with squeezed states of light.

It is convenient to use the quadrature operator components in order to describe squeezed states,

$$\begin{aligned}\mathcal{G}_1 &\equiv \mathcal{G}^+\exp[i(\omega_0 t-\phi)] + \mathcal{G}^-\exp[-i(\omega_0 t-\phi)],\\ \mathcal{G}_2 &\equiv \mathcal{G}^+\exp[i(\omega_0 t-\phi-\pi/2)] + \mathcal{G}^-\exp[-i(\omega_0 t-\phi-\pi/2)],\end{aligned} \tag{18}$$

where $\mathcal{G}^+$ and $\mathcal{G}^-$ are positive and negative frequency parts of the field strength with the commutators $[\mathcal{G}^+,\mathcal{G}^-] = c$, $[\mathcal{G}_1,\mathcal{G}_2] = 2ic$; also c is a positive number, and ϕ is a phase angle, that can be varied in experiments. Now we arrive at the following uncertainty relation:

$$\langle(\Delta\mathcal{G}_1)^2\rangle\langle(\Delta\mathcal{G}_2)^2\rangle \geq c^2, \tag{19}$$

which is minimized in a squeezed state, where the variance of one of the quadrature components is smaller than c, whereas the other one is larger than c.

A squeezed state can be generated from the vacuum state, under the action of the relevant evolution operator,

$$|\chi|^2\Big[1+2n_0(1-e^{-\gamma t})\Big] < 1. \tag{20}$$

The variance is calculated as follows:

$$\langle(\Delta\mathcal{G}_1)^2\rangle = \langle 0|[\mathcal{G}_1(t)-\langle\mathcal{G}_1(t)\rangle]^2|0\rangle < b, \quad \mathcal{G}_1 = U^+(t)\mathcal{G}_1 U(t), \quad \langle\mathcal{G}_1(t)\rangle = \langle 0|\mathcal{G}_1|0\rangle. \tag{21}$$

The Hamiltonian $\mathcal{H}_0$ with the $SU(1,1)$ dynamical symmetry describes the simplest optical process, which is able to generate squeezed states of light. The Hamiltonian $\mathcal{H}_1$ can also generate squeezed state of light under appropriate conditions. The combined Hamiltonian $\mathcal{H}_0+\mathcal{H}_1$ with the approximated dynamical symmetry $\mathcal{L}_5$ allows us to vary the squeezing conditions to increase the degree of squeezing.

Let us consider squeezing of light in the process of degenerate parametric amplification with the Hamiltonian $\mathcal{H}_0$ generating $SU(1,1)$ algebra. The classical pumping field is assumed to be harmonic one,

$$\alpha(t) = \alpha_0\exp(-2i\omega_0 t), \quad c=1. \tag{22}$$

Computing the quadrature component $\mathcal{G}_1$ with the $SU(1,1)$ evolution operator, in accordance with (21), we obtain

$$\mathcal{G}_1 \equiv a(t)\exp[i(\omega_0 t-\phi)] + a^\dagger\exp[-i(\omega_0 t-\phi)] = \chi(t)a + \chi^*(t)a^\dagger, \tag{23}$$

where

$$\chi(t) = e^{-i\phi}\cosh(\alpha_0 t) + ie^{i\phi}\sinh(\alpha_0 t). \tag{24}$$

The variance is calculated, in view of formula (21) the result reads

$$\langle(\Delta\mathcal{G}_1)^2\rangle = |\chi|^2, \tag{25}$$

where

$$|\chi|^2 = \cosh(2\alpha_0 t) - \sinh(2\alpha_0 t)\sin(2\phi). \tag{26}$$

The squeezing condition is fulfilled, if $\phi = \pi/4$. The final formula is

$$\langle(\Delta\mathcal{G}_1)^2\rangle_{\rm sq} = e^{-2\alpha_0 t}. \tag{27}$$

5. Account for Decay of the Field Mode

To take into account decay of the field mode, it requires to add the term

$$\mathcal{H}_3 = \sum_{j=1}^{\infty} \hbar\omega_j b_j^\dagger b_j + \hbar(\theta_j a^\dagger b_j + \theta_j^* a b_j^\dagger) \tag{28}$$

to the Hamiltonian $\mathcal{H}_0$ with the $SU(1,1)$ dynamical symmetry. This term describes a thermal "bath" $(b_j^\dagger, b_j)$ and the interaction of the field mode $(a^\dagger, a)$ with it. The solutions of the Heisenberg equations are well known; see, e.g., [20],

$$a(t) = u(t)a + \sum_j v_j(t)b_j, \quad b_j(t) = \sum_i x_{ij}(t)b_i + y_j(t)a, \quad u(0)=1, \quad v_j(0)=0, \quad x_{ij}(0)=\delta_{ij}, \quad y_j(0)=0. \tag{29}$$

In the Weisskopf–Wigner approximation, the functions $u(t)$ and $v_j(t)$ have the following forms:

$$\begin{gathered} u(t) = \exp\{-[(\gamma/2) + i\omega t]\}, \\ v_j(t) = \theta_j \frac{1 - \exp[i(\omega_j - \omega)t - \gamma t/2]}{\omega_j - \omega + i\gamma/2}\exp(-i\omega_j t). \end{gathered} \tag{30}$$

where $\gamma \equiv 2\pi\theta^2(\omega)\rho(\omega)$ is a damping constant, $\omega = \omega_0 + \delta\omega$, $\rho(\omega)$ is a form of line for the oscillators from the reservoir, and the frequency shift $\delta\omega$ is defined by the main value of the integral,

$$\delta\omega = -\int_{-\infty}^{\infty} |\theta_j|^2 \rho(\omega_j)(\omega_j - \omega_0)^{-1} d\omega_j. \tag{31}$$

The conservation law follows from the unitarity of the evolution operator

$$|u(t)|^2 + \sum_j |v_j(t)|^2 = 1. \tag{32}$$

In view of the solutions (30)–(32), now we can calculate the variance of the quadrature component (23) averaged over the vacuum state of the field mode and the Fock states $|n_j\rangle$ of the modes of the reservoir,

$$\langle(\Delta\mathcal{G}_1)^2\rangle = |\chi|^2\Big[1 + 2n_0(1 - e^{-\gamma t})\Big], \tag{33}$$

where the average number of reservoir's quanta n_0 with the frequency ω is given by the Plank distribution.

Finally, the condition for squeezing of one of the quadrature components of the field mode is given by

$$|\chi|^2\Big[1+2n_0(1-e^{-\gamma t})\Big] < 1. \tag{34}$$

This formula shows that decay of the field mode significantly impacts on squeezing. The corresponding variance can be lowered decreasing the number of photons n_0 and/or the decay coefficient γ.

Assuming that the contribution of the nonlinearity parameter β is small, the condition for squeezing in a quartic nonlinear optical process with the full Hamiltonian (13) can be found, using the same algorithm as in the $SU(1,1)$ case. In this process, the standard deviation of the field strength is given by

$$\begin{aligned} A_1(t) \qquad &= \chi(t)\left\{\Big[1-2i\beta t(a^\dagger a+1)\Big]\,a - i\frac{\beta t}{2}(a^\dagger a^2 + a^2 a^\dagger)\right\} \\ &+\chi^*(t)\left\{a^\dagger\,\Big[1+2i\beta t(a^\dagger a+1)\Big] + i\frac{\beta t}{2}\left({a^\dagger}^2 a + a{a^\dagger}^2\right)\right\} + O(\chi^2). \end{aligned} \tag{35}$$

The variance of this quantity in a squeezed state is calculated following (21).

6. Conclusions

In summary, we used the Lie-algebra approach to calculate variances of the quadrature components and standard deviations of the field strength in the squeezed states of light in the processes of interaction of light with a medium with quadratic and quartic nonlinearities, with the exact $SU(1,1)$ dynamical symmetry and approximated $\mathcal{L}_5$ dynamical symmetry, respectively. We found the explicit solution for the variance of the quadrature component of the field strength in the parametric process with the $SU(1,1)$ dynamical Lie algebra. We showed that decay of the field mode in these processes has strong impact on squeezing. The respective formula for the variance of the quadrature components of the field strength in the squeezed states allows for control of the squeezing. The solution for the standard deviation of the field strength in the quartic parametric process with the approximated $\mathcal{L}_5$ dynamical symmetry was obtained, using the contribution of small nonlinearity parameter.

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