DOI 10.1007/s10946-024-10197-3
Journal of Russian Laser Research, Volume 45, Number 2, March, 2024

CONDITIONAL TRANSFORMATION OPERATOR FOR NEAR STATE GENERATION

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Abstract

We show that by minor modification of the previously proposed procedure for the generation of near-coherent states, one can produce "near states" even from a mixed input state. In this study, we consider this procedure as a conditional quantum-state transformation and find the explicit form of the non-unitary transformation operator. We demonstrate that even in mixed states, this quantum process can produce nonclassical states with a smooth non-positive P-function.

Keywords: nonclassical states, near coherent states, conditional state transformation.

1. Introduction

Nonclassical states of light have wide applications in quantum optics [1,2]. State nonclassicality refers to the presence of statistical correlations that cannot be explained by any classical statistical model. Typically, a state is considered as nonclassical, if its representation in terms of the Glauber–Sudarshan P-function [3,4] is either non-positive or includes derivatives of the Dirac delta-function. In this sense, coherent states, which are the closest quantum states to a classical field, and any statistical mixture of them are considered as classical states, while photon-added and photon-subtracted thermal states, with regular non-positive P-functions, and various forms of cat states, squeezed states, and the similar ones, are nonclassical states. Single-mode state nonclassicality and quantum entanglement, which enable quantum information processing [5,6], quantum teleportation [7], and quantum communication [8], are closely related [9], making nonclassicality an important issue.

Some quantum states possess semi-classical features, similar to coherent states, and at the same time, they exhibit nonclassical behavior [10]. Some of them are photon-added (photon-subtracted) coherent states, which are nonclassical, even if they have a large number of photons [11,12], and the Schrödinger cat states [13,14]. One of the motivations for the generation of such states is the fact that they provide insight into quantum aspects of the wave–particle duality. The other interesting point is that, due to their semiclassical part, one can macroscopically control them without losing the nonclassical features of the states.

Recent examples of the aforementioned states include near-coherent [15] and Mth coherent states [16]. These states are essentially some kind of cat states formed by the superposition of two almost identical coherent states. A generation method has been introduced for these states, utilizing an optical setup that takes a coherent state as an input and conditionally generates a near-coherent state. However, all

Manuscript submitted by the authors in English on February 21, 2024.

previous investigations were focused on coherent states as input states, with no attention given to other pure or mixed states.

In this article, we use the generation method for the near-coherent states of the previous research [15], with minor modifications, to propose a potential new definition of near states for arbitrary (generally mixed) input states. Both experimentally (due to unavoidable quantum noises) and theoretically (because of the expanding domain of input states), this problem may be interesting.

This paper is organized as follows.

In Sec. 2, we review the definition of near-coherent states and some of their basic properties, while in Sec. 3 we introduce a useful modification to the generation method of near-coherent states. An optical setup is considered for generating a near-coherent state as a conditional quantum state transformation, and an explicit form of the corresponding transformation operator is derived in Sec. 4. Then, this transformation is examined as a generator of a more general form of near states, including near mixed states. Section 5 is devoted to the conclusions.

2. Near Coherent States and Derivative States

The near-coherent states are defined as a superposition of two almost identical coherent states [15]. If superposed states are denoted by complex parameters z and $z+\Delta z$, we obtain

$$|z,\Delta z\rangle = N\{|z+\Delta z\rangle - |z\rangle\}, \qquad \text{at} \qquad \Delta z \longrightarrow 0, \tag{1}$$

where N is a normalization factor that tends to infinity in the prescribed limit. One should note that such a state, as a limit of a function of complex parameter Δz, does not exist. To demonstrate the truth of this claim, within the framework of the photon number representation of coherent states, we have

$$\langle n|z\rangle = \exp\left(-\frac{1}{2}zz^*\right)\frac{z^n}{\sqrt{n!}}; \qquad n = 0,1,2,\ldots$$

which is not a differentiable function of complex parameter z. This means that the limit (1) depends on the phase of $\Delta z = |\Delta z|e^{i\theta_s}$, which is called the source term. The normalization factor obtained in the previous research [15] reads

$$N = \frac{1}{\sqrt{1+|z|^2\sin^2\delta\theta}}\frac{1}{|\Delta z|},$$

where $\delta\theta = \theta_s - \theta$ is defined as the relative phase of the source, and θ denotes the phase of complex parameter z. By expanding $|z+\Delta z\rangle$ up to the first order of the source term, we obtain

$$|z,\Delta z\rangle = \frac{1}{\sqrt{1+|z|^2\sin^2\delta\theta}}\left(e^{i\delta\theta}\frac{\partial|z\rangle}{\partial|z|} + i|z|\sin(\delta\theta)\,|z\rangle\right).$$

Hence, the normalized near-coherent state depends only on $\delta\theta$, the relative phase of the source term in the complex plane. The non-normalized state $\dfrac{\partial|z\rangle}{\partial|z|}$ is called a derivative coherent state, so near-coherent states are a superposition of a derivative coherent state and the coherent state itself. Moreover, when $\delta\theta = \pi/2$, we have

$$|z,\Delta z\rangle = \frac{i}{\sqrt{1+|z|^2}}\left(\frac{\partial|z\rangle}{\partial|z|} + |z||z\rangle\right),$$

and using the relation $\hat{n}|z\rangle = |z|\dfrac{\partial|z\rangle}{\partial|z|} + |z|^2|z\rangle$, for coherent states, we arrive at

$$|z, \Delta z\rangle = \frac{i}{|z|\sqrt{1+|z|^2}}\hat{n}|z\rangle.$$

Therefore, the generation of near-coherent states, when $\delta\theta = \pi/2$, leads to the generation of $\hat{n}|z\rangle$ – the first state of a class of nonclassical states, called Mth coherent states [16].

Mathematical, statistical, and nonclassical properties of near-coherent states were considered in [15]. These states exhibit nonclassical properties, such as sub-Poissonian statistics, squeezing, and a partially negative Wigner function. At the same time, they possess distribution functions for field quadratures and photon numbers that are highly similar to coherent states, when $|z| \gg 1$. Consequently, these states simultaneously demonstrate both classical and nonclassical behaviors.

3. Generation of Near Coherent States

In this section, we review a method for creating a near-coherent state described in [15], with some modifications depicted in Fig. 1. The Mach–Zehnder interferometer has two modes (c) and (d) in the photon number state $|1\rangle_c|0\rangle_d$. The counterclockwise arm of the interferometer is coupled with the other mode (b), which is in a coherent state $|\beta\rangle_b$, through the cross-Kerr interaction, which has a unitary operator $\hat{U}_K = \exp(-i\phi\hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c})$. The Mach–Zehnder interferometer has two identical 50/50 beam splitters ($BS1$ and $BS2$). An additional similar 50/50 beam splitter ($BS3$) is used to couple modes (a) in the coherent state $|\alpha\rangle_a$ with mode (b). Overall, this setup has four input ports and four output ports.

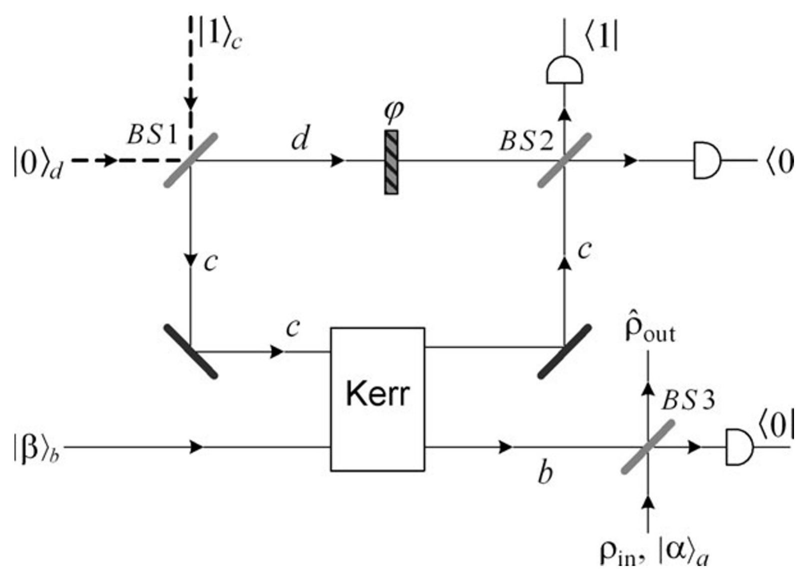

Fig. 1. An optical setup for generating the near coherent states.

It is assumed that all beam splitters are the same, with a transformation matrix T, and the related unitary operator $\hat{B}_T$ acts on the two-mode coherent states as follows:

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \longleftrightarrow \quad \hat{B}_T|\alpha, \beta\rangle = \left|\frac{\alpha+\beta}{\sqrt{2}}, \frac{-\alpha+\beta}{\sqrt{2}}\right\rangle. \tag{2}$$

The phase shifter φ, with the corresponding unitary operator $\exp(-i\varphi\hat{d}^\dagger\hat{d})$ in the clockwise arm of the interferometer, is essentially utilized to compensate for any undesired phase shift in the arms. Finally, we measure photon numbers in output ports (b), (c), and (d) to post-select state ${}_b\langle 0|_c\langle 1|_d\langle 0|$, while we do not touch the output port (a). Here, we show that, if the cross-Kerr interaction is weak ($|\phi| \ll 1$), the output state in port (a) is a near-coherent state.

Every beam splitter acts on the two-mode number states, in view of the rules,

$$\hat{B}|1,0\rangle = \frac{1}{\sqrt{2}}\{|1,0\rangle - |0,1\rangle\}, \qquad \hat{B}|0,1\rangle = \frac{1}{\sqrt{2}}\{|1,0\rangle + |0,1\rangle\}; \tag{3}$$

hence, $BS1$ transforms input states as follows:

$$\hat{B}_1|\alpha\rangle_a|\beta\rangle_b|1\rangle_c|0\rangle_d = \frac{1}{\sqrt{2}}|\alpha\rangle_a|\beta\rangle_b\{|1\rangle_c|0\rangle_d - |0\rangle_c|1\rangle_d\}.$$

Now, depending on the state (c), the operator $\hat{U}_K$ acts on the right-hand side, and the resulting state becomes

$$\frac{1}{\sqrt{2}}\{|\alpha\rangle_a|e^{-i\phi}\beta\rangle_b|1\rangle_c|0\rangle_d - |\alpha\rangle_a|\beta\rangle_b|0\rangle_c|1\rangle_d\}.$$

Next, $BS2$ acts, and the state evolves to

$$\frac{1}{2}\{|\alpha\rangle_a|e^{-i\phi}\beta\rangle_b - |\alpha\rangle_a|\beta\rangle_b\}|1\rangle_c|0\rangle_d - \frac{1}{2}\{|\alpha\rangle_a|e^{-i\phi}\beta\rangle_b + |\alpha\rangle_a|\beta\rangle_b\}|0\rangle_c|1\rangle_d.$$

Up to this point, all evolutions are unitary. Since $BS3$ only mixes modes (a) and (b), we can perform post-selection ${}_c\langle 1|_d\langle 0|$ to find

$$|\alpha\rangle_a|e^{-i\phi}\beta\rangle_b - |\alpha\rangle_a|\beta\rangle_b.$$

This two-mode state enters $BS3$ and is transformed to

$$\left|\frac{\alpha + e^{-i\phi}\beta}{\sqrt{2}}\right\rangle_a \left|\frac{-\alpha + e^{-i\phi}\beta}{\sqrt{2}}\right\rangle_b - \left|\frac{\alpha + \beta}{\sqrt{2}}\right\rangle_a \left|\frac{-\alpha + \beta}{\sqrt{2}}\right\rangle_b.$$

Finally, we have non-unitary post-selection of ${}_b\langle 0|$, and single-mode output state for mode (a) is found as follows:

$$|\psi_{\text{out}}\rangle_a = e^{-1/4|e^{-i\phi}\beta-\alpha|^2}\left|\frac{\alpha + e^{-i\phi}\beta}{\sqrt{2}}\right\rangle_a - e^{-1/4|\beta-\alpha|^2}\left|\frac{\alpha + \beta}{\sqrt{2}}\right\rangle_a. \tag{4}$$

By assumption, the cross-Kerr interaction is weak, $|\phi| \ll 1$, so we can write

$$|\psi_{\text{out}}\rangle_a \approx \left|\frac{\alpha + e^{-i\phi}\beta}{\sqrt{2}}\right\rangle_a - \left|\frac{\alpha + \beta}{\sqrt{2}}\right\rangle_a \equiv |z + \Delta z\rangle_a - |z\rangle_a. \tag{5}$$

Here, the complex parameters of two nearly identical coherent states z and $z + \Delta z$ are defined by

$$z = \frac{\alpha + \beta}{\sqrt{2}}, \qquad \Delta z \approx \frac{-i\phi\beta}{\sqrt{2}}. \tag{6}$$

In the previous research [15], the researchers left the output port (b) of $BS3$ unused and considered no post-selections for it. Such a quantum process will not result in a pure output state at all.

In the next section, we demonstrate that, if the input (generally mixed) state in port (a) is $\hat{\rho}_{\text{in}}$, then conditional quantum state transformation can be described by a rule in the form $\hat{\rho}_{\text{out}} \propto \hat{Y}\hat{\rho}_{\text{in}}\hat{Y}^{\dagger}$; while, without post-selection for ${}_b\langle 0|$, the resulting mixed state has only an infinite-sum Kraus-operator representation.

4. Calculation of Conditional Transformation Operator

Superposition of two neighboring states can be defined in various ways, depending on the specific criteria for neighborhood. For example, we can choose a Lie group of transformations, $\hat{g}(\alpha_1, \alpha_2 \ldots)$, and consider states $\hat{g}|\psi_{\rm in}\rangle$. Then, "near states" can be interpreted as the superposition of two states with almost identical parameters $(\alpha_1, \alpha_2, \ldots)$. Therefore, there are many different possible definitions for near states. In Sec. 3, we explain how the optical setup in Fig. 1, acting as a conditional quantum state transformer, can produce a superposition of two almost identical coherent states. Here, to find a possible definition for "near mixed states," we investigate the output state of the same setup, when the input state $\hat{\rho}_{\rm in}$ is arbitrary.

For any conditional quantum state transformation, up to a scalar factor, there exists a non-unitary operator $\hat{Y}$, such that

$$\hat{\rho}_{\rm out} = \frac{\hat{Y}\hat{\rho}_{\rm in}\hat{Y}^\dagger}{{\rm Tr}[\hat{Y}\hat{\rho}_{\rm in}\hat{Y}^\dagger]}. \tag{7}$$

The operator $\hat{Y}$ is the partial matrix element of unitary operator $\hat{U}$, which governs the time evolution of the system, between pre- and post-selected states [17]. As shown in Fig. 1, $|\beta\rangle_b|1\rangle_c|0\rangle_d$ and $|0\rangle_b|1\rangle_c|0\rangle_d$ are the pre- and post-selected states, respectively; thus,

$$\hat{Y} \propto {}_b\langle 0|_c\langle 1|_d\langle 0|\hat{U}|\beta\rangle_b|1\rangle_c|0\rangle_d.$$

It should be noted that this partial matrix element is not a scalar; rather, it is an operator $\hat{Y}(\hat{a}^\dagger, \hat{a})$ depending on the creation and annihilation operators of the unmeasured port.

Using unitary evolution operator $\hat{U} = \hat{B}_3\hat{B}_2 e^{-i\phi\hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c}}\hat{B}_1$ for the setup shown in Fig. 1, up to an unimportant scalar factor (which we always ignore), we arrive at

$$\hat{Y} = {}_b\langle 0|_c\langle 1|_d\langle 0|\hat{B}_3(\hat{a},\hat{b})\hat{B}_2(\hat{c},\hat{d})\, e^{-i\phi\hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c}}\hat{B}_1(\hat{c},\hat{d})|\beta\rangle_b|1\rangle_c|0\rangle_d.$$

By applying transformation rules, (3), we obtain

$$\hat{Y} = {}_b\langle 0|\hat{B}_3(\hat{a},\hat{b})_c\langle 1|_d\langle 0|\hat{B}_2(\hat{c},\hat{d})\, e^{-i\phi\hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c}}|\beta\rangle_b\{|1\rangle_c|0\rangle_d - |0\rangle_c|1\rangle_d\}.$$

Then, depending on the state of (c) on the right-hand side of $\hat{U}_K$, this operator acts on the state, and the resulting partial matrix element becomes

$$\hat{Y} = {}_b\langle 0|\hat{B}_3(\hat{a},\hat{b})_c\langle 1|_d\langle 0|\hat{B}_2(\hat{c},\hat{d})\,\{|e^{-i\phi}\beta\rangle_b|1\rangle_c|0\rangle_d - |\beta\rangle_b|0\rangle_c|1\rangle_d\}.$$

By the repeated application of Eq. (3) for $\hat{B}_2$, the transformation operator adopts the following form:

$$\hat{Y} = {}_b\langle 0|\hat{B}_3(\hat{a},\hat{b})|e^{-i\phi}\beta\rangle_b - {}_b\langle 0|\hat{B}_3(\hat{a},\hat{b})|\beta\rangle_b.$$

By applying the transformation rule given in Eq. (2), it is easy to show that $\hat{Y}|\alpha\rangle_a$ reproduces Eq. (4), which justifies our calculations.

Now, we find an explicit form of $\hat{Y}(\hat{a}^\dagger, \hat{a})$ as a function of the creation and annihilation operators of port (a). By invoking the completeness relation for two-mode coherent states and Eq. (2), we obtain

$${}_b\langle 0|\hat{B}_3(\hat{a},\hat{b})|\beta\rangle_b = \int \frac{d^2\lambda}{\pi}\frac{d^2\mu}{\pi}\left|\frac{\lambda+\mu}{\sqrt{2}}\right\rangle\langle\lambda|\left\langle 0\left|\frac{-\lambda+\mu}{\sqrt{2}}\right.\right\rangle\langle\mu|\beta\rangle\ .$$

Using identities $|\xi\rangle = e^{-|\xi|^2/2} e^{\xi \hat{a}^\dagger}|0\rangle$ and $|0\rangle\langle 0| =: e^{-\hat{a}^\dagger \hat{a}}$: on the right-hand side and integrating in the normally ordered product, we arrive at

$$ {}_b\langle 0|\hat{B}_3(\hat{a},\hat{b})|\beta\rangle_b = e^{-|\beta|^2/2} : e^{-\hat{a}^\dagger \hat{a}} e^{\hat{a}^\dagger \hat{a}/\sqrt{2}} e^{\beta \hat{a}^\dagger/\sqrt{2}}. $$

This can be simplified, in view of formula $e^{\eta \hat{a}^\dagger \hat{a}} =: e^{(e^\eta - 1)\hat{a}^\dagger \hat{a}}$:, and finally, we obtain

$$ \hat{Y} = e^{-|\beta|^2/2} \left(e^{e^{-i\phi}\beta \hat{a}^\dagger/\sqrt{2}} - e^{\beta \hat{a}^\dagger/\sqrt{2}} \right) \left(\frac{1}{\sqrt{2}} \right)^{\hat{a}^\dagger \hat{a}}. \tag{8} $$

In view of the assumption $|\phi| \ll 1$, employing the Taylor expansion, and neglecting any unimportant scalar factors, the transformation operator adopts the following form:

$$ \hat{Y} \equiv \hat{a}^\dagger e^{\beta \hat{a}^\dagger/\sqrt{2}} \left(\frac{1}{\sqrt{2}} \right)^{\hat{a}^\dagger \hat{a}}, \qquad \beta = i\sqrt{2}\Delta z/\phi. \tag{9} $$

Note that, for fixed parameters of the cross-Kerr medium, the conditional transformation operator depends on the source term Δz. In view of (7), this operator completely characterizes the quantum process realized by the setup in Fig. 1.

Equation 8 elucidates the physical interpretation of our new definition for near states. On the right-hand side of this equation, two nearly identical operators are subtracted. Hence, $\hat{Y}|\psi_{\rm in}\rangle$ can be interpreted as a superposition of two almost identical states. Moreover, although the output state is a superposition state for any input quantum state $|\psi_{\rm in}\rangle$, none of the superposed states are near to $|\psi_{\rm in}\rangle$. In fact, instead of $|\psi_{\rm in}\rangle$, they are near to the quantum state $e^{\beta \hat{a}^\dagger/\sqrt{2}} 2^{-\hat{a}^\dagger \hat{a}/2}|\psi_{\rm in}\rangle$. For example, $\hat{Y}|\alpha\rangle_a$ equals the superposition of coherent states near to $|(\alpha+\beta)/\sqrt{2}\rangle_a$.

To demonstrate some applications of formula (9), assume that the input state is a photon number state $|n\rangle$, then

$$ |\psi_{\rm out}\rangle \propto \hat{Y}|n\rangle = \hat{a}^\dagger e^{\beta \hat{a}^\dagger/\sqrt{2}} \left(\frac{1}{\sqrt{2}} \right)^{\hat{a}^\dagger \hat{a}} |n\rangle \propto \hat{a}^{\dagger\, n+1} e^{\beta \hat{a}^\dagger/\sqrt{2}}|0\rangle, $$
$$ |\psi_{\rm out}\rangle \propto \hat{a}^{\dagger\, n+1}|\beta/\sqrt{2}\rangle. $$

Thus, the output state is pure and can be considered as a photon-added coherent state.

The other example is the thermal state with the following P-function:

$$ \hat{\rho}_{\rm in} = \frac{1}{1+\bar{n}} \left(\frac{\bar{n}}{1+\bar{n}} \right)^{\hat{a}^\dagger \hat{a}}, \qquad P_{\rm th}(\alpha) = \frac{1}{\pi \bar{n}} \exp\left(-\frac{|\alpha|^2}{\bar{n}} \right), $$

with $\bar{n}$ denoting mean photon number of the state. Using Eqs. (7) and (9), we have

$$ \hat{\rho}_{\rm out} \propto \hat{Y}\hat{\rho}_{\rm in}\hat{Y}^\dagger = \frac{1}{1+\bar{n}} \hat{a}^\dagger e^{\beta \hat{a}^\dagger/\sqrt{2}} \left(\frac{1}{2} \frac{\bar{n}}{1+\bar{n}} \right)^{\hat{a}^\dagger \hat{a}} e^{\beta^* \hat{a}/\sqrt{2}} \hat{a}, $$

and then, applying operator identity $M^{\hat{a}^\dagger \hat{a}} =: e^{(M-1)\hat{a}^\dagger \hat{a}}$: we obtain

$$ \hat{\rho}_{\rm out} \propto \hat{a}^\dagger e^{\beta \hat{a}^\dagger/\sqrt{2}} : \exp\left[-\frac{1}{2} \left(\frac{\bar{n}+2}{\bar{n}+1} \right) \hat{a}^\dagger \hat{a} \right] : e^{\beta^* \hat{a}/\sqrt{2}} \hat{a}. $$

Now, we can calculate the Glauber–Sudarshan function of $\hat{\rho}_{\text{out}}$, using Fourier integral [18]; it reads

$$P(\alpha) = \frac{e^{|\alpha|^2}}{\pi} \int \frac{d^2u}{\pi} e^{\alpha u^* - \alpha^* u} e^{|u|^2} \langle -u|\hat{\rho}|u\rangle.$$

After some algebra, we obtain

$$P_{\text{out}}(\alpha) \propto \frac{e^{|\alpha|^2}}{\pi} \partial_\alpha \partial_{\alpha^*} \int \frac{d^2u}{\pi} \exp\left\{ -\frac{1}{2}\left(\frac{\bar{n}}{\bar{n}+1}\right)|u|^2 + \left(\frac{\beta}{\sqrt{2}} - \alpha\right)^* u - \left(\frac{\beta}{\sqrt{2}} - \alpha\right) u^* \right\}.$$

The above Gaussian integral can be calculated and, up to a normalization factor N, the Glauber–Sudarshan phase space representation of the output state is

$$P_{\text{out}}(\alpha) = N e^{|\alpha|^2} \left[2\frac{(\bar{n}+1)}{\bar{n}} \left|\alpha - \frac{\beta}{\sqrt{2}}\right|^2 - 1 \right] \exp\left[-2\frac{(\bar{n}+1)}{\bar{n}} \left|\alpha - \frac{\beta}{\sqrt{2}}\right|^2 \right].$$

This is a smooth but non-positive definite P-function that demonstrates the nonclassicality of the output state.

These examples show that the transformation operator given by Eqs. (8) and (9) can generate near states, according to our new definition.

5. Conclusions

In this article, we introduced a generalization of near-coherent states that can be generated from a generally mixed input state. We completed the previously proposed setup [15] for the generation of near-coherent states, by including an additional zero-photon detection post-selection step; thus, we achieved a closed form for the transformation operator. The explicit form of this non-unitary transformation operator was determined and subsequently applied to both pure and mixed input states. We demonstrated that, even for a mixed state, this quantum process can produce nonclassical states characterized by a smooth, non-positive P-function.

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