DOI 10.1007/s10946-022-10049-y
Journal of Russian Laser Research, Volume 43, Number 3, May, 2022

STATISTICAL PROPERTIES OF NONDEGENERATE THREE-LEVEL LASER

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Abstract

We study the squeezing and statistical properties of the light produced by nondegenerate three-level lasers coupled to a vacuum reservoir, in which two different nondegenerate three-level atoms are injected at constant rate into the cavity. Applying the pertinent master equation, we obtain the stochastic differential equations associated with the normal ordering. Making the use of the solutions of the resulting differential equations, the quadrature variances and the mean and variance of the photon number sum and difference are described. We see that the mean and variance of the photon number difference is positive; this fact indicates that the mean photon number of mode a, emitted from the top level, is greater than that of mode b, emitted from the intermediate level of the three-level atom. Moreover, we find that the mean and variance of the photon number difference decreases as η increases. We observe that the squeezing is higher for large values of linear gain coefficient, and the maximum squeezing occurs when the population of the atoms in the bottom level is slightly greater than that of the top level.

Keywords: stochastic differential equations, c-number Langevin equations, vacuum reservoir, mean photon number.

1. Introduction

There has been a considerable interest in the analysis of the squeezing and statistical properties of the light generated by three-level lasers [1–20]. A light mode to be in a squeezed state, if either the change in plus quadrature or the change in minus quadrature is less than one, with the product of the uncertainties in the two quadratures satisfying the uncertainty relation. Because of a smaller noise in one quadrature component, the squeezed states of light have important applications in information processing systems like quantum computations, photon detection, as well as in the field of high-precision measurements [10, 20].

A three-level laser may be defined as a quantum optical system, in which the injected three-level atoms in a cascade configuration are initially prepared in a coherent superposition of the top and bottom levels and coupled to a vacuum reservoir via a single port mirror. When a three-level atom in a cascade configuration makes a transition from the top to the bottom level via the intermediate level, two photons are generated. If the two photons have the same frequency, then the three-level atom is called degenerate three-level atom; otherwise, it is called nondegenerate one [3]. The two photons are highly correlated, and this correlation is responsible for the production of squeezed light.

Three-level lasers, in which the crucial role is played by the coherent superposition of the top and bottom levels of the injected atoms, have been studied by several authors [1–7]. These studies show that

Manuscript submitted by the author in English on February 2, 2022.

this quantum optical system can generate light in a squeezed state under certain conditions. Furthermore, Tesfa [2] has studied the squeezing property of the cavity modes produced by a nondegenerate three-level laser applying the solutions of stochastic differential equations. He has found that the two-mode cavity radiation exhibits squeezing, if the atoms are initially prepared with more atoms in the bottom level than in the upper level, and the degree of squeezing increases with the linear gain coefficient. He has shown that the maximum intracavity squeezing is 50% below the coherent-state level.

In addition, Fesseha has studied the squeezing and statistical properties of the light produced by a degenerate three-level laser, whose cavity contains a degenerate parametric amplifier [4]. His study indicates that a more squeezed light could be generated by a combination of these two quantum optical systems. On the other hand, Alebachew and Fesseha [10] have considered the same system with the injected atoms having equal probabilities to be in the upper and lower levels and with these two levels coupled by the pump mode emerging from the parametric amplifier. This study shows that the system generates light in a squeezed state with a maximum intracavity squeezing of 93% below the coherent-state level.

In this paper, we introduce a laser model in which bright and squeezed light from two nondegenerate three-level atoms is generated, where the cavity modes are coupled to a vacuum reservoir. The two atoms are different in preparation and injection rate; see Fig. 1. In order to determine the squeezing and statistical properties of the light produced by this quantum optical system, we first derive c-number Langevin equations using the pertinent master equation. Employing the solutions of the resulting c-number Langevin equations along with the properties of the Langevin forces, we calculate the quadrature variance of the cavity mode. Applying the same solutions, we also obtain the antinormally-ordered characteristic function with the aid of which the Q-function is determined. The resulting Q-function is then used to calculate the mean and variance of the photon number sum and difference of the cavity mode.

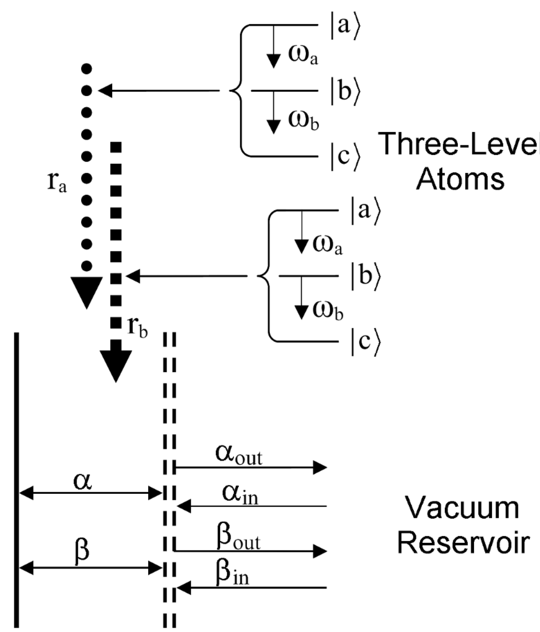

Fig. 1. Schematic representation of two nondegenerate three-level lasers.

2. Stochastic Differential Equations

As it is clearly indicated in Fig. 1, the top, intermediate, and bottom levels of a three-level atom are represented by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively. We prefer to call the light emitted from the top level light mode a and the one emitted from the intermediate level, light mode b. We assume the transitions between levels $|a\rangle$ and $|b\rangle$ and between levels $|b\rangle$ and $|c\rangle$ to be dipole allowed, with direct transitions between levels $|a\rangle$ and $|c\rangle$ to be dipole forbidden. We consider the case, for which the two cavity modes are at resonance with the two transitions $|a\rangle \to |b\rangle$ and $|b\rangle \to |c\rangle$ having transition frequencies ω_a and ω_b, respectively. The interaction of a nondegenerate three-level atom with two-mode cavity radiation can be expressed in the interaction picture with the rotating-wave approximation (RWA) by the Hamiltonian of the form [3]

$$\hat{H} = ig\left(|a\rangle\langle b|\hat{a} - \hat{a}^\dagger|b\rangle\langle a| + |b\rangle\langle c|\hat{b} - \hat{b}^\dagger|c\rangle\langle b|\right), \tag{1}$$

where g is the coupling constant assumed to be the same for both transitions, and $\hat{a}$ and $\hat{b}$ are the annihilation operators for the cavity modes. Similarly, the Hamiltonian describing the interaction of the cavity modes with the vacuum reservoir can be written as [20]

$$\hat{H}_{\mathrm{SR}}(t) = i\sum_k \lambda_k\left(\hat{a}^\dagger\hat{c}_k e^{i(\omega_a-\omega_k)t} - \hat{a}\hat{c}_k^\dagger e^{-i(\omega_a-\omega_k)t} + \hat{b}^\dagger\hat{d}_k e^{i(\omega_b-\omega_k)t} - \hat{b}\hat{d}_k^\dagger e^{-i(\omega_b-\omega_k)t}\right), \tag{2}$$

where λ_k is the coupling constant and $\hat{c}_k$ and $\hat{d}_k$ are the annihilation operators for a reservoir submode.

In this paper, we assume the state of a single three-level atom initially in the state

$$|\psi_A(0)\rangle = C_a|a\rangle + C_c|c\rangle, \tag{3}$$

and hence, the density operator of a single atom is

$$\hat{\rho}_A(0) = \rho_{aa}^{(0)}|a\rangle\langle a| + \rho_{ac}^{(0)}|a\rangle\langle c| + \rho_{ca}^{(0)}|c\rangle\langle a| + \rho_{cc}^{(0)}|c\rangle\langle c|, \tag{4}$$

where

$$\rho_{aa}^{(0)} = |C_a|^2 \qquad \text{and} \qquad \rho_{cc}^{(0)} = |C_c|^2 \tag{5}$$

are the initial probabilities of the atoms to be in the upper and lower levels, respectively, and

$$\rho_{ac}^{(0)} = C_a C_c^* \qquad \text{and} \qquad \rho_{ca}^{(0)} = C_c C_a^* \tag{6}$$

represent the atomic coherence at the initial time. We note that

$$|\rho_{ac}^{(0)}|^2 = \rho_{aa}^{(0)}\rho_{cc}^{(0)}. \tag{7}$$

The master equation corresponding to Eq. (1) takes the form [4]

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \frac{A_1\rho_{aa}^{(0)}}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + \frac{A_1\rho_{cc}^{(0)}}{2}\left(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\right)\\ &- \frac{A_1\rho_{ac}^{(0)}}{2}\left(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\right) - \frac{A_1\rho_{ca}^{(0)}}{2}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho}\right)\\ &+ \frac{\varkappa_1}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{\rho}\hat{b}^\dagger\hat{b}\right),\end{aligned} \tag{8}$$

where

$$A_1 = \frac{2g^2 r_a}{\gamma_1^2} \tag{9}$$

is the linear gain coefficient, $\varkappa_1$ is a cavity damping constant, and γ_1 is the atomic decay constant, which is considered to be the same for all the three levels. We note that Eq. (8) represents the master equation for the cavity mode corresponding the Hamiltonian given by Eq. (1), when one type of atoms is injected into the cavity at constant rate r_a.

In view of Eq. (8), we can also write the master equation for the cavity mode, in which two different types of atoms injected at rates r_a and r_b are

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} &= \frac{(A_1+A_2)}{2}\Big\{\rho_{aa}^{(0)}\left(2\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger\right) + \rho_{cc}^{(0)}\left(2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{\rho}\hat{b}^\dagger\hat{b} - \hat{b}^\dagger\hat{b}\hat{\rho}\right)\\ &- \rho_{ac}^{(0)}\left(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\right) - \rho_{ca}^{(0)}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho}\right)\Big\}\\ &+ \frac{\varkappa}{2}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger\hat{a} + 2\hat{b}\hat{\rho}\hat{b}^\dagger - \hat{b}^\dagger\hat{b}\hat{\rho} - \hat{\rho}\hat{b}^\dagger\hat{b}\right),\end{aligned} \tag{10}$$

where

$$A_2 = \frac{2g_2^2 r_b}{\gamma_2^2} \qquad \text{and} \qquad \varkappa = \varkappa_1 + \varkappa_2. \tag{11}$$

Equation (10) represents the stochastic master equation, which contains all necessary information on the dynamics of the system.

3. *C*-Number Langevin Equations

We now seek to obtain the *c*-number Langevin equations associated with the normal ordering for the cavity mode variables. To this end, employing the relation [5]

$$\frac{d}{dt}\langle \hat{A}\rangle = \mathrm{Tr}\left(\frac{d\hat{\rho}(t)}{dt}\hat{A}\right) \tag{12}$$

along with Eq. (10), we obtain that

$$\begin{aligned}\frac{d}{dt}\langle \hat{a}\rangle = {}& \frac{1}{2}(A\rho_{aa}^{(0)})\,\mathrm{Tr}\left(2\hat{a}^\dagger\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{a}^\dagger\hat{a}\right) + \frac{1}{2}(A\rho_{cc}^{(0)} + \varkappa)\,\mathrm{Tr}\left(2\hat{b}\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{b}\hat{a} - \hat{b}^\dagger\hat{b}\hat{\rho}\hat{a}\right)\\ &- \frac{1}{2}(A\rho_{ac}^{(0)})\,\mathrm{Tr}\left(2\hat{a}^\dagger\hat{\rho}\hat{b}^\dagger\hat{a} - \hat{b}^\dagger\hat{a}^\dagger\hat{\rho}\hat{a} - \hat{\rho}\hat{b}^\dagger\hat{a}^\dagger\hat{a}\right) - \frac{1}{2}(A\rho_{ca}^{(0)})\,\mathrm{Tr}\left(2\hat{b}\hat{\rho}\hat{a}\hat{a} - \hat{a}\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b}\hat{a}\right)\\ &+ \frac{\varkappa}{2}\left[\mathrm{Tr}\left(2\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}^\dagger\hat{a}\hat{a}\right)\right],\end{aligned} \tag{13}$$

where

$$A = A_1 + A_2. \tag{14}$$

Applying the cyclic property of the trace operation and taking into account the bosonic commutation relation

$$[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1, \tag{15}$$

the time evolution of the expectation value of the cavity mode variables is found to be

$$\frac{d}{dt}\langle \hat{a}\rangle = -\frac{1}{2}\mu_a\langle \hat{a}\rangle + \frac{1}{2}\nu_-\langle \hat{b}^\dagger\rangle, \qquad \frac{d}{dt}\langle \hat{b}\rangle = -\frac{1}{2}\mu_c\langle \hat{b}\rangle + \frac{1}{2}\nu_+\langle \hat{a}^\dagger\rangle, \tag{16}$$

$$\frac{d}{dt}\langle \hat{a}^2\rangle = -\mu_a\langle \hat{a}^2\rangle + \nu_-\langle \hat{b}^\dagger\hat{a}\rangle, \qquad \frac{d}{dt}\langle \hat{b}^2\rangle = -\mu_c\langle \hat{b}^2\rangle + \nu_+\langle \hat{a}^\dagger\hat{b}\rangle, \tag{17}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{a}\rangle = -\mu_a\langle \hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\nu_-\langle \hat{a}^\dagger\hat{b}^\dagger\rangle + \frac{1}{2}\nu_-^*\langle \hat{a}\hat{b}\rangle + A\rho_{aa}^{(0)}, \tag{18}$$

$$\frac{d}{dt}\langle \hat{b}^\dagger\hat{b}\rangle = -\mu_c\langle \hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+\langle \hat{b}^\dagger\hat{a}^\dagger\rangle + \frac{1}{2}\nu_+^*\langle \hat{a}\hat{b}\rangle, \tag{19}$$

$$\frac{d}{dt}\langle \hat{a}^\dagger\hat{b}\rangle = -\frac{1}{2}(\mu_a + \mu_c)\langle \hat{a}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+\langle \hat{a}^{\dagger 2}\rangle + \frac{1}{2}\nu_-^*\langle \hat{b}^2\rangle, \tag{20}$$

$$\frac{d}{dt}\langle \hat{a}\hat{b}\rangle = -\frac{1}{2}(\mu_a + \mu_c)\langle \hat{a}\hat{b}\rangle + \frac{1}{2}\nu_+\langle \hat{a}^\dagger\hat{a}\rangle + \frac{1}{2}\nu_-\langle \hat{b}^\dagger\hat{b}\rangle + \frac{1}{2}\nu_+, \tag{21}$$

where

$$\mu_a = \varkappa - A\rho_{aa}^{(0)}, \qquad \mu_c = \varkappa + A\rho_{cc}^{(0)}, \qquad \nu_- = -A\rho_{ac}^{(0)}, \qquad \nu_+ = +A\rho_{ac}^{(0)}. \tag{22}$$

We note that the operators in the above equations are in the normal order. The c-number equations corresponding to Eqs. (16)–(21) are [2]

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{1}{2}\mu_a\langle\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\rangle, \qquad \frac{d}{dt}\langle\beta\rangle = -\frac{1}{2}\mu_c\langle\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\rangle, \tag{23}$$

$$\frac{d}{dt}\langle\alpha^2\rangle = -\mu_a\langle\alpha^2\rangle + \nu_-\langle\beta^*\alpha\rangle, \qquad \frac{d}{dt}\langle\beta^2\rangle = -\mu_c\langle\beta^2\rangle + \nu_+\langle\alpha^*\beta\rangle, \tag{24}$$

$$\frac{d}{dt}\langle\alpha^*\alpha\rangle = -\mu_a\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\alpha^*\beta^*\rangle + \frac{1}{2}\nu_-^*\langle\alpha\beta\rangle + A\rho_{aa}^{(0)}, \tag{25}$$

$$\frac{d}{dt}\langle\beta^*\beta\rangle = -\mu_c\langle\beta^*\beta\rangle + \frac{1}{2}\nu_+\langle\beta^*\alpha^*\rangle + \frac{1}{2}\nu_+^*\langle\alpha\beta\rangle, \tag{26}$$

$$\frac{d}{dt}\langle\alpha^*\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha^*\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^{*2}\rangle + \frac{1}{2}\nu_-^*\langle\beta^2\rangle, \tag{27}$$

$$\frac{d}{dt}\langle\alpha\beta\rangle = -\frac{1}{2}(\mu_a+\mu_c)\langle\alpha\beta\rangle + \frac{1}{2}\nu_+\langle\alpha^*\alpha\rangle + \frac{1}{2}\nu_-\langle\beta^*\beta\rangle + \frac{1}{2}\nu_+. \tag{28}$$

On the basis of Eqs. (23), we can write

$$\frac{d}{dt}\alpha(t) = -\frac{1}{2}\mu_a\alpha(t) + \frac{1}{2}\nu_-\beta^*(t) + f_\alpha(t), \qquad \frac{d}{dt}\beta^*(t) = -\frac{1}{2}\mu_c\beta^*(t) + \frac{1}{2}\nu_+^*\alpha(t) + f_\beta^*(t), \tag{29}$$

where $f_\alpha(t)$ and $f_\beta(t)$ are Langevin forces, the properties of which remain to be determined, and $\alpha(t)$ and $\beta(t)$ are the c-number variables corresponding to the cavity mode operators $\hat{a}$ and $\hat{b}$.

The formal solutions of these equations can be put in the form [20]

$$\alpha(t) = \alpha(0)e^{-\mu_a t/2} + \int_0^t dt' e^{-\mu_a(t-t')/2}\left[\frac{1}{2}\nu_-\beta^*(t') + f_\alpha(t')\right], \tag{30}$$

$$\beta^*(t) = \beta^*(0)e^{-\mu_c t/2} + \int_0^t dt' e^{-\mu_c(t-t')/2}\left[\frac{1}{2}\nu_+^*\alpha(t') + f_\beta^*(t')\right]. \tag{31}$$

Making the use of Eqs. (23), the correlation properties of the Langevin forces can be readily put as [3]

$$\langle f_\alpha(t)\rangle = \langle f_\beta(t)\rangle = 0, \qquad \langle f_\alpha(t')f_\alpha(t)\rangle = 0, \tag{32}$$

$$\langle f_\beta(t')f_\beta(t)\rangle = \langle f_\alpha^*(t')f_\beta(t)\rangle = 0, \qquad \langle f_\alpha^*(t')f_\alpha(t)\rangle = A\rho_{aa}^{(0)}\delta(t-t'), \tag{33}$$

$$\langle f_\beta^*(t')f_\beta(t)\rangle = 0, \qquad \langle f_\alpha(t')f_\beta(t)\rangle = \frac{1}{2}\nu_+\delta(t-t'). \tag{34}$$

The results described by Eqs. (32)–(34) represent the correlation properties of the Langevin forces $f_\alpha(t)$ and $f_\beta(t)$ associated with the normal ordering.

4. Quadrature Variance of the Cavity Modes

Here, we seek to analyze the quadrature squeezing of the two-mode light in the cavity. The squeezing properties of the two-mode light in the cavity can be described by two quadrature operators defined by [20]

$$\hat{c}_\pm = \sqrt{\pm 1}(\hat{c}^\dagger \pm \hat{c}), \qquad \text{where} \qquad \hat{c} = \frac{1}{\sqrt{2}}(\hat{a}+\hat{b}), \tag{35}$$

with $\hat{a}$ and $\hat{b}$ representing the separate modes of cavity light emitted from the three-level atoms. The two-mode light is said to be in a squeezed state, if either $\Delta c_+^2 < 1$ and $\Delta c_-^2 > 1$ or $\Delta c_+^2 > 1$ and $\Delta c_1^2 < 1$, such that $\Delta c_+ \Delta c_- \geq 1$ [3,20].

The variances of the quadrature operator, defined by

$$\Delta c_\pm^2 = \langle \hat{c}_\pm^2 \rangle - \langle \hat{c}_\pm \rangle^2, \tag{36}$$

can be expressed in terms of c-number variables associated with the normal ordering as

$$\Delta c_\pm^2 = 1 + \langle \alpha^*(t)\alpha(t)\rangle + \langle \beta^*(t)\beta(t)\rangle \pm 2\langle \beta(t)\alpha(t)\rangle. \tag{37}$$

Using the fact that the Langevin force at time t does not affect the cavity mode variables at earlier time and taking the cavity modes to be initially in a vacuum state, one can easily establish that

$$\langle \alpha(t)\rangle = \langle \beta(t)\rangle = 0, \qquad \langle \alpha^2(t)\rangle = \langle \beta^2(t)\rangle = \langle \beta^*(t)\alpha(t)\rangle = 0, \tag{38}$$

$$\begin{aligned}
\langle \alpha^*(t)\alpha(t)\rangle = {}& \frac{A_+^*[A_+ A\rho_{aa}^{(0)} + 2\nu_-\nu_+^*/2] + 2\nu_-^* A_+\nu_+/2}{|2\lambda|^2(\lambda_-^* + \lambda_-)/2}\left[1 - e^{-(\lambda_-^*+\lambda_-)t/2}\right] \\
& - \frac{A_+^*[A_- A\rho_{aa}^{(0)} + 2\nu_-\nu_+^*/2] + 2\nu_-^* A_-\nu_+/2}{|2\lambda|^2(\lambda_-^* + \lambda_+)/2}\left[1 - e^{-(\lambda_-^*+\lambda_+)t/2}\right] \\
& - \frac{A_-^*[A_+ A\rho_{aa}^{(0)} + 2\nu_-\nu_+^*/2] + 2\nu_-^* A_+\nu_+/2}{|2\lambda|^2(\lambda_+^* + \lambda_-)/2}\left[1 - e^{-(\lambda_+^*+\lambda_-)t/2}\right] \\
& + \frac{A_-^*[A_- A\rho_{aa}^{(0)} + 2\nu_-\nu_+^*/2] + 2\nu_-^* A_-\nu_+/2}{|2\lambda|^2(\lambda_+^* + \lambda_+)/2}\left[1 - e^{-(\lambda_+^*+\lambda_+)t/2}\right],
\end{aligned} \tag{39}$$

$$\begin{aligned}
\langle \beta^*(t)\beta(t)\rangle = {}& \frac{A_+^* - 2\nu_+^*\nu_+/2 - 2\nu_+[A_+\nu_+^*/2 - 2\nu_+^* A\rho_{aa}^{(0)}]}{|2\lambda|^2(\lambda_+^* + \lambda_+)/2}\left[1 - e^{-(\lambda_+^*+\lambda_+)t/2}\right] \\
& - \frac{A_+^* - 2\nu_+^*\nu_+/2 - 2\nu_+[A_-\nu_+^*/2 - 2\nu_+^* A\rho_{aa}^{(0)}]}{|2\lambda|^2(\lambda_+^* + \lambda_-)/2}\left[1 - e^{-(\lambda_+^*+\lambda_-)t/2}\right] \\
& - \frac{A_-^* - 2\nu_+^*\nu_+/2 - 2\nu_+[A_+\nu_+^*/2 - 2\nu_+^* A\rho_{aa}^{(0)}]}{|2\lambda|^2(\lambda_-^* + \lambda_+)/2}\left[1 - e^{-(\lambda_-^*+\lambda_+)t/2}\right] \\
& + \frac{A_-^* - 2\nu_+^*\nu_+/2 - 2\nu_+[A_-\nu_+^*/2 - 2\nu_+^* A\rho_{aa}^{(0)}]}{|2\lambda|^2(\lambda_-^* + \lambda_-)/2}\left[1 - e^{-(\lambda_-^*+\lambda_-)t/2}\right],
\end{aligned} \tag{40}$$

and

$$\begin{aligned}\langle\alpha(t)\beta(t)\rangle = {} & \frac{A_+^*A_+\nu_+/2-2\nu_+[A_+A\rho_{aa}^{(0)}+2\nu_-\nu_+^*/2]}{|2\lambda|^2(\lambda_+^*+\lambda_-)/2}\left[1-e^{-(\lambda_+^*+\lambda_-)t/2}\right]\\ & -\frac{A_+^*A_-\nu_+/2-2\nu_+[A_-A\rho_{aa}^{(0)}+2\nu_-\nu_+^*/2]}{|2\lambda|^2(\lambda_+^*+\lambda_+)/2}\left[1-e^{-(\lambda_+^*+\lambda_+)t/2}\right]\\ & -\frac{A_-^*A_+\nu_+/2-2\nu_+[A_+A\rho_{aa}^{(0)}+2\nu_-\nu_+^*/2]}{|2\lambda|^2(\lambda_-^*+\lambda_-)/2}\left[1-e^{-(\lambda_-^*+\lambda_-)t/2}\right]\\ & +\frac{A_-^*A_-\nu_+/2-2\nu_+[A_-A\rho_{aa}^{(0)}+2\nu_-\nu_+^*/2]}{|2\lambda|^2(\lambda_-^*+\lambda_+)/2}\left[1-e^{-(\lambda_-^*+\lambda_+)t/2}\right].\end{aligned} \tag{41}$$

Now substitution of Eqs. (39)–(41) into Eq. (37) leads to

$$\begin{aligned}\Delta c_\pm^2 = {} & \pm\frac{1}{|2\lambda|^2}\left\{\frac{(A_+\pm 2\nu_+^*)[(A_+^*\pm 2\nu_+)A\rho_{aa}^{(0)}\mp(A_-^*\mp 2\nu_-^*)\nu_+/2]}{\lambda_-+\lambda_-^*}\right.\\ & \left.\frac{(A_-\mp 2\nu_-)\mp(A_+^*\pm 2\nu_+)\nu_+^*/2}{\lambda_-+\lambda_-^*}\right\}\left[1-e^{-(\lambda_-+\lambda_-^*)t/2}\right]\\ & \pm\frac{1}{|2\lambda|^2}\left\{\frac{(A_-\pm 2\nu_+^*)[(A_-^*\pm 2\nu_+)A\rho_{aa}^{(0)}\mp(A_+^*\mp 2\nu_-^*)\nu_+/2]}{\lambda_++\lambda_+^*}\right.\\ & \left.+\frac{(A_+\mp 2\nu_-)\mp(A_-^*\pm 2\nu_+)\nu_+^*/2}{\lambda_++\lambda_+^*}\right\}\left[1-e^{-(\lambda_++\lambda_+^*)t/2}\right]\\ & \mp\frac{2}{|2\lambda|^2}\left\{\frac{(A_-\pm 2\nu_+^*)[(A_+^*\pm 2\nu_+)A\rho_{aa}^{(0)}\mp(A_-^*\mp 2\nu_-^*)\nu_+/2]}{\lambda_++\lambda_-^*}\right.\\ & \left.+\frac{(A_+\mp 2\nu_-)\mp(A_+^*\pm 2\nu_+)\nu_+^*/2}{\lambda_++\lambda_-^*}\right\}\left[1-e^{-(\lambda_++\lambda_-^*)t/2}\right].\end{aligned} \tag{42}$$

Equation (42) takes at steady state the form

$$\begin{aligned}\Delta c_\pm^2 = 1 + {} & \frac{2}{|2\lambda|^2}\left\{\frac{|A_+\pm 2\nu_+^*|^2}{\lambda_-+\lambda_-^*}+\frac{|A_-\pm 2\nu_+^*|^2}{\lambda_++\lambda_+^*}-\frac{(A_+^*\pm 2\nu_+)(A_-\pm 2\nu_+^*)}{\lambda_++\lambda_-^*}\right.\\ & \left.-\frac{(A_+\pm 2\nu_+^*)(A_-^*\pm 2\nu_+)}{\lambda_+^*+\lambda_-}\right\}A\rho_{aa}^{(0)}\mp\frac{2}{|2\lambda|^2}\left\{\frac{(A_+\pm 2\nu_+^*)(A_-^*\mp 2\nu_-^*)}{\lambda_-+\lambda_-^*}+\frac{(A_+^*\mp 2\nu_-^*)(A_-\pm 2\nu_+^*)}{\lambda_++\lambda_+^*}\right.\\ & \left.-\frac{(A_-^*\mp 2\nu_-^*)(A_-\pm 2\nu_+^*)}{\lambda_++\lambda_-^*}-\frac{(A_+^*\mp 2\nu_-^*)(A_+\pm 2\nu_+^*)}{\lambda_+^*+\lambda_-}\right\}\nu_+/2\mp\frac{2}{|2\lambda|^2}\left\{\frac{(A_+^*\pm 2\nu_+)(A_-\mp 2\nu_-)}{\lambda_-+\lambda_-^*}\right.\\ & \left.+\frac{(A_+\mp 2\nu_-)(A_-^*\pm 2\nu_+)}{\lambda_++\lambda_+^*}-\frac{(A_+\mp 2\nu_-)(A_+^*\pm 2\nu_+)}{\lambda_++\lambda_-^*}-\frac{(A_-^*\pm 2\nu_+)(A_-\mp 2\nu_-)}{\lambda_+^*+\lambda_-}\right\}\nu_+^*/2.\end{aligned} \tag{43}$$

In order to have a mathematically manageable analysis, we take $\rho_{ac}=\rho_{ca}$. Now, in view of this and Eq. (7), we have

$$\begin{aligned}& 2\nu_\pm = 2\nu_\pm^* = \pm A\sqrt{1-\eta^2}, && \lambda=\lambda^*=A\eta,\\ & A_\pm = A_\pm^* = A\pm A\eta, && \lambda_\pm=\lambda_\pm^*=\frac{1}{2}(2\varkappa+A\eta\pm A\eta), \qquad \rho_{aa}^{(0)}=\frac{1}{2}(1-\eta).\end{aligned} \tag{44}$$

So that with the aid Eqs. (43) and (44), we get

$$\Delta c_{\pm}^{2} = 1 \pm \frac{A\sqrt{1-\eta^{2}}(2\varkappa + A\eta + A) \pm A(1-\eta)(2\varkappa + 2A\eta + A)}{2(\varkappa + A\eta)(2\varkappa + A\eta)}, \tag{45}$$

where $\eta = \rho_{cc}^{(0)} - \rho_{aa}^{(0)}$ describes the initial preparation of a three-level atom.

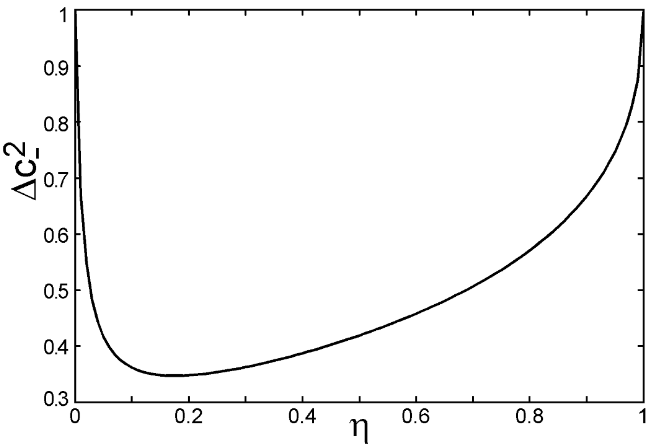

Fig. 2. Quadrature variance, Eq. (45), versus η for $A = 100$ and $\varkappa = 0.8$.

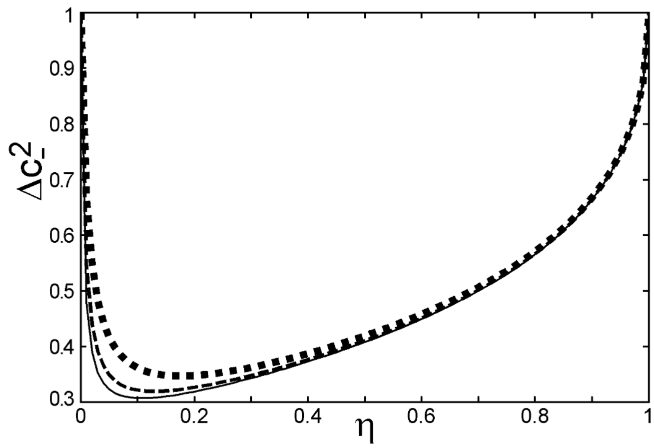

Fig. 3. Quadrature variances, Eq. (45), versus η for different values of the total linear gain coefficient $A = 100$ (the solid curve), $A = 200$ (the dashed curve), and $A = 300$ (the dotted curve) and $\varkappa = 0.8$.

Equation (45) represents the variances of the cavity mode at steady state for two nondegenerate three-level atoms coupled to a vacuum reservoir. In Fig. 2, we plot the minus quadrature variance of the two-mode light, Eq. (45), versus η, where the minimum value of the quadrature variance for $A = 100$ and $\varkappa = 0.8$ is found to be $\Delta c_{-}^{2} = 0.3467$ and occurs at $\eta = 0.18$. This result implies that the maximum intracavity squeezing for the above values is 65.3% below the coherent-state level. This result is greater than the one obtained by Tesfa [2].

In Fig. 3, we represent the variances of the minus quadrature, Eq. (45), versus η for different values of A. Here, one can see that the degree of squeezing increases with the total linear gain coefficient and almost perfect squeezing can be obtained for large values of the linear gain coefficient and for small values of η. Thus, we realize that better squeezing can be achieved by preparing the atoms initially in such a way that slightly more atoms are in the lower level than in the upper level. We also see that the degree of squeezing increases with the total linear gain coefficient, which is in a complete agreement with previous studies [2, 4, 11].

5. Photon Statistics

In order to know about the brightness of the generated light, it is necessary to study the mean number of photon pairs describing the two-mode cavity radiation that can be defined as [18]

$$\bar{n} = \langle \hat{c}^{\dagger}\hat{c} \rangle. \tag{46}$$

It is possible to put this expression in terms of c-number variables associated with the normal ordering, namely,

$$\bar{n} = \frac{1}{2}\Big[\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle + \langle\beta^*(t)\alpha(t)\rangle + \langle\alpha^*(t)\beta(t)\rangle\Big]. \tag{47}$$

In view of Eq. (38), Eq. (47) is reduced to

$$\bar{n} = \frac{1}{2}\Big[\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle\Big]; \tag{48}$$

this equation represents the mean photon number pair of the system.

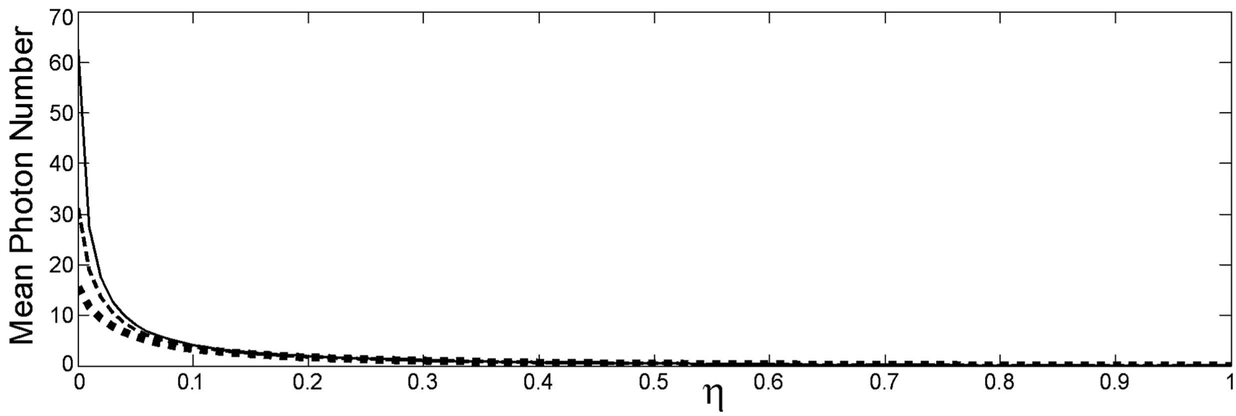

Fig. 4. The mean photon number, Eq. (48), versus η for different values of the total linear gain coefficient $A = 100$ (the solid curve), $A = 50$ (the dashed curve), and $A = 25$ (the dotted curve) and $\varkappa = 0.8$.

In Fig. 4, we plot the mean photon number of the two-mode light versus η for different values of the total linear gain coefficient. It is very easy to see from Fig. 4 that this system generates a bright and highly-squeezed light. We also notice that the mean number of photons is larger for small values of η, at which the squeezing is found to be relatively higher.

6. The Q-Function

Using the solutions of the c-number Langevin equations, one can readily establish the antinormally-ordered characteristic function defined in the Heisenberg picture for the cavity modes. With the aid of the resulting characteristic function, we obtain the Q-function, which is then used to calculate the mean and variance of the photon number sum and difference for the cavity modes.

The Q-function for a two-mode light can be expressed as [3]

$$Q(\alpha,\beta,t) = \frac{1}{\pi^2}\int \frac{d^2z}{\pi}\frac{d^2w}{\pi}\,\Phi_A(z,w,t)e^{z^*\alpha - z\alpha^* + w^*\beta - w\beta^*}, \tag{49}$$

with the characteristic function $\Phi_A(z,w,t)$ defined in the Heisenberg picture by

$$\Phi_A(z,w,t) = \mathrm{Tr}\left[\rho(0)e^{-z^*\hat{a}(t)}e^{z\hat{a}^\dagger(t)}e^{-w^*\hat{b}(t)}e^{w\hat{b}^\dagger(t)}\right]. \tag{50}$$

Employing the Baker–Hausdorff identity, we can rewrite Eq. (50) in the normal order as follows:

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w}\,\mathrm{Tr}\left[\rho(0)e^{z\hat{a}^\dagger(t)}e^{-z^*\hat{a}(t)}e^{w\hat{b}^\dagger(t)}e^{-w^*\hat{b}(t)}\right], \tag{51}$$

so that the corresponding c-number equation is

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w}\left\langle e^{z\alpha^*(t)-z^*\alpha(t)+w\beta^*(t)-w^*\beta(t)}\right\rangle. \tag{52}$$

We recall that

$$\frac{d}{dt}\langle\alpha(t)\rangle = -\frac{1}{2}\mu_a\langle\alpha(t)\rangle + \frac{1}{2}\nu_-\langle\beta^*(t)\rangle, \qquad \frac{d}{dt}\langle\beta(t)\rangle = -\frac{1}{2}\mu_c\langle\beta(t)\rangle + \frac{1}{2}\nu_+\langle\alpha^*(t)\rangle. \tag{53}$$

We see that Eqs. (53) are linear differential equations for $\alpha(t)$ and $\beta(t)$. On account of Eqs. (53) and (38), we observe that $\alpha(t)$ and $\beta(t)$ are Gaussian variables with a vanishing mean. In view of this, Eq. (52) can be rewritten as follows [20]:

$$\Phi_A(z,w,t) = e^{-z^*z-w^*w}\exp\left[\left\langle\frac{1}{2}\left(z\alpha^*(t)-z^*\alpha(t)+w\beta^*(t)-w^*\beta(t)\right)^2\right\rangle\right]. \tag{54}$$

Hence on account of Eqs. (38)–(41), the characteristic function can be put in the form

$$\Phi_A(z,w,t) = e^{-a_\alpha z^*z+z^*w^*b+zwb^*}e^{-a_\beta w^*w}, \tag{55}$$

where

$$a_\alpha = 1+\frac{A(1-\eta)(4\varkappa+3A\eta+A)}{4(\varkappa+A\eta)(2\varkappa+A\eta)}, \qquad a_\beta = 1+\frac{A^2(1-\eta^2)}{4(\varkappa+A\eta)(2\varkappa+A\eta)}, \tag{56}$$

and

$$b = \frac{A\sqrt{1-\eta^2}(2\varkappa+A\eta+A)}{4(\varkappa+A\eta)(2\varkappa+A\eta)}. \tag{57}$$

Now inserting (55) into Eq. (49) and carrying out the integration with the help of

$$\int\frac{d^2z}{\pi^2}\exp(-azz^*+bz+cz^*+Az^2+Bz^{*2}) = \frac{1}{\sqrt{a^2-4AB}}\exp\left[\frac{abc+Ac^2+Bb^2}{a^2-4AB}\right], \quad a>0, \tag{58}$$

we obtain

$$Q(\alpha,\beta,t) = \frac{u_\alpha u_\beta - v^*v}{\pi^2}\exp\big[-u_\beta\alpha^*\alpha+\alpha v^*\beta+\alpha^*v\beta^*-u_\alpha\beta^*\beta\big], \tag{59}$$

where

$$u_\alpha = \frac{a_\alpha}{a_\alpha a_\beta - b^*b}, \qquad u_\beta = \frac{a_\beta}{a_\alpha a_\beta - b^*b}, \qquad v = \frac{b}{a_\alpha a_\beta - b^*b}. \tag{60}$$

7. Mean of the Photon Number Sum and Difference

We define the operators representing the photon number sum and difference of mode a and mode b by

$$\hat{n}_\pm = \hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b}. \tag{61}$$

The mean of the photon number sum and difference can be written in terms of the Q-function as follows:

$$\overline{n}_{\pm} = \int d^2\alpha\, d^2\beta\, Q(\alpha,\beta,t)\,(\alpha^*\alpha \pm \beta^*\beta - 1 \mp 1)\,. \tag{62}$$

Now applying the Q-function of Eq. (59) to Eq. (62) and performing the integration with the help of Eq. (58), we arrive at

$$\overline{n}_{\pm} = (u_\alpha u_\beta - v^* v)\left(\frac{\partial}{r\partial u_\beta} \pm \frac{\partial}{\partial u_\alpha} - 1 \mp 1\right) \times \left(\frac{1}{u_\alpha u_\beta - v^* v}\right), \tag{63}$$

from which follows

$$\overline{n}_{\pm} = \overline{n}_a \pm \overline{n}_b, \tag{64}$$

where

$$\overline{n}_a = \frac{u_\alpha}{u_\alpha u_\beta - v^* v} - 1 \qquad \text{and} \qquad \overline{n}_b = \frac{u_\beta}{u_\alpha u_\beta - v^* v} - 1 \tag{65}$$

are the mean photon numbers of mode a and mode b. With the aid of Eqs. (60) and (56), we obtain

$$\overline{n}_a = \frac{A(1-\eta)(4\varkappa + 3A\eta + A)}{4(\varkappa + A\eta)(2\varkappa + A\eta)} \qquad \text{and} \qquad \overline{n}_b = \frac{A^2(1-\eta^2)}{4(\varkappa + A\eta)(2\varkappa + A\eta)}. \tag{66}$$

On account of Eqs. (66), the mean of the photon number sum and difference can be written as follows:

$$\overline{n}_{\pm} = A(1-\eta)2(2\varkappa + A\eta) + (1 \pm 1)A(1+\eta)4(\varkappa + A\eta)(2\varkappa + A\eta). \tag{67}$$

This is the mean of the photon number sum and difference for the cavity modes produced by two different nondegenerate three-level atoms coupled to a vacuum reservoir. We see from Eq. (67) that the mean of the photon number difference is positive. This fact shows that the mean photon number of mode a is greater than that of mode b due to the three-level laser.

8. Variances of the Photon Number Sum and Difference

The variances of the photon number sum and difference defined by

$$\Delta n_{\pm}^2 = \langle(\hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b})^2\rangle - \langle\hat{a}^\dagger\hat{a} \pm \hat{b}^\dagger\hat{b}\rangle^2 \tag{68}$$

can be expressed as

$$\Delta n_{\pm}^2 = \Delta n_a^2 + \Delta n_b^2 \pm 2n_{ab}, \tag{69}$$

in which $\Delta n_a^2 = \langle(\hat{a}^\dagger\hat{a})^2\rangle - \overline{n}_a^2$ is the photon number variance of mode a, $\Delta n_b^2 = \langle(\hat{b}^\dagger\hat{b})^2\rangle - \overline{n}_b^2$ is the photon number variance of mode b, and $n_{ab} = \langle\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\rangle - \overline{n}_a\overline{n}_b$, with $\overline{n}_a = \langle\hat{a}^\dagger\hat{a}\rangle$ and $\overline{n}_b = \langle\hat{b}^\dagger\hat{b}\rangle$.

Using the commutation relation $[\hat{a},\hat{a}^\dagger] = 1$, we can write

$$\Delta n_a^2 = \langle\hat{a}^2\hat{a}^{\dagger 2}\rangle - \overline{n}_a^2 - 3\overline{n}_a - 2. \tag{70}$$

The first term on the right side of Eq. (70) can be expressed in terms of the Q-function as [3]

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle = \int d\alpha^2 d\beta^2 Q(\alpha,\beta,t)\alpha^{*2}\alpha^2. \tag{71}$$

Now applying the Q-function of Eq. (59) to Eq. (71) and performing the integration, we obtain

$$\langle\hat{a}^2\hat{a}^{\dagger 2}\rangle = 2(\overline{n}_a + 1)^2. \tag{72}$$

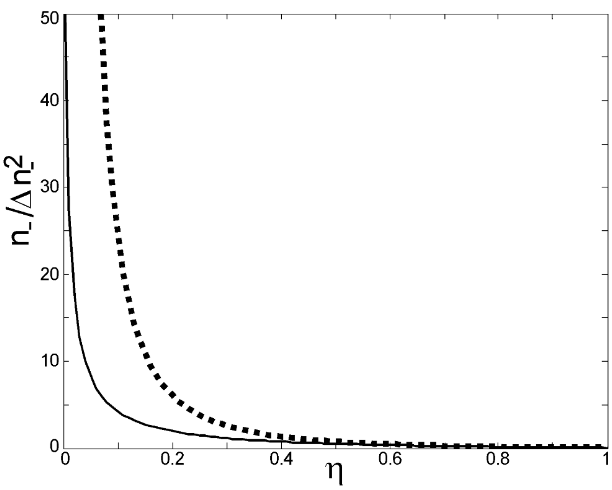

Fig. 5. The mean of the photon number difference, Eq. (67), versus η (the solid curve) and the variances of the photon number difference, Eq. (75), versus η (dotted curve) for $A = 100$ and $\varkappa = 0.8$.

Therefore, substitution of Eq. (72) into Eq. (70) yields

$$\Delta n_a^2 = \overline{n}_a^2 + \overline{n}_a. \tag{73}$$

Following the same procedure, we easily obtain

$$\Delta n_b^2 = \overline{n}_b^2 + \overline{n}_b \qquad \text{and} \qquad n_{ab} = |b|^2. \tag{74}$$

Hence combination of Eqs. (69), (73), and (74) results in

$$\Delta n_\pm^2 = \overline{n}_a^2 + \overline{n}_a + \overline{n}_b^2 + \overline{n}_b \pm 2|b|^2. \tag{75}$$

In Fig. 5, we see that the mean and variances of the photon number difference is positive. This indicates that the cavity radiation exhibits a super-Poissonian photon statistics [20].

9. Conclusions

In this paper, we studied the squeezing and statistical properties of the cavity modes produced by two nondegenerate three-level atoms, with the cavity mode coupled to a vacuum reservoir. We obtained the c-number Langevin equations associated with the normal ordering, using the master equation. Applying the solutions of the resulting Langevin equations, we calculated the quadrature variances. The light produced by the system under consideration is in a squeezing state with a maximum intracavity squeezing of 65.3% below the coherent-state level. This result is greater than the one obtained by Tesfa [2]. We found that the degree of squeezing increases with the total linear gain coefficient and almost perfect squeezing can be obtained for large values of the total linear gain coefficient and for small values of η, which is in a complete agreement with previous studies.

We determined the mean and variances of the photon number sum and difference for the cavity modes employing the Q-function. The result shows that the mean photon number of mode a is greater than that of mode b. Furthermore, we also observed that the photon number statistics is super-Poissonian.

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