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THE MACROSCOPIC EVOLUTION IN THE MEASUREMENT SPACE

C. Muñoz and Andrei B. Klimov*

Departamento de Fisica
Universidad de Guadalajara
Guadalajara 44420, Mexico

*Corresponding author e-mail: klimov @ cencar.udg.mx

Abstract

We analyze the dynamics of N-qubit systems in the measurement space under the action of symmetric Hamiltonians. We show that the evolution of the discrete distribution function, representing the global properties of multipartite states, becomes quasicontinuous in the macroscopic limit $N \gg 1$. The short-time dynamics can be approximately described as a propagation along "classical" trajectories in the measurement space.

Keywords: discrete phase-space, quasidistribution functions, quantum and semiclassical evolution.

1. Introduction

The phase-space methods are a natural framework for studying semiclassical dynamics of macroscopic quantum systems. For quantum systems with a continuous dynamical symmetry, the density matrix ρ is mapped into a quasidistribution function $W_\rho(\Omega)$ defined on the corresponding classical phase-space $\mathcal{M}$, $\Omega \in \mathcal{M}$; see, e.g. [1] and references therein. The evolution of $W_\rho(\Omega, t)$ is then governed by the so-called Moyal differential) equation [2], which usually has a complicated form and contains higher-order derivatives on the phase-space coordinates.

The main tool for the analysis of quantum–classical transition consists in expanding the exact evolution equation in powers of a small semiclassical parameter. The leading order of such a procedure usually leads to the reduction of the Moyal bracket between $W_\rho(\Omega, t)$ and the corresponding phase-space symbol of the Hamiltonian to the Poisson brackets between them.

The situation becomes significantly more involved in the case of discrete symmetries, which are appropriate to use for a nonredundant description of finite-dimensional systems as, for instance, a finite number of qudits. For p, a prime number, a faithful phase-space mapping exists for p^N; $N = 1, 2, \ldots$ dimensional systems [3–9], yet the dynamics of the corresponding quasidistributions are described by extremely complicated finite-difference operators [10, 11]. In fact, the main drawback in the description of the evolution in a discrete phase space is the absence of a natural ordering of the coordinates, and their reordering (permutations) may significantly change the shape of the distribution [18, 19]. As a consequence, a typical "picture" of the evolution in discrete representation consists in an irregular sequence of jumps of almost randomly distributed peaks [10, 11]. Thus, even if discrete quasidistributions contain full information on the corresponding N-qudit system but, unfortunately, they are not very convenient for representation purposes [18, 19].

In macroscopic quantum systems, only correlation functions of collective observables can frequently be efficiently assessed, both theoretically and experimentally [12, 13]. Such permutationally invariant

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functions, containing only partial information on the whole system, describe their global properties and can be "gathered" in a single discrete quasidistribution, the so-called projected $\widetilde{Q}$-function [14–16]. This distribution function "lives" in a three-dimensional macroscopic measurement space $\mathcal{M}$ and comprises complete and nonredundant information on all collective observables. The nonredundancy implies the possibility of reconstructing the $\widetilde{Q}$-distribution by collecting only a finite number of measured data.

In this paper, we analyze the dynamics of the discrete $\widetilde{Q}$-function generated by permutationally invariant Hamiltonians, focusing on the particular case of N-qubit systems. We show that the evolution in the measurement space becomes smooth (quasicontinuous) in the macroscopic limit, $N \gg 1$, and obtain approximate evolution equations for some simple Hamiltonians describing both interacting and noninteracting qubits. A fundamental difference of our treatment from the standard $SU(2)$ approach [17] consists in describing the dynamics of arbitrary, not necessarily symmetric, initial multipartite states, in a nonredundant way.

2. Preliminaries: $\widetilde{Q}$-Distribution in the Measurement Space

Operators acting in N-qubit Hilbert space $\mathcal{H}_{2^N} = \mathcal{H}_2^{\otimes N}$ can be put in the one-to-one correspondence with discrete distributions in the $2^N \times 2^N$ phase space, where each point is labeled by a pair of $\mathbb{Z}_2$-strings (α, β) and $\alpha = (a_1, \ldots, a_N)$ and $\beta = (b_1, \ldots, b_N)$ and $a_i, b_i \in \mathbb{Z}_2$. In particular, one can introduce analogs of the Husimi Q-function and Glauber–Sudarshan P-function according to the maps [18–23],

$$\hat{f} \Leftrightarrow \begin{cases} Q_f(\alpha,\beta) = \mathrm{Tr}\left[\hat{\Delta}^{(-1)}(\alpha,\beta)\hat{f}\right], \\ P_f(\alpha,\beta) = \mathrm{Tr}\left[\hat{\Delta}^{(1)}(\alpha,\beta)\hat{f}\right], \end{cases} \tag{1}$$

where $\hat{\Delta}^{(1)}(\alpha,\beta)$ and $\hat{\Delta}^{(-1)}(\alpha,\beta)$ are dual kernels,

$$\hat{\Delta}^{(-1)}(\alpha,\beta) = |\alpha,\beta\rangle\langle\alpha,\beta|, \quad |\alpha,\beta\rangle = \otimes\Pi_{i=1}^N \hat{\sigma}_z^{(i)a_i}\hat{\sigma}_x^{(i)b_i}|\mathbf{n}_0\rangle_i, \tag{2}$$

$$\hat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|, \quad \hat{\sigma}_x = |0\rangle\langle 1| + |1\rangle\langle 0|, \tag{3}$$

$$\mathrm{Tr}\left[\hat{\Delta}^{(1)}(\alpha,\beta)\hat{\Delta}^{(-1)}(\alpha',\beta')\right] = 2^N \delta_{\alpha\alpha'}\delta_{\beta\beta'}, \tag{4}$$

and the fiducial single-qubit states $|\mathbf{n}_0\rangle_i$ are fixed by $\mathbf{n}_0 = (1,1,1)/\sqrt{3}$ [24], so that the product $|\mathbf{n}_0\rangle_1$, $\ldots$ $|\mathbf{n}_0\rangle_N$ is a spin coherent state,

$$\bigotimes_{i=1}^N |\mathbf{n}_0\rangle_i = \left|\xi = \frac{\sqrt{3}-1}{\sqrt{2}}e^{i\pi/4}\right\rangle. \tag{5}$$

The average value of any N-qubit operator is computed as a convolution

$$\langle\hat{f}\rangle = \sum_{\alpha,\beta} P_f(\alpha,\beta)Q_\rho(\alpha,\beta), \tag{6}$$

where $Q_\rho(\alpha,\beta)$ is the symbol of the density matrix $\hat{\rho}$.

The kernels (2)–(4) satisfy the symmetry condition,

$$\mathcal{P}\hat{\Delta}^{(\pm1)}(\alpha,\beta)\mathcal{P}^\dagger = \hat{\Delta}^{(\pm1)}(\pi\alpha,\pi\beta),$$

where $\mathcal{P}$ is the permutation operator and $\pi\alpha$ is a permutation of the string α corresponding to $\mathcal{P}$. Thus, the P-symbol of any collective observable $\hat{s}$, invariant under permutations,

$$\hat{s} = \mathcal{P}^\dagger \hat{s} \mathcal{P}, \tag{7}$$

depends only on the symmetric functions of pairs (α, β), the so-called weights,

$$(\alpha, \beta) \Rightarrow \mathbf{h} = \big(h(\alpha), h(\alpha+\beta), h(\beta) \big), \tag{8}$$

where $h(\alpha)$ is the length of the string α,

$$0 \le h(\alpha) = \sum_{i=1}^{N} a_i = h(\pi\alpha) \le N, \tag{9}$$

i.e., the kernel $\hat{\Delta}^{(1)}(\alpha, \beta)$ maps collective operators (7) into functions of weights (8),

$$P_s(\alpha, \beta) = \mathrm{Tr}\left[\hat{\Delta}^{(1)}(\alpha, \beta) \hat{s} \right] |_{h(\alpha)=m, h(\alpha+\beta)=n, h(\beta)=k} \equiv P_s(\mathbf{m}), \quad \mathbf{m} \equiv (m, n, k). \tag{10}$$

Then, expression (6) for average values of collective operators (7) can be rewritten in the following form:

$$\langle\ \hat{s}\ \rangle = \sum_{\mathbf{m}} P_s(\mathbf{m}) \widetilde{Q}_\rho(\mathbf{m}), \tag{11}$$

where

$$\widetilde{Q}_\rho(\mathbf{m}) = \sum_{\alpha,\beta} Q_\rho(\alpha, \beta) \delta_{h(\alpha),m} \delta_{h(\beta),n} \delta_{h(\alpha+\beta),k}, \ \mathbf{m} = (m, n, k), \tag{12}$$

and $m, n = 0, \ldots, N$, while k runs in steps of two from $k = |m - n|$ to $\min(m + n, N, 2N - m - n)$.

In other words, the $\widetilde{Q}$-function (12) is a map of the density operator into a discrete function in a three-dimensional $N \times N \times N$ measurement space $\mathcal{M}$ of weights $\mathbf{h}$ contained inside a tetrahedron with the vertices $(0,0,0)$, $(N,0,N)$, $(0,N,N)$, and $(N,N,0)$ [14]. As it follows from (11), the $\widetilde{Q}$-function contains all necessary information required for the determination of any collective property of a N-qubit system. Thus, the $\widetilde{Q}$-function can be considered as a collective distribution corresponding to an arbitrary N-qubit state. The $\widetilde{Q}$-function can be plotted as a collection of spheres, whose size is proportional to the density of the distribution at a given point $(m,n,k) \in \mathcal{M}$. In the macroscopic limit, $N \gg 1$, the $\widetilde{Q}$-functions tend to acquire smooth shapes and provide an intuitive representation of N-qubit states as seen by collective observables.

In addition, since $\widetilde{Q}_{\mathbf{S}\cdot\mathbf{n}}(\mathbf{m}) \sim N/2 - \mathbf{h} \cdot \mathbf{n}$, where $\mathbf{n}$ is a unit three-dimensional vector and

$$\hat{S}_{x,y,z} = \sum_{i=1}^{N} \hat{\sigma}^{(i)}_{x,y,z} \tag{13}$$

is a collective spin operator, the directions in the measurement space $\mathcal{M}$ can be associated with vectors in the real configuration space, i.e.,

$$(m, n, k) \Leftrightarrow (x, z, y). \tag{14}$$

It is important to stress that the mapping

$$\begin{aligned} \hat{\rho} \quad &\rightarrow \quad \widetilde{Q}_{\rho}(\mathbf{m}) = \mathrm{Tr}\left(\hat{\rho}\hat{\Delta}^{(-1)}(\mathbf{m})\right), && (15)\\ \hat{\Delta}^{(-1)}(\mathbf{m}) \quad &= \quad \sum_{\alpha,\beta}\hat{\Delta}^{(-1)}(\alpha,\beta)\,\delta_{h(\alpha),\,m}\,\delta_{h(\beta),\,n}\,\delta_{h(\alpha+\beta),\,k} && (16)\end{aligned}$$

is not one-to-one (actually, it is a specific averaging), and thus $\widetilde{Q}_{\rho}(\mathbf{m})$ does not contain the whole microscopic information on the system.

3. The Evolution in the Measurement Space

The Schrödinger equation can be, in principle, exactly mapped onto the evolution equation for discrete distribution functions (1), which solution, $Q_{\rho}(\alpha,\beta|t)$, would contain full information on the dynamics of N-qubit systems for an arbitrary Hamiltonian. Unfortunately, the corresponding equations are extremely complicated and can hardly be used in practice for studying the discrete phase-space evolution [10, 11]. On the other hand, the structure of the map (15) allows one to obtain the evolution equation for $\widetilde{Q}_{\rho}\,(\mathbf{m}|t)$ in the case of symmetric Hamiltonians. Such equations are suitable for describing the global dynamics of both symmetric and nonsymmetric initial states. Here, we analyze the evolution of $\widetilde{Q}_{\rho}(\mathbf{m})$ under the action of linear and quadratic spin Hamiltonians.

3.1. The Linear Evolution

For the simplest linear Hamiltonian

$$\hat{H} = \sum_{j=1}^{N}\hat{\sigma}_z^{(j)} = \hat{S}_z, \tag{17}$$

the evolution equation in the measurement space takes the following form; see Appendix A:

$$\begin{aligned} \partial_t\tilde{Q}_{\rho}(m,n,k) \quad = \quad & \frac{1}{2}(2\,N+2-m-n-k)\left[\tilde{Q}_{\rho}(m,n-1,k-1)-\tilde{Q}_{\rho}(m-1,n-1,k)\right]\\ &+\frac{1}{2}(2+n+k-m)\left[\tilde{Q}_{\rho}(m-1,n+1,k)-\tilde{Q}_{\rho}(m,n+1,k+1)\right]\\ &+\frac{1}{2}(2-n+m+k)\left[\tilde{Q}_{\rho}(m,n-1,k+1)-\tilde{Q}_{\rho}(m+1,n-1,k)\right]\\ &+\frac{1}{2}(2+m+n-k)\left[\tilde{Q}_{\rho}(m+1,n+1,k)-\tilde{Q}_{\rho}(m,n+1,k-1)\right]. \end{aligned} \tag{18}$$

One can exactly solve the above equation, but it is more instructive to analyze its continuous limit. Restricting the expansion of the finite difference operators to the first-order derivatives with respect to the corresponding variables, we arrive at the following intuitively clear equation:

$$\partial_t\tilde{Q}_{\rho}(m,n,k) \approx (N-2\,k)\,\frac{\partial}{\partial m}\tilde{Q}_{\rho}\,(m,n,k) - (N-2\,m)\,\frac{\partial}{\partial k}\tilde{Q}_{\rho}(m,n,k).$$

The above equation describes a solid rotation of an initial distribution with respect to the center of the tetrahedron $(N/2, N/2, N/2)$ in the plane (m,k),

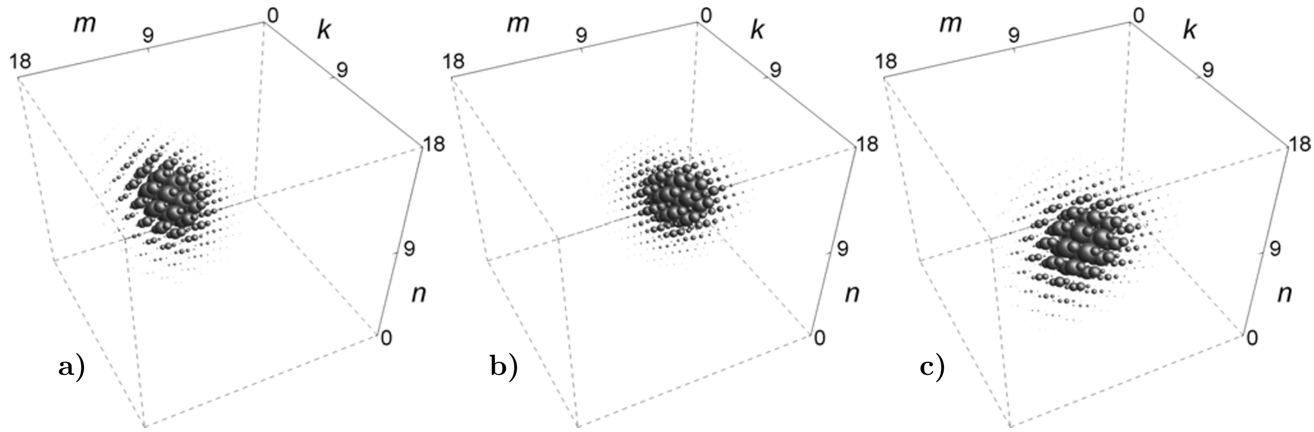

Fig. 1. The evolution of $\widetilde{Q}_\rho$-function for the initial GHZ state (22) under the action of Hamiltonian $\hat{H} = \hat{S}_z$ at $t = 0$ (a), $t = 0.5$ (b), and $t = 0.9$ (c); $N = 18$.

$$\begin{aligned} \tilde{Q}_\rho\,(m, n, k|t) &\approx \tilde{Q}_\rho\,(m(t), n, k(t)|t = 0)\,, && (19)\\ m(t) &= (\,m - N/2) \cos 2\,t + (N/2 -\, k) \sin 2\,t + N/2, && (20)\\ k(t) &= (k - N/2) \cos 2\,t + (m - N/2) \sin 2\,t + N/2, && (21) \end{aligned}$$

which is in a direct correspondence with the association (14). The above solution resembles the familiar Liuovilian approximation, common in the standard semiclassical treatment of the phase-space evolution for quantum systems with continuous dynamical symmetries [1, 17]. In such approximation, the phase-space elements move along the classical trajectories, preserving their respective volumes. The existence of the continuous limit of the evolution equation (18) indicates that the dynamics in the measurement space becomes smooth in the macroscopic limit $N \gg 1$.

It can be appreciated from Fig. 1 that the distribution corresponding to the initial (symmetric) GHZ state in the x basis,

$$|\mathrm{GHZ}\rangle_x = \frac{|0\dots0\rangle_x + |1\dots1\rangle_x}{\sqrt{2}}, \quad \hat{\sigma}_x^{(j)}|0\rangle_x^{(j)} = |0\rangle_x^{(j)}, \quad \hat{\sigma}_x^{(j)}|1\rangle_x^{(j)} = -|1\rangle_x^{(j)}, \qquad (22)$$

approximately rotates around the axis n (z) that passes through the center of the tetrahedron.

A similar behavior (see Fig. 2) can be observed in the evolution of the following factorized nonsymmetric initial state in the x basis:

$$|\psi\rangle = |\underbrace{0\dots0}_{k}\underbrace{1\dots1}_{N-k}\rangle_x. \qquad (23)$$

In comparison, in Fig. 3, we plot the evolution of $Q_\rho(\alpha, \beta|t)$ corresponding to the initial GHZ state (22) generated by the Hamiltonian (17) in the whole $2^N \times 2^N$ discrete phase space.

3.2. The Nonlinear Evolution

The quadratic Hamiltonian

$$\hat{H} = \hat{S}_z^2 = \sum_{j \neq k}^{N} \hat{\sigma}_z^{(j)} \hat{\sigma}_z^{(k)} + N \qquad (24)$$

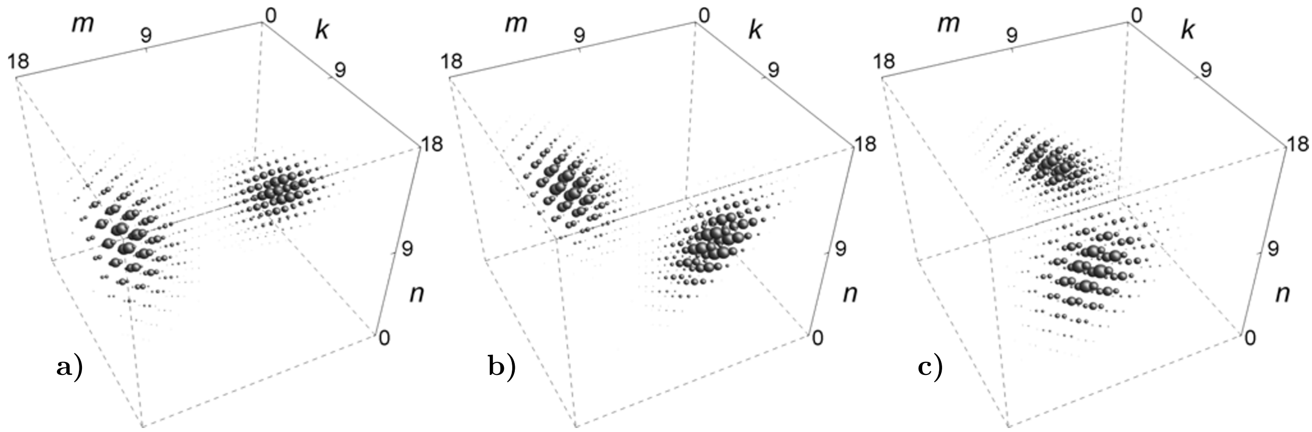

Fig. 2. The evolution of $\widetilde{Q}_\rho$-function for the initial nonsymmetric factorized state (23), $k = 4$, under the action of Hamiltonian $\hat{H} = \hat{S}_z$ at $t = 0$ (a), $t = 1$ (b), and $t = 2$ (c); $N = 18$.

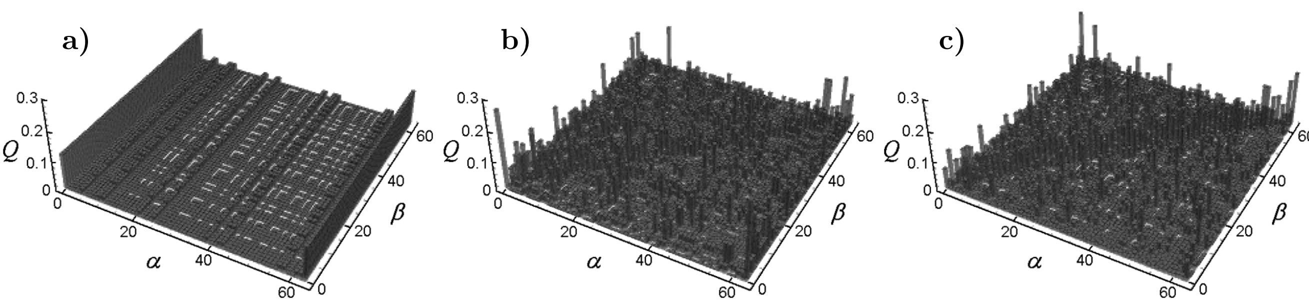

Fig. 3. The evolution of full discrete $Q(\alpha, \beta)$ distribution (1) for the initial GHZ state (22) under the action of Hamiltonian $\hat{H} = \hat{S}_z$ at $t = 0$ (a), $t = 0.5$ (b), and $t = 0.9$ (c); $N = 6$.

generates spin–spin correlations for a wide class of initial N-qubit states. The corresponding evolution equation for $\widetilde{Q}_\rho(\mathbf{m}|t)$ is significantly more involved than for the linear Hamiltonian (17); see Eq. (41) in Appendix B, the first-order approximation to this equation in the continuous limit reads

$$\partial_t \tilde{Q}_\rho(m,n,k) \approx 2\sqrt{3}u(n,k,m)\frac{\partial}{\partial m}\tilde{Q}_\rho(m,n,k) - 2\sqrt{3}u(m,n,k)\frac{\partial}{\partial k}\tilde{Q}_\rho(m,n,k), \tag{25}$$

where $u(m,n,k) = (N-2m)(N-2n) + N - 2k$. Its solution

$$\tilde{Q}_\rho(m,n,k|t) \approx \tilde{Q}_\rho(m(t),n,k(t)|t=0) \tag{26}$$

$$m(t) = \frac{N}{2} + \frac{2\sqrt{3}u(n,k,m)}{\Omega(n)}\sin(\Omega(n)t) - \frac{N-2m}{2}\cos(\Omega(n)t), \tag{27}$$

$$k(t) = \frac{N}{2} - \frac{2\sqrt{3}u(m,n,k)}{\Omega(n)}\sin(\Omega(n)t) - \frac{N-2k}{2}\cos(\Omega(n)t), \tag{28}$$

where $\Omega(n) = 4\sqrt{3}\sqrt{(N-2n)^2-1}$, describes rotations with the position-dependent frequency and amplitudes, allowing to visualize the short-time deformations of the initial distribution. In practice, the solution (26) is valid for times $\sim N^{-1/2}$ as a consequence of neglecting the higher derivatives in the

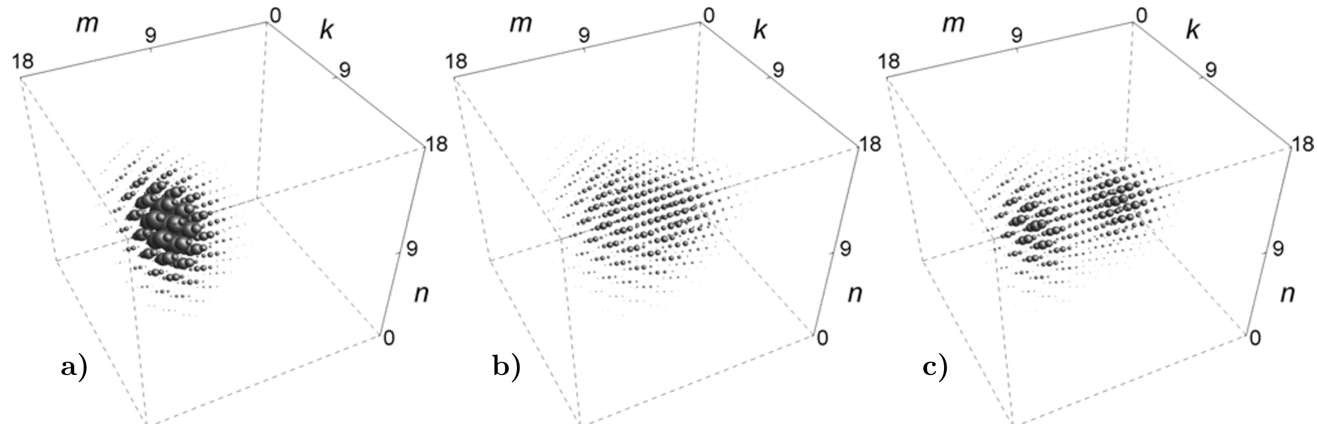

Fig. 4. The evolution of $\widetilde{Q}_\rho$-function for the initial GHZ state (22) under the action of Hamiltonian $\hat{H} = \hat{S}_z^2$ at $t = 0$ (a), $t = 0.04$ (b), and $t = 0.09$; $N = 18$ (c).

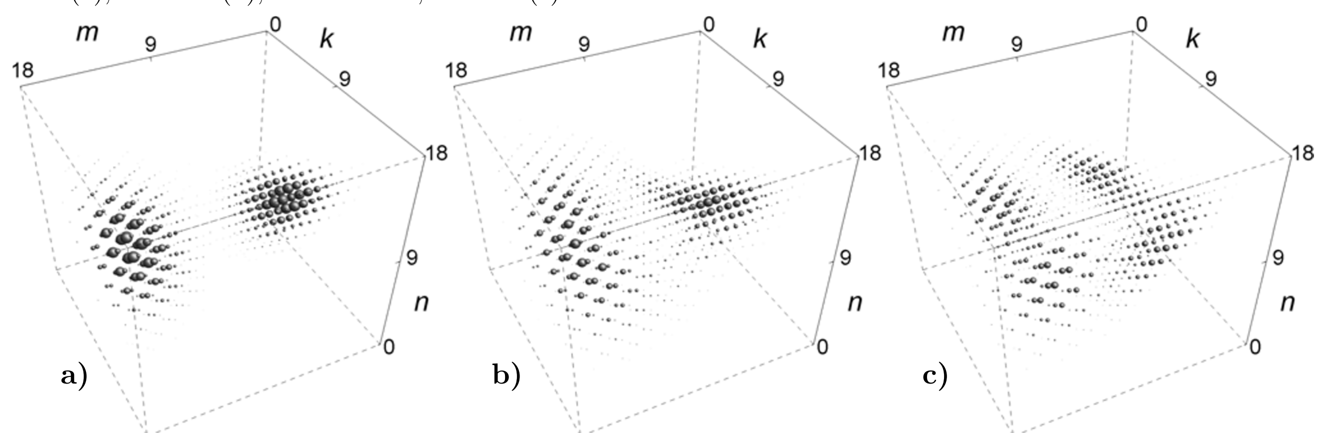

Fig. 5. The evolution of $\widetilde{Q}_\rho$-function for the initial nonsymmetric state (23); $k = 4$, under the action of Hamiltonian $\hat{H} = \hat{S}_z^2$ at $t = 0$ (a), $t = 0.2$ (b), and $t = 0.4$; $N = 18$ (c).

expansion of the exact finite-difference equation (41). Thus, Eqs. (27)–(28) can be approximated as

$$m(t) \approx m + 2\sqrt{3}h(n,k,m)t, \tag{29}$$

$$k(t) \approx k - 2\sqrt{3}h(m,n,k)t, \tag{30}$$

which is quite similar to the semiclassical dynamics of quasidistributions over the sphere S_2 [17] generated by (24).

In Figs. 4 and 5, we plot $\tilde{Q}_\rho\,(m,n,k|t)$ for states (22) and (23) for different times, where the initial diffusion described by the semiclassical approximation (26) is followed by a splitting of each maxima, i.e., generation of Schrödinger-like cat states [25]. The splitting of the distribution corresponding to the GHZ state occurs at times $t \sim \pi/32$ due to the approximate permutational symmetry of the corresponding $\tilde{Q}_\rho(m,n,k)$. The $\tilde{Q}_\rho(t)$-function of state (23) splits into two pieces at $t \sim \pi/8$, as expected [25].

4. Conclusions

The analysis of the evolution in the measurement space provides a useful insight into the global properties of multipartite dynamics. The time-evolved $\tilde{Q}\,(\mathbf{m}|t)$-function represents the complete "collective dynamics" and not only the evolution of a particular observable. The framework provided here allows one, in particular, to visualize the evolution of nonsymmetric states in the macroscopic limit $N \gg 1$ as a quasicontinuous motion in the (discrete) measurement space $\mathcal{M}$. In addition, the short-time dynamics can be described in a "semiclassical" approximation, where every point of the initial discrete distribution propagates along a "classical" trajectory in $\mathcal{M}$. Such a behavior of $\tilde{Q}_\rho\,(\mathbf{m}|t)$-functions resembles the dynamics of continuous distributions (on classical manifolds) and is fundamentally different from that of the distribution functions (1) in the full $2^N\times 2^N$ discrete phase space. It is worth noting that it is possible to obtain a closed form of the (discrete) evolution equation for $\tilde{Q}_\rho\,(\mathbf{m}|t)$-function only for symmetric Hamiltonians.

Acknowledgments

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5. Appendix A

The explicit form of the kernel (16) is [14]

$$\hat{\Delta}^{(-1)}(\mathbf{m}) = \frac{1}{2^N}\sum_{\gamma,\delta,\alpha,\beta}\chi(\alpha\delta+\beta\gamma+\gamma\delta)\langle\hat{Z}_\gamma\hat{X}_\delta\rangle\hat{Z}_\gamma\hat{X}_\delta\delta_{\mathbf{h},\mathbf{m}}, \quad (31)$$

$$\delta_{\mathbf{h},\mathbf{m}} = \delta_{h(\alpha),m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta),k}, \qquad \langle\hat{Z}_\gamma\hat{X}_\delta\rangle = \langle\xi|\hat{Z}_\gamma\hat{X}_\delta|\xi\rangle, \quad (32)$$

where $\hat{Z}_\alpha = \hat{\sigma}_z^{a_1}\otimes\ldots\otimes\hat{\sigma}_z^{a_N}$; $\alpha=(a_1,\ldots,a_N)$ and $\hat{X}_\beta = \hat{\sigma}_x^{b_1}\otimes\ldots\otimes\hat{\sigma}_x^{b_N}$; $\beta=(b_1,\ldots,b_N)$, with $a_j,b_j\in\mathbb{Z}_2$ operators acting in the standard way in the computational basis $\{|\kappa\rangle = |k_1,\ldots,k_N\rangle, k_i\in\mathbb{Z}_2\}$ in $\mathcal{H}_{2^N} = \mathcal{H}_2^{\otimes N}$ [5],

$$\hat{Z}_\alpha|\kappa\rangle = (-1)^{\alpha\kappa}|\kappa\rangle, \quad \hat{X}_\beta|\kappa\rangle = |\kappa+\beta\rangle, \quad (33)$$

the multiplication and sum are mode 2 operations, $\alpha\kappa = a_1k_1+\cdots+a_Nk_N\in\mathbb{Z}_2$ and $\kappa+\beta = (b_1+k_1,\ldots,b_N+k_N)$. It is convenient to consider the indices that label both states and operators acting in $\mathcal{H}_{2^N}$ with elements of the finite field $\mathbb{F}_{2^N}$.

The right multiplication of the collective operator $\hat{S}_z = \sum_{p=1}^N \hat{Z}_{\sigma_p}$, where $\{\sigma_p, p=1,\ldots,N\}$ are elements of a self-dual basis in $\mathbb{F}_{2^N}$, on the kernel (31) is transformed in the following way:

$$\begin{aligned}\hat{\Delta}^{(-1)}(\mathbf{m})\hat{S}_z &= \frac{1}{2^N}\sum_{\gamma,\delta,\alpha,\beta}\sum_p\chi\left[\alpha\delta+\beta\gamma+(\gamma+\sigma_p)\delta\right]\langle\hat{Z}_\gamma\hat{X}_\delta\rangle\hat{Z}_{\gamma+\sigma_p}\hat{X}_\delta\delta_{\mathbf{h},\mathbf{m}}\\ &= \frac{1}{2^N}\sum_{\gamma,\delta,\alpha,\beta}\sum_p\chi(\alpha\delta+\beta\gamma+\gamma\delta+\beta\sigma_p)\langle\hat{Z}_{\gamma+\sigma_p}\hat{X}_\delta\rangle\hat{Z}_\gamma\hat{X}_\delta\delta_{\mathbf{h},\mathbf{m}} \quad (34)\\ &= \frac{1}{2^N}\sum_{\gamma,\delta,\alpha,\beta}\sum_p\chi(\alpha\delta+\beta\gamma+\gamma\delta+\beta\sigma_p)\langle\hat{Z}_\gamma\hat{X}_\delta\rangle,\left[(3)^{\gamma_p}\frac{(i\sqrt{3})^{\delta_p}}{\sqrt{3}}\left(-\frac{1}{3}\right)^{\gamma_p\delta_p}\right]\hat{Z}_\gamma\hat{X}_\delta\delta_{\mathbf{h},\mathbf{m}}\end{aligned}$$

$$= \frac{1}{2^N\sqrt{3}} \sum_{\gamma,\delta,\alpha,\beta} \chi(\alpha\delta+\beta\gamma+\gamma\delta)\langle \hat{Z}_\gamma \hat{X}_\delta\rangle \hat{Z}_\gamma \hat{X}_\delta \sum_p (-1)^{\beta_p}\delta_{\mathbf{h},\mathbf{m}} \tag{35}$$

$$+ \frac{1-i\sqrt{3}}{2\sqrt{3}}\frac{1}{2^N} \sum_{\gamma,\delta,\alpha,\beta} \chi(\alpha\delta+\beta\gamma+\gamma\delta)\langle \hat{Z}_\gamma \hat{X}_\delta\rangle \hat{Z}_\gamma \hat{X}_\delta \times \sum_p (-1)^{\beta_p}\delta_{h(\alpha),m}\delta_{h(\beta+\sigma_p),n}\delta_{h(\alpha+\beta+\sigma_p),k} \tag{36}$$

$$+ \frac{1}{2^N\sqrt{3}} \sum_{\gamma,\delta,\alpha,\beta} \chi(\alpha\delta+\beta\gamma+\gamma\delta)\langle \hat{Z}_\gamma \hat{X}_\delta\rangle \hat{Z}_\gamma \hat{X}_\delta \times \sum_p (-1)^{\beta_p}\delta_{h(\alpha+\sigma_p),m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta+\sigma_p),k} \tag{37}$$

$$+ \frac{1+i\sqrt{3}}{2\sqrt{3}}\frac{1}{2^N} \sum_{\gamma,\delta,\alpha,\beta} \chi(\alpha\delta+\beta\gamma+\gamma\delta)\langle \xi| Z_\gamma X_\delta|\xi\rangle \hat{Z}_\gamma \hat{X}_\delta \times \sum_p (-1)^{\beta_p}\delta_{h(\alpha+\sigma_p),m}\delta_{h(\beta+\sigma_p),n}\delta_{h(\alpha+\beta),k}. \tag{38}$$

Here, δ_p and γ_p denote the expansion coefficients of elements $\delta, \gamma \in \mathbb{F}_{2^N}$ over a self-dual basis $\{\sigma_p,\, p = 1,\ldots,N\}$; $\gamma = \sum_p \gamma_p\sigma_p$.

The sum in (35) can be easily computed,

$$\sum_p (-1)^{\beta_p}\delta_{h(\alpha),m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta),k} = (N-2h(\beta))\delta_{h(\alpha),m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta),k}.$$

The sum in (36) is transformed as follows:

$$\sum_p (-1)^{\beta_p}\delta_{h(\alpha),m}\delta_{h(\beta+\sigma_p),n}\delta_{h(\alpha+\beta+\sigma_p),k} = \sum_p (1-2\beta_p)\delta_{h(\alpha),m}\delta_{h(\beta)+1-2\beta_p,n}\delta_{h(\alpha+\beta)+1-2(\alpha_p+\beta_p-2\alpha_p\beta_p),k}$$

$$= \sum_p (1-\alpha_p)(1-\beta_p)(1-2\beta_p)\delta_{h(\alpha),m}\delta_{h(\beta)+1,n}\delta_{h(\alpha+\beta)+1,k} \tag{39}$$

$$+\sum_p \alpha_p(1-\beta_p)(1-2\beta_p)\delta_{h(\alpha),m}\delta_{h(\beta)+1,n}\delta_{h(\alpha+\beta)-1,k} + \sum_p (1-\alpha_p)\beta_p(1-2\beta_p)\delta_{h(\alpha),m}\delta_{h(\beta)-1,n}\delta_{h(\alpha+\beta)-1,k}$$

$$+\sum_p \alpha_p\beta_p(1-2\beta_p)\delta_{h(\alpha),m}\delta_{h(\beta)-1,n}\delta_{h(\alpha+\beta)+1,k}$$

$$= N\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k} + \frac{1}{2}h(\alpha)\delta_{h(\alpha)-1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)-1,k} - \frac{1}{2}h(\alpha)\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k}$$

$$-\frac{1}{2}h(\beta)\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)-1,k} - \frac{1}{2}h(\beta)\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k} - \frac{1}{2}h(\alpha)\delta_{h(\alpha)-1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k}$$

$$+\frac{1}{2}h(\alpha)\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)-1,k} - \frac{1}{2}h(\beta)\delta_{h(\alpha)-1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)-1,k} - \frac{1}{2}h(\beta)\delta_{h(\alpha)-1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k}$$

$$+\frac{1}{2}h(\alpha+\beta)\delta_{h(\alpha)-1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)-1,k} + \frac{1}{2}h(\alpha+\beta)\delta_{h(\alpha)-1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k}$$

$$-\frac{1}{2}h(\alpha+\beta)\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)-1,k} - \frac{1}{2}h(\alpha+\beta)\delta_{h(\alpha)+1,m}\delta_{h(\beta),n}\delta_{h(\alpha+\beta)+1,k}.$$

Computing in a similar way the sum in (37)–(38) and repeating the procedure for the left multiplication $\hat{S}_z\hat{\Delta}^{(-1)}(\mathbf{m})$, we arrive at the following form of the commutator:

$$\begin{aligned} & [\hat{\Delta}^{(-1)}(\mathbf{m}), \hat{S}_z] \\ = \; & \frac{i}{2}(2\,N - m - n - k + 2)(\hat{\Delta}_{m,n-1,k-1} - \hat{\Delta}_{m-1,n-1,k}) + \frac{i}{2}(n + k - m + 2)(\hat{\Delta}_{m-1,n+1,k} - \hat{\Delta}_{m,n+1,k+1}) \\ + \; & \frac{i}{2}(m + n - k + 2)(\hat{\Delta}_{m+1,n+1,k} - \hat{\Delta}_{m,n+1,k-1}) + \frac{i}{2}(m + k - n + 2)(\hat{\Delta}_{m,n-1,k+1} - \hat{\Delta}_{m+1,n-1,k}). \quad (40) \end{aligned}$$

Observe that, in the case of nonsymmetric Hamiltonians, the sums over the elements of the basis in $\mathbb{F}_{2^N}$ appearing in (36)–(37) would not be functions of the weights (9). Thus, multiplication of nonsymmetric operators on the mapping kernel (31) is not reduced to the operations over its indices, as in (40).

6. Appendix B

Proceeding in a similar way as in the case of the linear Hamiltonian, we obtain the following equation for $\hat{H} = \hat{S}_z^2$:

$$\partial_t \tilde{Q}_\rho(m,n,k) = \sum_{a,b,c=-2}^{2} g_{mnk}(a,b,c)\tilde{Q}_\rho(m+a,n+b,k+c), \quad (41)$$

where

$$g_{m,n,k}(-2,-2,0) = -\frac{\sqrt{3}}{12}(2\,N - n + 4 - m - k)(2\,N - m - n + 2 - k),$$

$$\begin{aligned} & g_{m,n,k}(-2,-1,-1) = 2g_{m,n,k}(-2,-2,0), && g_{m,n,k}(-2,1,1) = 2g_{m,N-n,N-k}(-2,-2,0), \\ & g_{m,n,k}(-1,-1,-2) = -2g_{m,n,k}(-2,-2,0), && g_{m,n,k}(0,-2,-2) = -g_{m,n,k}(-2,-2,0), \\ & g_{m,n,k}(0,-2,2) = -g_{N-m,n,N-k}(-2,-2,0), && g_{m,n,k}(2,1,-1) = -2g_{N-m,N-n,k}(-2,-2,0), \\ & g_{m,n,k}(-2,2,0) = g_{m,N-n,N-k}(-2,-2,0), && g_{m,n,k}(-1,1,2) = -2g_{m,N-n,N-k}(-2,-2,0), \\ & g_{m,n,k}(0,2,-2) = g_{N-m,N-n,k}(-2,-2,0), && g_{m,n,k}(0,2,2) = -g_{m,N-n,N-k}(-2,-2,0), \\ & g_{m,n,k}(1,-1,2) = -2g_{m,n,k}(-2,-2,0), && g_{m,n,k}(1,1,-2) = 2g_{N-m,N-n,k}(-2,-2,0), \\ & g_{m,n,k}(2,-2,0) = g_{N-m,n,N-k}(-2,-2,0), && g_{m,n,k}(2,-1,1) = 2g_{N-m,n,N-k}(-2,-2,0), \\ & g_{m,n,k}(2,2,0) = -g_{N-m,N-n,k}(-2,-2,0), \end{aligned}$$

$$g_{m,n,k}(-2,-1,1) = \frac{\sqrt{3}}{6}(2\,N - m - n + 2 - k)(n + 2 + k - m),$$

$$\begin{aligned} & g_{m,n,k}(-2,0,0) = g_{m,n,k}(-2,-1,1), && g_{m,n,k}(-2,1,-1) = g_{m,n,k}(-2,-1,1), \\ & g_{m,n,k}(-1,-1,2) = -g_{N-k,n,N-m}(-2,-1,1), && g_{m,n,k}(-1,1,-2) = -g_{k,n,m}(-2,-1,1), \\ & g_{m,n,k}(2,-1,-1) = g_{N-m,n,N-k}(-2,-1,1), && g_{m,n,k}(0,0,-2) = -g_{k,n,m}(-2,-1,1), \\ & g_{m,n,k}(0,0,2) = -g_{N-k,n,N-m}(-2,-1,1), && g_{m,n,k}(1,-1,-2) = -g_{k,n,m}(-2,-1,1), \\ & g_{m,n,k}(1,1,2) = -g_{N-k,n,N-m}(-2,-1,1), && g_{m,n,k}(2,0,0) = g_{N-m,n,N-k}(-2,-1,1), \\ & g_{m,n,k}(2,1,1) = g_{N-m,n,N-k}(-2,-1,1), \end{aligned}$$

$$g_{m,n,k}(-1,-1,0) = -\frac{\sqrt{3}}{3}(N-m-n+1)(2N-m-n+2-k),$$

$$g_{m,n,k}(-1,1,0) = g_{m,N-n,N-k}(-1,-1,0), \qquad g_{m,n,k}(0,-1,-1) = -g_{k,n,m}(-1,-1,0),$$

$$g_{m,n,k}(0,-1,1) = -g_{N-k,n,N-m}(-1,-1,0), \qquad g_{m,n,k}(0,1,-1) = -g_{k,N-n,N-m}(-1,-1,0),$$

$$g_{m,n,k}(0,1,1) = -g_{N-k,N-n,m}(-1,-1,0), \qquad g_{m,n,k}(1,-1,0) = g_{N-m,n,N-k}(-1,-1,0),$$

$$g_{m,n,k}(1,1,0) = g_{N-m,N-n,k}(-1,-1,0), \qquad g_{m,n,k}(0,0,0) = -\frac{2\sqrt{3}}{3}(N-k-m)(k-m).$$

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