AGAINST IDENTIFICATION OF CONTEXTUALITY WITH VIOLATION OF THE BELL INEQUALITIES: LESSONS FROM THEORY OF RANDOMNESS

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Abstract

Nowadays, contextuality is the hottest topic of quantum foundations and, especially, foundations of quantum information theory. This notion is characterized by the huge diversity of approaches and interpretations. One of the strongest trends in contextual research is to identify contextuality with *Bell test contextuality* (BTC). In this paper, we criticize the BTC approach. It can be compared with an attempt to identify the complex and theoretically nontrivial notion of randomness with a test for randomness (or a batch of tests, as the NIST test). We advertise *Bohr contextuality* taking into account all experimental conditions (context). In the simplest case, the measurement context of an observable A is reduced to joint measurement with a compatible observable B. The latter definition was originally considered by Bell in relation to his inequality. We call it *joint measurement contextuality* (JMC). Although JMC is based on the use of counterfactuals, by considering it within the general Bohr's framework, it is possible to handle JMC on physical grounds. We suggest (similarly to randomness) to certify JMC in experimental data with Bell tests, but only certify and not reduce.

Keywords: contextuality, Bell inequality, Bohr contextuality, randomness, tests for randomness, tests for contextuality.

1.   Introduction

Already Bell pointed out [1] that, in explanation of violation of Bell-type inequalities [1–3], contextuality is an important alternative to nonlocality. The measurement context of an observable A was coupled to joint measurement with a compatible observable B – *joint measurement contextuality* (JMC). Thus, by struggling with the nonlocal interpretation of quantum mechanics[1] one can follow Bell and appeal to contextuality [1], in the JMC meaning.

Unfortunately, in modern quantum foundations, especially, in quantum information theory, contextuality is typically not identified with original Bell's JMC. The theoretically complex and rich notion of contextuality was reduced to one special empirical test, namely, violation of the Bell-type inequalities – *Bell test contextuality* BTC). Within this framework, they are known as noncontextual inequalities [7–10]. Of course, one can proceed in this way and be completely fine just by deriving and testing new Bell-type

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[1]Quantum nonlocality and spooky action at a distance mystify quantum theory. From author's viewpoint [4,5], quantum nonlocality is only apparent, and one can get rid of this misleading notion from quantum theory through the consistent use of Bohr's principle of complementarity [6].

(noncontextual) inequalities. However, I am critical to this pragmatic handling of quantum contextuality. It can be compared with an attempt to identify the complex and theoretically nontrivial notion of randomness with one concrete test of randomness (or a few tests) and ignoring deep studies on the notion of randomness, started by von Mises and continued by Church, Solomonoff, Kolmogorov, Chatin, and Martin-Löv; see review [11] or book [12] for the physicists' friendly presentation.

Since long time ago (see, e.g., [13–21]), I have advertised the notion of contextuality, which was considered by Bohr in his formulation of the *complementarity principle* [22–24]. Bohr did not use the word "contextuality" (neither Bell); he wrote about complexes of experimental physical conditions. JMC, which role was emphasized by Bell, is a special case of *Bohr's contextuality*. And within the Bohr framework, JMC can be handled on the physical (and not metaphysical) grounds. In turn, Bohr's contextuality can be rigorously formulated within the theory of open quantum systems; see my recent paper [21]. Then BTC is just one special class of tests for JMC and generally Bohr's contextuality as, say, NIST tests for randomness. (In this paper, our aim is not to diminish the role of experimental tests, neither in theory of randomness nor contextuality.) We shall discuss randomness vs. contextuality tests in Secs. 5 and 6.

We also recall that Bell appealed to contextuality in an attempt to suggest an explanation of violation of his inequality, different from nonlocality; see Sec. 3 for details. Therefore, *it is meaningless to reduce contextuality to BTC, i.e., to explain violation of the Bell-type inequalities by their violation.*

Quantum contextuality is characterized by diversity of approaches, which are not reduced to Bohr, joint measurement, and Bell test contextualities. However, in this paper, we restrict our analysis only to these three approaches. In particular, we do not consider Kohen–Specker contextuality; see the recent paper of Svozil [25] for detailed review on the notion of contextuality.

2.   Joint Measurement Contextuality (JMC)

In the discussion on possible seeds of violation of his inequality, Bell argued [1] that "the result of an observation may reasonably depend not only upon the state of the system, including the hidden variables, but also on the complete disposition of the apparatus." Then Shimony [26] emphasized that this is the first statement about contextuality, although Bell did not use this terminology.[2] In fact, Bell's statement is closely coupled with Bohr's emphasis of the role of experimental arrangement; see Sec. 3. Shimony concreted of the Bell statement on the role of experimental arrangement [26]:

"John Stewart Bell (1928–90) gave a new lease on life to the program of hidden variables by proposing contextuality. In the physical example just considered,[3] the complete state λ in a contextual hidden variables model would indeed ascribe an antecedent element of physical reality to each squared spin component s_n^2 but in a complex manner: the outcome of the measurement of s^{2n} is a function $s_n^2(\lambda, C)$ of the hidden variable λ and the context C, which is the set of quantities measured along with s_n^2. ......, a minimum constraint on the context C is that it consists of quantities that are quantum-mechanically compatible, which is represented by self-adjoint operators which commute with each other ..."

In modern literature, the latter sentence is formulated as follow:

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[2]In quantum theory, the word "contextualistic" was invented by Shimony [27] and a shortening to "contextual" was made by Beltrametti and Cassinelli [28]. It is surprising that neither Bell nor Shimony even mention Bohr. Did they read Bohr? At least Shimony, as a philosopher, should do this ...

[3]The Bohm version of the EPR experiment – the spin-projection measurements.

Definition (JMC). *If A, B, and C are three observables, such that A is compatible with B and C, a measurement of A might give different result depending upon whether A is measured with B or with C.*

Triple of observable $(A; B, C)$ with given experimental JPDs $- p_A^{\exp}, \; p_A^{\exp}, \; p_A^{\exp}, \; p_C^{\exp}, \; p_{AB}^{\exp}, \; p_{AC}^{\exp} -$ is called the JMC scenario.

Using the word "might" makes this statement counterfactual. It seems to be difficult, if possible at all, to test JMC experimentally. Nevertheless, there were published at least two articles, where authors claimed that JMC can be tested experimentally; they presented the schemes of experiments [29,30] which, unfortunately, have never been performed.

3.   Contextual Viewpoint on Bohr's Complementarity Principle

Typically physicists, even the experts in quantum foundations, consider Bohr's writings as difficult for understanding and try to "simplify" his statements. In particular, the Bohr's complementarity principle is widely known as "wave–particle duality." First, we note that Bohr by himself had never used this notion. In some degree, it is relevant to the earliest attempts of Bohr (1925) to invent complementarity to quantum physics. However, later Bohr had generalized the notion of complementarity in terms of characteristic properties of quantum measurements. In a long series of publications [6, 31, 32], I emphasized the contextual basis of this "measurement-complementarity" principle. It seems that this, in fact, basic contextual component of the complementarity principle was not understood by other experts, and quantum contextuality and complementarity go through quantum theory relatively independently. Let cite Bohr [22] (vol. 2, pp. 40-41):

"This crucial point ... implies *the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments, which serve to define the conditions under which the phenomena appear.* In fact, the individuality of the typical quantum effects finds its proper expression in the circumstance that any attempt of subdividing the phenomena will demand a change in the experimental arrangement introducing new possibilities of interaction between objects and measuring instruments, which, in principle, cannot be controlled. Consequently, evidence obtained under different experimental conditions cannot be comprehended within a single picture, but must be regarded as complementary in the sense that only the totality of the phenomena exhausts the possible information about the objects."

The contextual component of this statement can be formulated as the following principle:

Principle 1 (Contextuality) *The output of any quantum observable is indivisibly composed of the contributions of the system and the measurement apparatus. Hence, the whole experimental arrangement (context $\mathcal{C}$) should be taken into account.*

Logically, one has no reason to expect that all experimental contexts can be combined and all observables can be jointly measured. Hence, incompatible observables (complementary experimental contexts) can exist. Moreover, they should exist, otherwise the content of the contextuality principle would be empty. Really, if all experimental contexts can be combined into single context $\mathcal{C}$ and all observables can be jointly measured in this context, then the outputs of such joint measurements can be assigned directly to a system. To be more careful, we have to say: "assigned to a system and context $\mathcal{C}$." But, the latter can be omitted, since this is the same context for all observables. Thus, *contextuality is meaningful only in combination with incompatibility.*

Principle 2 (Incompatibility) *There exist observables based on complementary experimental contexts. Such observables cannot be jointly measured.*

Principle 2 is slightly modified comparing with my previous papers. Observables existing due to Principle 2 are called *incompatible*. Principles 1 and 2 can be treated as the integral **Contextuality–Incompatibility principle**. This is my understanding of Bohr's complementarity principle. In this paper, we discuss mainly the contextual component of the Contextuality–Incompatibility principle.

As was mentioned, JMC considered by Bell is a special case of the Bohr contextuality.

4. Experimental and Kolmogorovian Joint Probability Distributions

We restrict considerations to observables with finitely many values.

4.1. Experimental Joint Probability Distribution

Consider a system of physical observables $A, B, C, \ldots, D$. As we know from quantum physics [22–24, 33] and cognitive psychology (as well as decision making) [34–37], some observables can be incompatible, i.e., their joint measurement is impossible; see Sec. 3 for author's reformulation of Bohr's complementarity principle. If some subsystem of observables is jointly measurable then experimenters can determine their *joint probability distribution* (JPD); for example, let A and B be compatible, then $p_{AB}^{\exp}(a, b)$ can be determined from experiment (as the frequency of the outcome ($A = a$, $B = b$) in a long trial of measurements).[4] It is assumed that, for each observable, measurement probabilities can be determined as $p_A^{\exp}(a)$, $p_B^{\exp}(b), \ldots$

Generally, there is no consistency between the JPDs of different orders. For example, even if the observables in each pair A, B and A, C are compatible and JPDs for $p_{AB}^{\exp}(a, b)$ and $p_{AC}^{\exp}(a, c)$ are defined, there is no guarantee that the following equalities:

$$p_A^{\exp}(a) = \sum_b p_{AB}^{\exp}(a, b) = \sum_c p_{AC}^{\exp}(a, c) \qquad (1)$$

hold. If they hold, one says that *there is no-signaling* (for measurement of A jointly with B and with C) – measurements of observable A jointly with B and with C generate the same probability distribution as in measurement of solely A. (The use of the terminology "signaling" in such probabilistic formulation might be misleading, but it is widely used in quantum physics.)

We stress that *theoretical probabilities of quantum theory always satisfy to the no-signaling condition.* The role of this condition in the EPR–Bohm–Bell experiments was highlighted in our article with Adenier [38]. We found that none of the first experiments demonstrating the violation of Bell inequalities satisfied to this condition, e.g., [39–41]. It was also found that the first experiment, which was claimed to be totally loophole free [42] also suffers of statistically non-negligible signaling [43]. Theoretical analysis of this condition was performed in detail in articles of Dzhafarov et al. [44]. The classical conditional probabilistic analysis of the condition of no-signaling was done in [45].

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[4]To make this definition mathematically rigorous, one must consider infinite sequence of trials; see Sec. 5.

4.2.   Classical Probability: Observables as Random Variables

Consider a system of observables, e.g., physical, $A, B, C, \ldots, D$ with given experimental probability distributions for some of its subsystems. The natural questions arises: *Is it possible to describe them by classical probability theory (CP)?* Here, "to describe" means "in such a way that all experimental probability distributions would match the theoretical ones given by CP."

For reader's convenience, we recall the notion of a *probability space.* It was invented by Kolmogorov [46] in 1933; it serves as the mathematical basis of the classical probability theory. The probability space is a triple $\mathcal{P} = (\Lambda, \mathcal{F}, p)$, where Λ is the set of random parameters,[5] $\mathcal{F}$ is the collection of subsets of Λ representing events (this is an event algebra), and p is the probability measure on $\mathcal{F}$. A *random variable* is a function $\xi : \Lambda \to \mathbf{R}$ such that, for each half-interval $(a, b]$, the set $\{\lambda \in \Lambda : a < \xi(\lambda) \le b\}$ is an event, i.e., it belongs to the set system $\mathcal{F}$.

In applications of CP to physics, an observable A is described by a random variable $A = A(\lambda)$. Consider now compatible, e.g., physical, observables $B_1, \ldots, B_n$. They can be represented by random variables $B_1 = B_1(\lambda), \ldots, B_n = B_n(\lambda)$. Their joint probability distribution (JPD) is well defined,

$$p_{B_1 \cdots B_n}(b_1, \ldots, b_n) = p(\lambda \in \Lambda : B_1(\lambda) = b_1, \ldots, B_n(\lambda) = b_n\}.$$

Any subsystem of these observables is also jointly measurable, and the corresponding experimental JPDs approximately are equal to the theoretical ones,

$$p_{B_1}(b_1) \approx p_{B_1}^{\exp}(b_1), \ldots,$$

$$p_{B_1 B_2}(b_1, b_2) \approx p_{B_1 B_2}^{\exp}(b_1, b_2), \ldots,$$

and so on.

Theoretical and, hence, experimental JPDs for subsystems of observables can be obtained from $p_{B_1 \cdots B_n}(b_1, \ldots, b_n)$ as the marginal distribution. For example,

$$p_{B_1}(b_1) = \sum_{b_2 \cdots b_n} p_{B_1 \cdots B_n}(b_1, \ldots, b_n), \ldots$$

$$p_{B_1 B_2}(b_1, b_2) = \sum_{b_3 \cdots b_n} p_{B_1 \cdots B_n}(b_1, \ldots, b_n), \ldots$$

and so on. We remark that the CP description implies consistency of JPDs of different orders; for example,

$$p_{B_1}(b_1) = \sum_{b_2} p_{B_1 B_2}(b_1, b_2) = \cdots = \sum_{b_2} p_{B_1 B_n}(b_1, b_n).$$

Hence, there is no-signaling in the CP model for observables.

Now, turn to the general scheme of Sec. 4.1. If physical observables $A, B, C, \ldots, D$ are compatible, they are CP representable and have JPD, which approximately coincides with the experimental JPD. Consider now the situation, where only some groups of these observables are compatible and for them experimental JPDs can be determined. We are interested in the following question: *Is it possible to*

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[5]In mathematical literature, typically symbol Ω is used, instead of symbol Λ. Points of Ω are called elementary events. They can be interpreted as realizations of random parameters. In quantum physics, random parameters are known as hidden variables.

represent observables $A, B, C, \ldots, D$ by random variables $A = A(\lambda)$, $B = B(\lambda)$, $C = C(\lambda)$, $\ldots, D = D(\lambda)$ on the same probability space consistently with experimental JPDs? This is the problem of the existence of hidden variables. It is very complicated. Its solution for the special case (CHSH framework) was given by Fine's theorem [47, 48]; see Sec. 6.1. The Suppes–Zanotti theorem [49] gives the solution for the other special case – the original Bell framework for correlated observables [1, 2]; see Sec. 6.2.

4.3. Existence of Triple JPD as Noncontextuality Test

Consider now JMC scenario $(A; B, C)$, which can realized within the CP framework, i.e., observables can be represented by random variables $A = A(\lambda)$, $B = B(\lambda)$, $C = C(\lambda)$, and their JPD p_{ABC} is consistent with the experimental probabilities, $p_A^{\exp}$, $p_B^{\exp}$, $p_C^{\exp}$; $p_{AB}^{\exp}$, $p_{AC}^{\exp}$. Within the CP framework, the vector of two random variables (A, B) mathematically represents the joint measurement of the corresponding observables with the outcome (a, b), where $(a, b) = \big(A(\lambda), B(\lambda)\big)$.

We point out that the value $a = A(\lambda)$ depends only on λ ("hidden variable"). It does not depend on whether random variable A is considered as a coordinate of the random vector (A, B) or the random vector (A, C). Hence, the possibility of CP representation consistent with experimental probabilities is a sufficient condition of noncontextuality. This framework will be considered (Secs. 6.1 and 6.2) for testing the JPD existence – to reject the JMC hypothesis.

5. Randomness: Notion versus Test

The randomness' studies were initiated by von Mises [50–52]. He introduced the notion of a collective, which later on was formalized as the notion of a random sequence. Let $L = \{\alpha_1, \ldots, \alpha_m\}$ be a set of all possible outcomes of some random experiment – labels in von Mises' terminology; for example, coin tossing with $L = \{0, 1\}$. A sequence

$$x = (x_1, x_2, \ldots, x_n, \ldots), x_j \in L \tag{2}$$

of experiment's outcomes in a long series of trials is called a collective, if it satisfies the following two principles:

- statistical stabilization;

- randomness.

By the first principle for each $\alpha \in L$, there exists the limit

$$p(\alpha; x) = \lim_{N \to \infty} n_N(\alpha)/N, \tag{3}$$

where $n_N(\alpha)$ is the number of $x_j = \alpha$ in the initial block of x of the length N; $(x_1, x_2, \ldots, x_N)$. Per definition, this limit is the probability of the outcome α. But, in this paper, we are mainly interested in the notion of randomness.

Randomness was defined as the limit stability w.r.t. to the special class of selection of subsequences in x, so-called place selections [50]: "a subsequence has been derived by a place selection, if the decision to retain or reject the nth element of the original sequence depends on the number n and on label values

$x_1, \ldots, x_{n-1}$ of the $n-1$ preceding elements, and not on the label value of the nth element or any following element."

Thus, a place selection can be defined by a set of functions

$$F = \{f_1, f_2(x_1), f_3(x_1, x_2), f_4(x_1, x_2, x_3), \ldots, f_n(x_1, \ldots, x_{n-1}), \ldots\}; \tag{4}$$

each function yielding the values 0 (rejecting the nth element) or 1 (retaining the nth element). Since any place selection should produce from an infinite input sequence also an infinite output sequence, it should also satisfy the following restriction:

$$f_n(x_1, \ldots, x_{n-1}) = 1 \text{ for infinitely many } n. \tag{5}$$

Here, there are some examples of place selections:

- choose those x_n for which n is prime;

- choose those x_n which follow the word 01;

- toss a (different) coin; choose x_n if the nth toss yields heads.

Each place selection F is a test of randomness. If a sequence x does not pass some F test, it should be rejected, such a sequence is non-random. To be random, sequence x should pass all place selection tests.

However, it did not work so simply; one should formalize the notion of place selection rule more rigorously; see [11, 12]. This was done by Church and led to the theory of algorithms. However, some Mises–Church random sequences have counter-intuitive properties. The final theory of randomness tests was proposed by Martin Löf.[6] Kolmogorov suggested another approach to randomness based on the notion of algorithmic complexity. This approach was formalized by Chatin. As was shown by Schnorr, a sequence is Martin Löf random if and only if it is Kolmogorov–Chatin random.

Of course, one cannot apply to the concrete sequence x all possible tests for randomness. In applications, people use some batch of tests, e.g., NIST tests.

Mathematically situation is more interesting. Martin Löf showed that there exists the *universal test for randomness*. And a sequence is random per definition, if it passes this test. However, this proof is not constructive. The result on the existence of the universal test is similar to the result of Solomonoff and Kolmogorov on the existence of the optimal algorithm used to define the algorithmically random sequence. But, this proof is neither constructive.

6.   Contextuality: Notion versus Test

Now we turn to JMC. One can find similarity between testing JMC with various Bell-type inequalities and testing randomness with various tests for randomness. The crucial difference is that, in the latter, Solomonoff, Kolmogorov, and Martin Löf were able to prove the existence of the optimal algorithm and the universal test. This makes the theory mathematically rigorous. (The impossibility of constructive proofs reminds me the counterfactual nature of JMC.)

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[6]In 1964 and 1965, Martin Löf studied in Moscow under the supervision of Kolmogorov. After coming back to Sweden, Martin Löf formalized the discussions with his supervisor.

6.1. CHSH Test

Consider now dichotomous observables taking values ± 1 and the quadrupole JMC scenario given by four triples (A_i, B_1, B_2), (B_i, A_1, A_2); $i = 1, 2$. Each observable A_i is compatible with both observables B_i, $i = 1, 2$, and each observable B_i is compatible with both observables A_i; $i = 1, 2$. Thus, their pairwise JPDs are well defined, as well as the probability distribution for each observable. *No-signaling condition is assumed.*

Now, let us compose the CHSH correlation

$$\mathcal{B}_{A_1 A_2; B_1 B_2} = \langle A_1, B_1 \rangle + \langle A_1, B_2 \rangle + \langle A_2, B_1 \rangle - \langle A_2, B_2 \rangle. \tag{6}$$

Denote by σ some permutation inside A and B blocks or permutation of the blocks and consider the corresponding σ-correlations $\mathcal{B}_{\sigma(A_1 A_2; B_1 B_2)}$. Consider the inequality

$$\max_\sigma |\mathcal{B}_{\sigma(A_1 A_2; B_1 B_2)}| \le 2. \tag{7}$$

Due to Fine's theorem [47, 48], there exists the quadrupole JPD $p_{A_1 A_2; B_1 B_2}(a_1, a_2, b_1, b_2)$ matching the given experimental probabilities if and only if inequality (7) holds true.

Matching of experimental probabilities has the form

$$p_{A_1 B_1}^{\exp}(a_1, b_1) = \sum_{a_2, b_2} p_{A_1 A_2; B_1 B_2}(a_1, a_2, b_1, b_2), \dots$$

$$p_{A_1}^{\exp}(a_1) = \sum_{a_2, b_1, b_2} p_{A_1 A_2; B_1 B_2}(a_1, a_2, b_1, b_2), \dots$$

and so on.

Existence of this JPD can be considered as mathematical confirmation of noncontextuality.

The Bell test in the CHSH form generates the four pairs of sequences of outcomes ± 1

$$(x_{A_1 j}, y_{B_1 j}), \; (x_{A_1 i}, y_{B_2 i}), \; (x_{A_2 k}, y_{B_1 k}), \; (x_{A_2 m}, y_{B_2 m}). \tag{8}$$

One puts these four sequences in the expression for the CHSH correlation and its permutations and check condition (7). If it is satisfied, then the experimentally generated quadrupole sequence (8) is rejected, it is not contextual (in the JMC sense). If inequality (7) is violated, then we say that quadrupole sequence (8) passed the CHSH test for JMC contextuality (but nothing more). Thus, the pioneer experiments of Aspect et al. [39, 40] and Weihs [41] as well as the 2015 experiments [42, 55, 56] showed that quadrupole sequences obtained in them passed the CHSH test for contextuality.

6.2. Original Bell's Inequality Test

6.2.1. Suppes–Zanotti Theorem

Following Suppes and Zanotti [49], consider three observables X_1, X_2, and X_3 taking values ± 1 and having zero averages, $EX_j = 0$. Assume that they are pairwise compatible. Observables in each pair X_1, X_2; X_1, X_3; X_2, X_3 can be jointly measurable and, hence, their JPDs are well defined, $p_{X_1 X_2}^{\exp}$, $p_{X_1 X_3}^{\exp}$, $p_{X_2 X_3}^{\exp}$; of course, it is assumed that they are separately measurable, and their probability distributions are well defined $p_{X_1}^{\exp}$, $p_{X_2}^{\exp}$, $p_{X_3}^{\exp}$. *Condition of no-signaling is assumed.*

Under the assumption that JPD $p_{X_1 X_2 X_3}$ consistent with the experimental probabilities exists, Suppes and Zanotti [49] derived the following inequality:

$$-1 \leq \langle X_1 X_2 \rangle + \langle X_2 X_3 \rangle + \langle X_1 X_3 \rangle. \tag{9}$$

Typically, it is identified with the original Bell inequality [1, 2]. But such straightforward interpretation is ambiguous. If the observables B_1, B_2, ..., B_n are quantum and pairwise jointly measurable, then they are jointly measurable, and their JPD always exists, being given by Born's rule,

$$p^{\text{theor}}_{B_1 \cdots B_n}(b_1, \ldots, b_n) = \operatorname{Tr} \hat{\rho} \; \hat{P}^{b_1}_{B_1} \cdots \hat{P}^{b_n}_{B_n}, \tag{10}$$

where $(\hat{P}^{b_j}_{B_j})$ are the projectors onto the corresponding eigensubspaces of the operators $\hat{B}_j$, with eigenvalues b_j.

We will be back to the analysis of "original Bell versus Suppes–Zanotti inequalities." Now we formulate the Suppes–Zanotti theorem on the existence of triple JPD.

Theorem. *Under the above conditions on observables* X_1*,* X_2*, and* X_3*,, a necessary and sufficient condition for the existence of a triple JPD is that the following two inequalities should satisfy* [49]:

$$-1 \leq \langle X_1 X_2 \rangle + \langle X_2 X_3 \rangle + \langle X_1 X_3 \rangle \leq 1 + 2\min\{\langle X_1 X_2 \rangle, \langle X_2 X_3 \rangle, \langle X_1 X_3 \rangle\}. \tag{11}$$

6.2.2. From Suppes–Zanotti Inequality to Original Bell Inequality

The *original Bell inequality* has the form [1, 2]

$$\langle A_1 B_1 \rangle - \langle A_2 B_2 \rangle + \langle A_1 B_2 \rangle \leq 1. \tag{12}$$

Here, observable A_1 should be compatible with observables B_1 and B_2 and observable A_2, with observable B_2 (this is the minimal constraint; typically one assumes that A_2 is also compatible with B_1, but the latter is not needed). The crucial condition for its derivation is the precise correlation condition [1, 2],

$$\langle A_2, B_1 \rangle = 1 \tag{13}$$

(or precise anticorrelation). This inequality differs crucially from other Bell-type inequalities, because of the correlation constraint (13).

To match with the Suppes–Zanotti inequality, we first write (12) as

$$-1 \leq -\langle A_1 B_1 \rangle + \langle A_2 B_2 \rangle - \langle A_1 B_2 \rangle, \tag{14}$$

then we change A_1 to $-A_1$ and obtain

$$-1 \leq \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_2 \rangle. \tag{15}$$

Finally, using the correlation condition (13), we transform (14) into the Suppes–Zanotti inequality (9). But in an experimental test, we should operate with four observables A_i, B_j; $i, j = 1, 2$. So, we reformulate the Suppes–Zanotti condition (12) as

$$-1 \leq \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_2 \rangle \leq 1 + 2\min\{\langle A_1 B_1 \rangle, \langle A_1 B_2 \rangle, \langle A_2 B_2 \rangle\}. \tag{16}$$

Thus, the triple of observables A_1, B_1, and B_2 has JPD consistent with experimental probabilities if and only if inequality (14) holds.

The Bell test in the CHSH form generates the three pairs of sequences of outcomes ± 1

$$(x_{A_1 j}, y_{B_1 j}), \quad (x_{A_1 i}, y_{B_2 i}), \quad (x_{A_2 m}, y_{B_2 m}). \tag{17}$$

One puts these three sequences in $\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_2 \rangle$ and check inequality (16). If it is satisfied, then the experimentally generated triple sequence (17) is rejected as noncontextual (in the JMC sense). If inequality (16) is violated, then we say that quadrupole sequence (17) passed the CHSH test for JMC contextuality (but nothing more).

Testing of the original Bell inequality is essentially more complicated than the CHSH inequality; additional constraint (13) makes the state preparation procedure more complicated. The possibility of such an experimental test was analyzed in our paper [53], including analysis of the needed efficiency of detectors. In this paper, an analog of the Tsirelson bound for the original Bell inequality (equal to 3/2) was also found; see [54] as well. One can consider a variety of tests based on different Bell-type inequalities; see, e.g., [57–61].

7.   Concluding Remarks

In this paper, we criticize identification of BTC with quantum contextuality. In the same way, as the notion of randomness is not reduced to a concrete test or a batch of tests, the notion of contextuality cannot be reduced to the batch of tests based on the Bell-type inequalities. They can only be used to reject the hypothesis on noncontextuality for sequences of outcomes generated by quantum experiment. Although this is important from practical viewpoint, it cannot serve as the basis of the theory of contextuality.

The BTC pragmatism is misleading, and this way of thinking slows down development of contextuality theory. *The value of BTC for quantum foundations is questionable; BTC explains violation of the Bell inequalities by their violation.*

Acknowledgments

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