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CENTER-OF-MASS TOMOGRAPHY OF COHERENT STATES OF TWO FREE PARTICLES†

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Abstract

We study tomographic probability distributions of quantum states of two free particles. We construct the conditional tomographic probability distribution of one random variable determining any quantum state of the system of two free particles. We obtain explicit forms of the center-of-mass tomograms for coherent states of the system of moving free particles and discuss the generalization of the results to the case of several free particles.

Keywords: probability representation of quantum mechanics, symplectic tomography, quanrizer and dequantizer operators.

1. Introduction

The quantum system states are described by wave functions [1, 2] and density operators acting in the Hilbert space [3–5]. It turned out that they can be also described by standard probability distribution functions; see [6–8]. In such probability representations of quantum states for systems with continuous variables like photons, the methods of quantum tomography like optical tomography [9–12] and symplectic tomography [13] are used. For multimode systems, the center-of-mass tomography was introduced [14,15]; this probability representation of quantum states was studied in [16, 17].

Usually quantum states of free particles are described by de Broglie waves, and recently for free-particle coherent states symplectic tomograms were introduced. These tomograms are conditional probability distributions of the free particle positions measured in the phase-space reference frames with scaled and rotated axes. For two particles, these symplectic tomograms depend on two random positions measured in common phase space with two scaled and rotated axes. The center-of-mass tomograms, being dependent only on one random position determining the quantum state and introduced in [14, 15], were discussed for oscillator systems.

The aim of this work is to study the center-of-mass tomography for coherent and Fock states of two free particles usually associated with systems of two-dimensional oscillators. We follow the method of integrals of motion of free particles recently employed to consider coherent states in the probability representation of a single particle [18]. We apply this method for coherent and Fock states of two free particles, using the approach based on center-of-mass tomography.

This paper is organized as follows.

In Sec. 2, we present the quantizer–dequantizer formalism [19] for center-of-mass tomography of quantum states of two free particles, calculate the center-of-mass tomograms for coherent states of two

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free particles, obtain the center-of-mass tomographic-probability distribution of one random variable for two free particles, and present the generalization of the problem to several free-particle states. In Sec. 3, we give our conclusions and prospectives.

2. Quantuzer–Dequantizer Operators for Center-of-Mass Tomography of Two Particles

Let us introduce the operators called dequantizers

$$\hat{U}_2\left(X,\mu_1,\mu_2,\nu_1,\nu_2\right)=\delta\left(X\hat{1}-\mu_1\hat{q}_1-\mu_2\hat{q}_2-\nu_1\hat{p}_1-\nu_2\hat{p}_2\right), \tag{1}$$

where X is a random position related to the center-of-mass of the system, $\hat{q}_1$ ($\hat{p}_1$) and $\hat{q}_2$ ($\hat{p}_2$) are the position (momentum) operators of the first and second particles, respectively.

We discuss the meaning of formula (1) considering classical particles.

In classical mechanics, the classical state is described by two variables – position $q_0(t)$ and momentum $p_0(t)$ obeying to the Newton equation of motion.

In classical statistical mechanics, the states are described by the probability densities in the phase space $f(q,p,t)$ obeying to the Liouville equation; if there is no fluctuation, the function is

$$f(q,p,t)=\delta\left(q-q_0(t)\right)\,\delta\left(p-p_0(t)\right). \tag{2}$$

The symplectic tomogram reads

$$w(X\mid\mu,\nu,t)=\int\delta\big(q-q_0(t)\big)\,\delta\big(p-p_0(t)\big)\,f(q,p,t)\,dq\,dp=\delta\left(X-\mu q_0(t)-\nu p_0(t)\right), \tag{3}$$

corresponding to the Radon transform [20] of the probability density $f(q,p,t)$ in the phase space describing the state in classical statistical mechanics. The inverse Radon transform maps the tomogram, which is the conditional probability distribution of position X measured in an ensemble of reference frames in the phase space labeled by real parameters $\mu=s\cos\theta$ and $\nu=s^{-1}\sin\theta$, where θ is rotation angle of the initial axes of the position and momentum in the particle's phase space and s is the scaling parameter of these axes.

There exists the following expression for the probability density given by the inverse Radon transform:

$$f(q,p,t)=\frac{1}{(2\pi)^2}\int e^{i(X-\mu q-\nu p)}w(X\mid\mu,\nu,t)\,dX\,d\mu\,d\nu. \tag{4}$$

This relation shows that the state of classical particle can be described by tomographic map (3) since there exists an invertible map of the probability distribution of two random variables q and p onto the conditional probability distribution of one random variable X given by the Radon transform. For two particles, there exist two possibilities to construct such maps. The first possibility is given by the following formula available for symplectic tomogram:

$$w\left(X_1,X_2\mid\mu_1,\mu_2,\nu_1,\nu_2,t\right)=\int f\left(q_1,p_1,q_2,p_2,t\right)\left\{\prod_{j=1}^{2}\delta\left(X_j-\mu_jq_j-\nu_jp_j\right)\,dq_j\,dp_j\right\}. \tag{5}$$

The second possibility is provided by the formula corresponding to the new notion of state in classical mechanics – the state of two classical particles can be described by the center-of-mass tomogram discussed in quantum mechanics [14–17]. The center-of-mass tomogram of two classical particles reads

$$w\left(X \mid \mu_1,\nu_1,\mu_2,\nu_2,t\right)=\int f\left(q_1,p_1,q_2,p_2,t\right)\,\delta\left(X-\mu_1q_1-\nu_1p_1-\mu_2q_2-\nu_2p_2\right)\,dq_1\,dp_1\,dq_2\,dp_2. \tag{6}$$

If one considers the function

$$w\left(X \mid \mu_1,\nu_1,\mu_2,\nu_2,t\right)=\mathrm{Tr}\left[\hat\rho(t)\hat U_2\left(X,\mu_1,\mu_2,\nu_1,\nu_2\right)\right] \tag{7}$$

for quantum-mechanical system of two particles with dequantizer given by Eq. (1), this function is center-of-mass tomogram of the quantum state of a system of two particles; the density operator of this state is determined by quantizer

$$\hat D_2\left(X,\mu_1,\nu_1,\mu_2,\nu_2\right)=\frac{1}{(2\pi)^2}e^{i\left(X\hat 1-\mu_1\hat q_1-\mu_2\hat q_2-\nu_1\hat p_1-\nu_2\hat p_2\right)}, \tag{8}$$

in view of the relation

$$\hat\rho(t)=\int w_2\left(X \mid \mu_1,\nu_1,\mu_2,\nu_2,t\right)\hat D_2\left(X,\mu_1,\nu_1,\mu_2,\nu_2\right)\,dX\,d\mu_1 d\nu_1\,d\mu_2\,d\nu_2. \tag{9}$$

This means that the center-of-mass tomogram, being the conditional probability distribution of one random variable X which depends on extra parameters μ_1, μ_2, ν_1, and ν_2, completely determines the quantum state.

For coherent states of two free particles, the center-of-mass tomogram at $t=0$ reads

$$\begin{aligned} w_2\left(X \mid \mu_1,\nu_1,\mu_2,\nu_2,0\right) &= \mathrm{Tr}\left[\hat U_2\left(X,\mu_1,\nu_1,\mu_2,\nu_2\right)\hat\rho(0)\right] \\ &= \frac{1}{\sqrt{\mathrm{Tr}\left(\mu_1^2+\mu_2^2+\nu_1^2+\nu_2^2\right)}}\exp\left[-\frac{(X-\langle X\rangle)^2}{\mu_1^2+\mu_2^2+\nu_1^2+\nu_2^2}\right], \end{aligned} \tag{10}$$

where $\langle X\rangle=\sqrt{2}\left[\mu_1(\mathrm{Re}\,\alpha_1)+\mu_2(\mathrm{Re}\,\alpha_2)+\nu_1(\mathrm{Im}\,\alpha_1)+\nu_2(\mathrm{Im}\,\alpha_2)\right]$ and $\hat\rho(0)=\mid\alpha_1\rangle\mid\alpha_2\rangle\langle\alpha_1\mid\langle\alpha_2\mid$ is the coherent-state density operator. The tomogram of coherent states of two free particles at time t is

$$w_2\left(X \mid \mu_1,\nu_1,\mu_2,\nu_2,t\right)=\mathrm{Tr}\left[\hat u(t)\hat\rho(0)\hat u^\dagger(t)\hat U_2\left(X,\mu_1,\nu_1,\mu_2,\nu_2\right)\right], \tag{11}$$

where the evolution operator of two-free-particle system reads

$$\hat u(t)=\exp\left[-it\left(\hat p_1^2+\hat p_2^2\right)/2\right]; \tag{12}$$

this means that, according to results of [18], one has the evolution of tomogram (10) of the same form but with replacement of parameters

$$w_2\left(X \mid \mu_1,\nu_1,\mu_2,\nu_2,t\right)\to w_2\left(X \mid \mu_1,\nu_1+\mu_1t,\mu_2,\nu_2+\mu_2t,0\right). \tag{13}$$

For N free particles in the coherent state $\mid\psi_{\vec\alpha}\rangle=\mid\alpha_1\rangle\mid\alpha_2\rangle\cdots\mid\alpha_N\rangle$, the evolution of tomogram is given by the Gaussian probability distribution,

$$w_N\left(X \mid \vec\mu,\vec\nu,t\right)=\frac{1}{\sqrt{\pi\sum_{j=1}^N\left[\mu_j^2+(\nu_j+\mu_jt)^2\right]}}\exp\left[-\frac{(X-\bar X)^2}{\sum_{j=1}^N\left[\mu_j^2+(\nu_j+\mu_jt)^2\right]}\right], \tag{14}$$

where

$$\bar{X} = \sqrt{2}\,\sum_{j=1}^{N}\left(\mu_j\,\mathrm{Re}\,\alpha_j + \nu_j\,\mathrm{Im}\,\alpha_j\right). \tag{15}$$

Introducing two vectors $\vec{\mu} = (\mu_1, \ldots, \mu_N)$ and $\vec{\nu} = (\nu_1, \ldots, \nu_N)$, we have the dequantizer operator, which reads

$$\hat{U}_N\left(X, \vec{\mu}, \vec{\nu}\right) = \delta\left[X - \sum_{j=1}^{N}\left(\mu_j \hat{q}_j + \nu_j \hat{p}_j\right)\right]. \tag{16}$$

Center-of-mass tomograms of Fock states of two as well as N particles can be also obtained, in view of the approach elaborated.

3. Conclusions

To conclude, we point out the main results of our work.

We introduced the notion of tomography of classical particle state and applied this notion to the particle motion obeying to the Newton law. There are various possibilities to introduce classical state tomograms of several particles, employing either symplectic tomography or center-of-mass tomography. We discussed free motion of quantum free particles and introduced the center-of-mass tomograms of two (and N) free particles, in view of the method where tomograms for oscillators with frequencies equal to zero are considered.

Thus, for N free particles of arbitrary masses, we elaborated the description of states of free particle systems by the conditional probability distribution of only one random variable. An analogous consideration can be provided for photon systems in quantum optics, where coherent states of several photons can be described by tomographic-probability distributions of vibrating electric and magnetic fields.

We pointed out that the suggested construction of quantum states of free particles is related to the old problem of quantum mechanics, namely, to find the possibility to use the probability distribution instead of the wave function or the density matrix. Thus problem is solved, in view of the method of quantizer–dequantuzer operators on the example of free-particle motion.

We obtained the evolution of center-of mass tomograms of coherent states of two and N particles given by the Gaussian probability distribution Eq. (11) and the corresponding quantizer and dequantizer operators. The method can be extended to the description of the evolution of photon states. It is worth mentioning that the center-of-mass tomogram was also introduced for systems of classical particle states. In the literature [21–23], the problem of photons interacting with axions was discussed. One can apply the approach developed here to study the properties of photon–axion systems and their evolution

Also the center-of-mass tomograms of free-particle states can be connected with Lie algebras of the $SU(N)$ groups, using the approach elaborated in [24]; it will be studied in the future publication.

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