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PROPERTIES OF PANCHARATNAM PHASE AND ENTANGLEMENT OF A FIVE-LEVEL ATOM INTERACTING WITH A SQUEEZED FIELD

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Abstract

We introduce a quantum scheme where a single five-level atom interacts with a single-mode cavity field by a time-dependent coupling. During the interaction, the temporal behavior of the quantum entropy in the atomic basis is compared with that of the Mandel parameter used to quantify the nonclassical properties of the field. With the field prepared in a squeezed coherent state, the atomic quantum entropy is then used to quantify the entanglement or the nonlocal correlation of the five-level atom (5 LA) − field system. The influence of one- and two-photon transitions and the atomic motion on the degree of entanglement and the Pancharatnam phase is analyzed. The analysis emphasizes that both the time dependence and photon multiplicity play an important role in the evolution of the degree of entanglement, the Pancharatnam phase, and nonclassical properties. This insight may be very useful in various applications in quantum physics and quantum optics.

Keywords: five-level atom, linear entropy, atomic motion, squeezing parameter, photon multiplicity.

1. Introduction

Essential features in quantum mechanics, namely, the Pancharatnam phase (PP) and the geometric phase (GP) have been studied by many physicists [1–5]. Michael Berry demonstrated that the quantum object – the wave function (WF) – maintains its evolution in the complex-valued argument of the WF, namely, the GP factor. In regard to its dynamical influence, the GP factor depends on the path geometry of the scheme that the quantum object traverses [6]. This factor is stable despite the uncertainties in control and environmental perturbations. Therefore, researchers pay it close attention while conducting fault-tolerant quantum computations. Focusing on the generalized Heisenberg algebra coherent state (CS), we have recently explored the PP and the purity of the field for several quantum systems [7,8].

In this context, the link between the GP and field purity is highly sensitive to the photon transition number and the initial atomic state. In regard to the atomic motion, the influence of a time-dependent

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coupling on the enolution of the PP has been examined recently [9]. In addition, the dynamical behavior of the GP and the entanglement between a field and a three-level atom was analyzed with and without the rotating wave approximation [10]. The results obtained using the rotating wave approximation clarified the dynamics of the scheme and showed a richer structure compared with those obtained without applying this approximation. Abdel-Khalek et al. [11] discussed the impact of cavity damping on attributes of the PP and the entanglement of the three-level atom. In outline, the GP features are very sensitive to the various initial states of the atom. This feature unleashes a diversity of phenomena stemming from cavity damping.

The squeezed coherent state (SCS) of light was introduced as a development of the CS for the quantum harmonic oscillator and attracted the attention of many researchers in quantum optics [12–14]. An advancement of the SCS concept is widely applied in quantum information sciences [15, 16] having potential relevance to two-mode entangled SCSs [17,18]. Moreover, much effort has been spent in theoretical investigations into the commonly overlooked impact of thermal radiation at room temperature. In contrast, the investigations into squeezed thermal states primarily focus on nonclassical aspects [19], the quantitative measurements of entanglement based on the Bures distance [20], and the examination of phase estimations [21]. Additionally, a quantum scheme was introduced involving the coupling of two entangled atoms, each accessing three levels, in the cascade configuration interacting with an SCS with and without the influence of the atomic motion [22]. The results obtained featured phenomena such as sudden birth and sudden death of entanglement over the duration of the interaction. The dynamics of the quantum Fisher information for the two atoms and the nonclassical properties of a radiation field in the SCS were investigated [23]. Both the quantum Fisher information and the nonclassical properties of the field were strongly affected and exhibited a dependence on the squeezing parameter.

A primary goal of quantum technologies is to control quantum correlations between subsystems [24–26]. These correlations have been studied for each qubit in a system of two atoms and an environment that governs the interatomic distance [27]. Recently, various devices, e.g., beam splitters [28–30], nanoresonators [31], interactions involving superconducting circuit (QED systems) [32, 33], and optomechanical interactions [34] have been developed to obtain quantum entanglement. In this context, the dynamical properties of quantum entanglement for a scheme describing the interaction between a four-level atomic system in the Ξ-configuration were presented [35]. This scheme has been extended to the five-level atoms (5 LAs) in different configurations, for which the M and K types have garnered attention in quantum optics and quantum information.

The spontaneous emission in a 5 LA driven by four fields was studied in [36]. Moreover, the phase-dependent optical characteristics of a 5 LA in the K-type configuration were discussed in [37]. Decoherence effects in a 5 LA in the Ξ-configuration were also studied [38]. Controllable entanglement of light in a 5 LA was investigated in [39]. An early study had addressed the level-population numbers in a 5 LA for different photon distributions [40].

Given the above situation, examining the entanglement and PP for a multilevel atomic scheme as a promising and distinctive idea relies on the initial quantum preparation of the optical field. Therefore, we examine in detail the evolution and physical content of the PP, the Mandel parameter, and the entanglement of a 5 LA interacting with the optical field initially prepared in an SCS. In outline, we describe the physical structure of the interaction between a 5 LA and a field initially prepared in the SCS (Sec. 2) and discuss the key terms of the atom–field entanglement, nonclassical properties of the field via the Mandel parameter, and the PP and its dynamical behavior (Secs. 3 and 4). In Sec. 5, we summarize our main findings.

2. Quantum Scheme of a Ladder-Type Five-Level Atom

Large sets of quantum structures may be identified by the SCS through an appropriate choice of the squeezing parameter denoted by r. This parameter governs the physical and nonclassical properties of the radiation field. Setting $r = 0$ places the quantum system in a CS field. We assume the field is initially in the SCS given by [41, 42]

$$|\Theta_F(0)\rangle = |\alpha, r\rangle = \sum_{n=0}^{\infty} \Upsilon_n(r)|n\rangle, \tag{1}$$

where $\Upsilon_n(r)$, the amplitude of the SCS, reads

$$\Upsilon_n(r) = \sqrt{\frac{(\tanh r)^n}{2^n n! \cosh r}} \exp\left\{\frac{\alpha^2}{2}[\tanh r - 1]\right\} \mathrm{H}_n\left(\sqrt{\frac{\alpha^2}{\sinh(2r)}}\right),$$

and $\mathrm{H}_n(*)$ is the Hermit polynomial.

The interaction Hamiltonian of the 5 LA – field structure within the rotating wave approximation takes the form

$$\widehat{H}_{\rm int} = \sum_{j=1}^{4} \beta_j(t)\left[\hat{A}^{\ell}\ |j\rangle\langle j+1| + \hat{A}^{\dagger\ell}|j+1\rangle\langle j|\right], \tag{2}$$

where ℓ represents the photon multiplicity number, $\hat{A}$ ($\hat{A}^{\dagger}$) is the annihilation (creation) operator of the quantized field that acts on photon states $|n\rangle$ and $|j\rangle$, the $j^{\rm th}$ levels of the atomic system. The time-dependent 5 LA – field coupling is defined by $\beta_j(t) = \varepsilon_j \cos(p_2 + p_1 t)$, which tends to ε_j in the constant-coupling case. We set $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon$ and have a scaled time $T = \varepsilon t$.

Next, we provide a solution to the model for the atomic system being investigated. The 5 LA – field WF at an arbitrary scaled time is then expressed as follows:

$$|\Theta(T)\rangle = \sum_{m=0} \Upsilon_m(r)\left[\sum_{j=1}^{5} B_j(m,T)|m+j-1\rangle \otimes |j\rangle\right]. \tag{3}$$

The coefficients $B_j(n,T)$ of the amplitude are determined from the time-dependent Schrödinger equation in the interaction picture, $\frac{d}{dT}|\Theta(T)\rangle = -i\widehat{H}_{\rm int}|\Theta(T)\rangle$, which leads to the ordinary differential equation,

$$\frac{d}{dT}B(n,T) = -iUB(n,T), \qquad j = 1, 2, \ldots 5, \tag{4}$$

where

$$B(n,T) = \begin{pmatrix} B_1(n,T) \\ B_2(n,T) \\ B_3(n,T) \\ B_4(n,T) \\ B_5(n,T) \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & \Gamma_1(n) & 0 & 0 & 0 \\ \Gamma_1(n) & 0 & \Gamma_2(n) & 0 & 0 \\ 0 & \Gamma_2(n) & 0 & \Gamma_3(n) & 0 \\ 0 & 0 & \Gamma_3(n) & 0 & \Gamma_4(n) \\ 0 & 0 & 0 & \Gamma_4(n) & 0 \end{pmatrix}, \tag{5}$$

$$\Gamma_j(n) = \beta_j(T)\sqrt{m+j}. \tag{6}$$

At time $T = 0$, the 5 LA initially is in an entangled state composed of two upper states $|1\rangle$ and $|2\rangle$. Consequently, the initial WF decomposed into its atomic and field components is expressed as

$$|\Theta(0)\rangle = \frac{1}{\sqrt{2}} \sum_m \Upsilon(m,r) \left\{ |m,1\rangle + |m+1,2\rangle \right\}. \tag{7}$$

The 5LA–field WF (3) can be obtained as a numerical solution of the system of equations (4) under the specified initial condition (7).

Next, we investigate the influence of squeezing and time-dependent coupling on the evolution of the PP, entanglement, and the field's nonclassical characteristics.

3. Nonclassical Properties of the Field and 5 LA – Field Disentanglement

We analyze the degree of entanglement by employing the atomic quantum version of von Neumann entropy [43], because it is the main criterion by which the entanglement of system components is measured [44–48]. The quantum entropy as a measure of the 5 LA – field entanglement is given by

$$S_A(T) = -\operatorname{Tr} \hat{\rho}_{5\mathrm{LA}}(T) \ln(\hat{\rho}_{5\mathrm{LA}}(T)), \tag{8}$$

in which the reduced density operator $\hat{\rho}_{5LA}$ takes the form

$$\hat{\rho}_{5\mathrm{LA[F]}} = \mathrm{Tr}_{F[5\mathrm{LA}]}\{\hat{\rho}_{5\mathrm{LA}}(T)\}, \tag{9}$$

$$\hat{\rho}_{5LA}(t) = |\Theta(T)\rangle\langle\Theta(T)| = \sum_j \sum_k \left[\sum_{n=0} \Upsilon_n(r) \bar{\Upsilon}_{n+j-k}(r) B_j(n) \bar{B}_k(n+k-j) \right] |j\rangle\langle k|. \tag{10}$$

Given equations (8) and (10), the atomic quantum entropy is formulated in terms of the eigenvalues $\varpi_j(T)$ of the atomic density matrix $\hat{\rho}_{5LA}(t)$ as follows:

$$\mathrm{S}_A(T) = -\sum_{j=1}^{5} \operatorname{Tr} \varpi_j(T) \ln \varpi_j(T). \tag{11}$$

To define the nonclassicality of an arbitrary quantum state, the Mandel parameter is adopted. Mandel made earlier attempts to highlight the nonclassicality of a quantum condition [49, 50] investigating radiation fields and introducing a parameter measuring the characteristics of the coherence condition and the deviation of the photon-number statistics from the Poisson distribution. The Mandel parameter concerning the mean (variance) photon occupation number of the field denoted by $\langle N \rangle$ ($\langle (\Delta N)^2 \rangle$) is

$$M_Q(t) = \frac{\langle (\Delta N)^2 \rangle - \langle N \rangle}{\langle N \rangle}, \tag{12}$$

where $\langle N \rangle = \langle \Theta(t) | \hat{A}^\dagger \hat{A} | \Theta(t) \rangle$. This parameter is convenient in classifying nonclassical states as sub-Poissonian if $-1 \le M_Q \le 0$, or super-Poissonian if $M_Q > 0$. The distribution of the field becomes Poissonian when $\langle N^2 \rangle = \langle N \rangle^2 + \langle N \rangle$ corresponds to semiclassical conditions.

4. Results and Discussion

Now we discuss the influence of the squeezing parameter and the sum of the photon multiplicities on the dynamics of the atomic quantum entropy with and without time-dependent coupling. We assume that the field is prepared in an SCS with $\alpha = 5$ and $r = 0.5$. In Fig. 1 a, we show plots of the atomic quantum entropy for constant coupling in the 5 LA – squeezed-field (SF) system corresponding to the standard case. With the atomic motion neglected, the associated entropy was not determined from the pure state $S_A(T) \neq 0$ and was not periodic. Furthermore, the oscillations about the saturation level range from 1 and ln 5 and result in a semilong-living entanglement between the field and the 5 LA. The field entropy begins from zero at $T = 0$ and achieves a maximum (minimum) value at $T \approx \pi$ ($T \approx 2\pi$). With $\ell = 2$, for which the system is more nonlinear, $S_A(T)$ exhibits a more chaotic behavior with a small increase in the 5 LA – SF entanglement. In Fig. 1 b, d, we see that the atomic quantum entropy is periodic with decreasing amplitude when an atomic motion of the form $\beta_j(t) = \sin(t)^2$ is considered. Similarly, a pure state emerges as the atomic motion increases, with the evolving entropy appearing more periodic with period $\ell = 2$.

The time evolution of the Mandel parameter (Fig. 2) characterizes the statistics of the photon field. The nonclassical characteristics of the field are sub-Poissonian with a constant coupling or a time-dependent coupling assuming the one-photon transitions. The oscillations in the PP are more regular and periodic when $\beta_j(t) = \sin(t)$. Interestingly, the behavior of the field undergoing the two-photon transitions cycles between sub-Poissonian, Poissonian, and super-Poissonian statistics corresponding to negative, zero, and positive values of the Mandel parameter. The field becomes more super-Poissonian

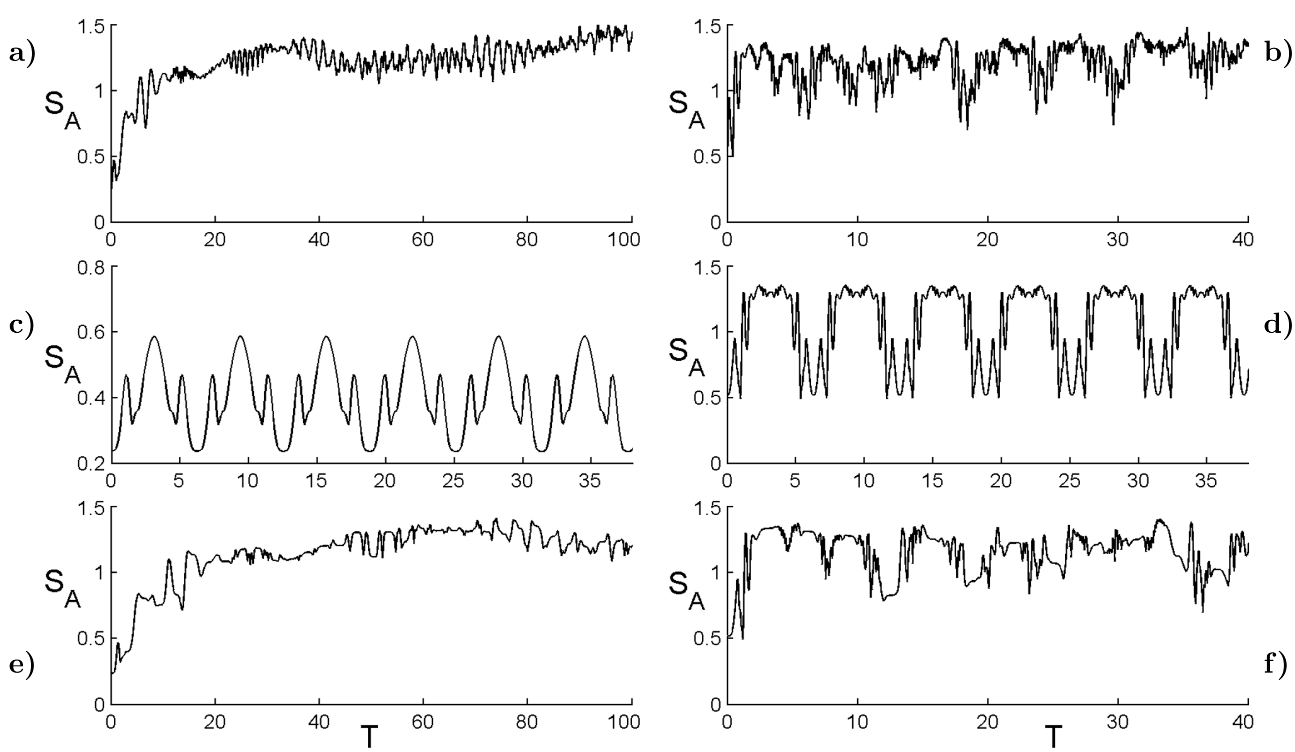

Fig. 1. The 5 LA – field entanglement measured by the atomic quantum entropy for the field in the SCS with the parameters $\alpha = 5$ and $r = 0$, where $\beta_j(t) = 1$ (a, b), $\beta_j(t) = \sin(t)$ (c, d), and $\beta_j(t) = \sin(t)^2$ (e, f) for the one-photon transitions $\ell = 1$ (a, c, e) and the two-photon transitions $\ell = 2$ (b, d, f).

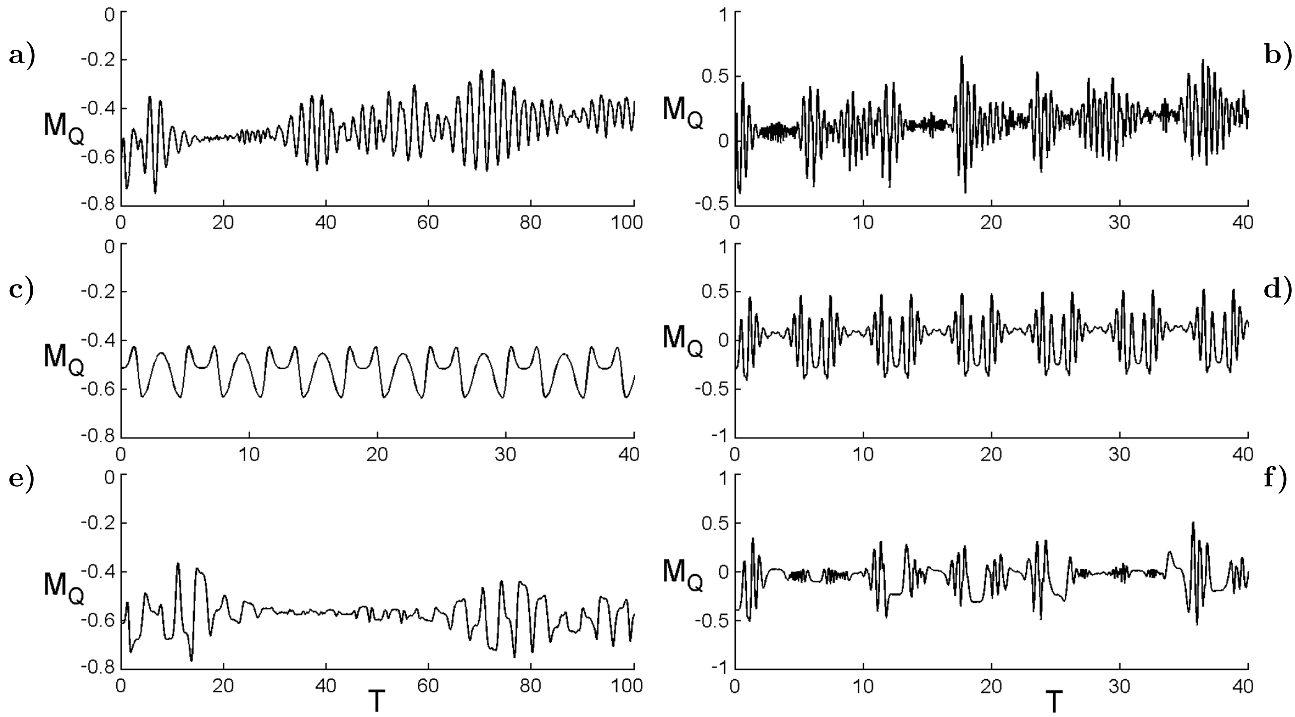

Fig. 2. Nonclassical properties of the photon field quantified by the Mandel parameter using the same parameter settings and conditions as stated in Fig. 1.

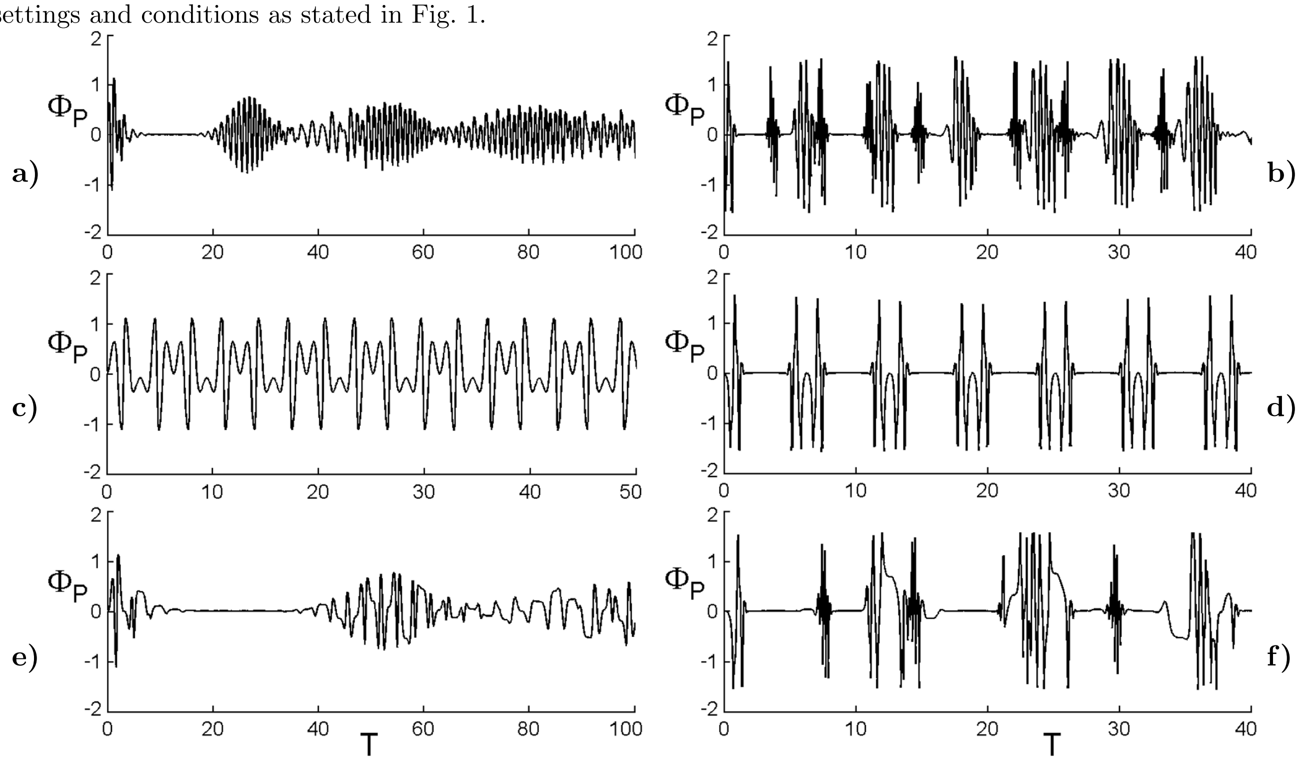

Fig. 3. Pancharatnam phase of the 5 LA – field system with the same parameter settings and conditions as stated in Fig. 1.

as time elapses. Therefore, with a time-dependent coupling, we arrive at nonclassical field characteristics that are influenced considerably by the number of photon transitions.

The dynamical performance of the PP between the initial and final states of the system is defined by [51, 52]

$$\Phi_P(t) = \arg\left(\langle\Theta(0)|\Theta(T)\rangle\right). \tag{13}$$

For the 5LA prepared in an initial state (7), the dynamical performance is given by

$$\Phi_P(T) = \arg\left\{|\Upsilon(m,r)|^2 B_j(m,T) + \Upsilon(m+1,r)\Upsilon(m,r)B_j(m+1,T)\right\}. \tag{14}$$

In Fig. 3, we plot the dynamical performance of the PP given by Eq. (13), using the same settings as stated in Fig. 1 and considering only one-photon transitions $\ell = 1$ (a, c, e) and diverse forms for the time-dependent coupling. For constant couplings, the dynamical behavior of the PP appears within the death and revival periods. After including atomic motion effects, these deaths and revivals develop a regular and periodic behavior, suggesting that our system is very sensitive to the time-dependent coupling having an influence on the atomic motion. Using the new time-dependent coupling $\beta_j(t) = \sin(t)^2$, the PP performance returns to that for the constant coupling case but over half the entanglement time, thereby creating a time delay. This new behavior in the PP performance is observed in two-photon processes $\ell = 1$, for which the PP has a richer structure.

5. Summary

In this paper, we presented the quantum scheme of a field initially prepared in the SCS and the 5 LA accessing ladder-type states. Numerical solutions describing the time and squeezing-parameter dependences of the PP, the nonclassical characteristics of the field, and the nonlocal correlation between the 5 LA and field were investigated. The findings hold promise for applications in quantum physics and quantum optics. The time evolution of the three quantifiers with and without the presence of a time-varying coupling depended strongly on the number of photon transitions. In addition, the enhancement and preservation of the PP occurred through the control of the strength of the interaction coupling in the whole system. Finally, the findings highlighted the relationship between the dynamical behavior of the PP and the nonclassical properties of the field for both stationary and moving 5 LAs. We obtained a monotonic correlation between the Pancharatnam and Mandel parameters in the absence of a time-varying coupling; a time-varying coupling breaks this relationship.

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