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EMISSION SPECTRUM AND FIDELITY OF AN ATOMIC SYSTEM COUPLED TO FIELDS WITH LEVEL ENERGY DIFFERENCES

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Abstract

We study the emission spectrum and quantum-state fidelity for an atomic system interacting with radiation fields of quantum systems described by power-law potentials. We consider a set of manifold fields that are characterizable by their level-energy differences. They include triangular, infinite, and harmonic potentials as special cases. We present the time evolution of the spectrum and fidelity for this class of potentials with respect to model parameters of the physical system. The results indicate that the spectrum and fidelity during the dynamics are highly sensitive to the value of the photon-transition number and exponent parameter.

Keywords: emission spectrum, coherent states of power-law potential, quantum-state fidelity, triangular potential, infinite potential, harmonic potential.

1. Introduction

Studies of different phenomena related to the field of quantum information have performed an extremely important role in the development of new applications based on nonclassical effects, such as quantum cryptography, quantum teleportation, and quantum computation [1–5]. Comprehending the time evolution of a quantum emitter interacting with an electromagnetic field has for a long time been an essential aspect and is presently of considerable technological interest. In particular, the potential to control and manipulate the absorption or emission spectrum (ES) of the emitter will be of ultimate importance in the domain of, for example, quantum metrology [6–11], quantum control [12–14], information processing [15, 16], and quantum tomography [17–22]. In this context, several quantum systems

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are considered favorable candidates to serve as quantum emitters in areas such as quantum optics and quantum information.

The spontaneous emissions of fields are commonly known to be chief sources of energy relaxation for open systems. The dynamical features of quantum electrodynamics for emission, absorption, and scattering from Rydberg atoms have been discussed [23]. In this context, the effectiveness of the intensity-dependent coupling in the Jaynes–Cummings model has been considered in regard to the comportment of the emission spectra [24]. Recently, multidimensional photon correlation spectroscopy for cavity polaritons has been considered with respect to the impact of the atomic spontaneous emissions. This scheme was found to provide high-precision measurements of atom nanooptics and of the center-of-mass wave functions of nonstationary atoms [25]. Moreover, the derivation of the analytical solution for the peak value and line width of the absorption spectrum has been considered [26]. The Rabi frequency of the radiation field for an efficient coupling was shown to tend to zero, forcing the fluorescence to vanish in the absence of an interaction of the atom–field system. Furthermore, the spectra of the discontinuous resonance fluorescence for a quantum system comprising a laser and a three-level atom as a result of electron shelving were examined [27].

Fidelity, a basic notion of quantum information, is related to the measure of the similarity of quantum states. In practice, there exist several useful means to compare quantum states. As an example, given that any empirical establishment of a quantum state is restricted by noise and imperfections, one is either producing or transmitting specific quantum states in an environment of noise or other sources of errors, where one is often concerned with knowing how close the state truly generated is to the state prescribed. This is a standard topic in quantum computing and communication. Another application appears within the framework of entanglement quantification [28–31]: the farther a given state is from the set of factorizable states, the more entangled the state is, and vice versa. Evaluating and quantifying fidelity between quantum states is at the heart of various tasks of quantum information science. Recently, analyzes of quantum-phase transitions have been performed using measures of fidelity.

The Schrödinger equation involving a central potential is used to treat and analyze a wide set of significant processes in many branches of quantum physics. For example, the Hellmann potentials [32] have been considered in examining the atomic inner-shell ionization, the electron–ion interactions, and the electron core. The short-range Hulthen potentials [33] were studied in particle and nuclear physics. The exponential potential and Morse potential [34] have been applied in the solid-state physics. Almost since the beginning of quantum mechanics, a large number of significant formalisms have been suggested by many researchers to evaluate the eigenvectors and eigenvalues of spherical potentials. Considerable advances have been taken over the years and thus research field remains active. The power-law potentials (PLPs) have provided many applications in the field of particle physics [35–40]. These potentials have been examined from different points of view by researchers utilizing a number of approximations, such as the Wentzel–Kramers–Brillouin (WKB) approximation [41] and the variational technique [42], as well as by numerical integration methods [43]. On the basis of the aforementioned issues, the examination of the ES and fidelity for an atomic system as a promising and distinctive idea depends on the initial quantum state of the quantized field with the effect of level-energy differences considered.

The aim of the present paper is to study the ES and fidelity for an atomic system that interacts with a radiation field for quantum systems with PLPs. We consider a class of manifold fields that is characterized by multiphoton transitions. The variation of the ES behavior is found to be influenced significantly by the exponent parameter and the photon transitions during the dynamics.

The outline of this paper is as follows.

In Sec. 2, we present the physical model that describes the interaction between a two-level atom and a field in the PLP. The main concepts of the spectrum and atom–field state fidelity and its dynamical behavior are discussed in Sec. 3. Finally, the main results are concluded in Sec. 4.

2. Physical Model

Large sets of quantum systems can be characterized by PLPs through a suitable choice of exponent parameter denoted by l. This parameter characterizes and dictates the level energy differences. For $l < 2$, the level energy differences $\Delta E_n = E_{n+1} - E_n$ increase with energy level n, but inversely so for $l > 2$. For $l = 2$, all $\Delta E_n = E_{n+1} - E_n$ are independent of n, the energy levels being equally spaced. Here, we consider a quantized field for which the potential and its corresponding energies are given by [44, 45]

$$U(x,l) = U_0 \left|\frac{x}{a}\right|^l, \qquad E_n = \left(n + \frac{\gamma}{4}\right)^{2l/(l+2)}, \tag{1}$$

where $U_0(a)$ denotes the dimension of energy (length), and l defines the exponent parameter. The total Hamiltonian of the system takes the form

$$\hat{H}_T = \hat{H}_0 + \hat{H}_{\text{int}}, \tag{2}$$

where $\hat{H}_0$ ($\hat{H}_{\text{int}}$) describes the scheme of the free (interaction) part of the two-level atom, its upper (lower) state being denoted by $|\uparrow\rangle$ ($|\downarrow\rangle$). For a multiphoton transition within the rotating-wave approximation, the Hamiltonian of Eq. (2) is prescribed as

$$\hat{H}_T = \omega_f \hat{A}^\dagger \hat{A} + \frac{\omega_A}{2}\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow| \right) + \beta\left(\hat{A}^k\, |\uparrow\rangle\langle\downarrow| + \hat{A}^{\dagger k}\, |\downarrow\rangle\langle\uparrow|\right), \tag{3}$$

where β denotes the atom–field coupling constant, k is the transition-photon number, and $\hat{A}$ ($\hat{A}^\dagger$) is the generalized annihilation (creation) operator of the quantized field that act on photon states $|n\rangle$,

$$\hat{A}^\dagger\, |dn\rangle = M_n\, |n+1\rangle, \qquad \hat{A}\, |n\rangle = M_{n-1}\, |n-1\rangle, \qquad \text{with} \qquad M_n^2 = E_{n+1} - E_0. \tag{4}$$

We consider the two-level atom to be initially in the excited state (i.e., $\rho_A(0) = |\uparrow\rangle\langle\uparrow|$) and the quantized field is described by the coherent state (CS) of the PLP (i.e., $\rho_F(0) = |z,l\rangle\langle z,l|$). Consequently, the state of the whole system is $\rho_A F(0) = |z,l,\uparrow\rangle\langle z,l,\uparrow|$. The CSs of a PLP read [44–46]

$$|z,l\rangle = \left[\sum_{j=0}^{\infty} \frac{|z|^{2j}}{\chi(j,l)}\right]^{-1/2} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{\chi(n,l)}} |n\rangle, \tag{5}$$

where

$$\chi(n,l) = \prod_{j=1}^{n} \left[\left(j + \frac{\gamma}{4}\right)^{2l/(l+2)} - \left(\frac{\gamma}{4}\right)^{2l/(l+2)}\right], \qquad \chi(0,l) = 1. \tag{6}$$

Here, we limit our study to three potential wells for fields with level-energy differences; specifically, we consider the triangular potential $l = 1$, harmonic oscillator potential with $l = 2$, and the infinite well potential with $l \to \infty$. In the next section, we discuss the dynamical properties of the ES and fidelity for a two-level atom–field system for each of these potentials.

3. Emission Spectrum and Fidelity

First, we present an overview of the ES obtained using the Fourier transform of the correlation function $\langle u(0)|\sigma_+(t_1)\sigma_-(t_2)|u(0)\rangle$. The ES of the field, denoted $S(\nu)$, emitted by a cavity-bound two-level atom with respect to the interaction time t and the bandwidth of a filter γ is [47]

$$S(\nu) = \gamma \int_0^\infty dt_1 \int_0^\infty dt_2 \exp[-(\gamma - iv)(t - t_1) - (\gamma + iv)(t - t_2)]\langle u(0)|\sigma_+(t_1)\sigma_-(t_2)|u(0)\rangle, \qquad (7)$$

where the time dependence of the atomic raising (lowering) operator $\hat{\sigma}_+(t)$ ($\hat{\sigma}_-(t)$) is obtained using the Heisenberg equation

$$i\,\frac{\partial \hat{\sigma}_\pm(t)}{\partial t} = [\hat{\sigma}_\pm(t), \hat{H}_{\text{tot}}]. \qquad (8)$$

For a two-level atom, the atomic raising (lowering) operator at $t = 0$ is $|\uparrow\rangle\langle\downarrow| \otimes \sum_m |m\rangle\langle m|$. Here, $\hat{\sigma}_+(t_1)$ and $\hat{\sigma}_-(t_2)$ indicate that the atomic operators are obtained at different times; moreover, $\hat{\sigma}_-(t)$ denotes the complex conjugate of $\hat{\sigma}_+(t)$. In performing the required calculations of the ES, Eq. (7), we find that it depends on the exponent parameter l and photon number k.

To discuss the effect of the various physical parameters on the atomic ES, we plot the variation of $S(\nu)$ with respect to different parameter values l and k (Fig. 1). They show that the atom has a substantial probability to emit photons with the atomic ES exhibiting peaks that are very sensitive to the values of l and k. Generally, we find the symmetry and weight of the peaks of the atomic ES are dependent on

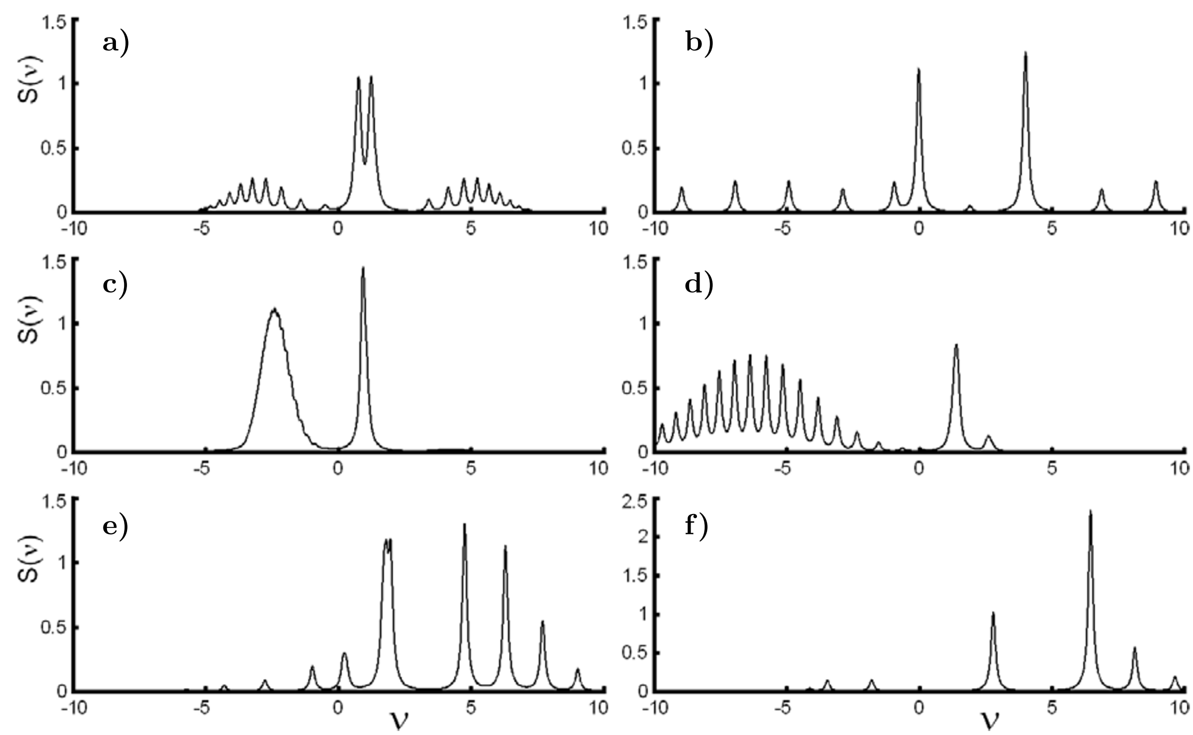

Fig. 1. Variation of the atomic emission spectra as a function of ν for $z = 2$ for various values of parameters l and k. Here, the harmonic oscillator potential for $k = 1$ (a) and $k = 2$ (b), the triangular potential for $k = 1$ (c) and $k = 2$ (d), and the infinite well potential for $k = 1$ (e) and $k = 2$ (f), respectively.

the potentials in the presence and absence of multiphoton transitions. Interestingly, for the harmonic oscillator ($l = 2$), the ES features two asymmetric peaks around $\nu = 0$ for a one-photon transition ($k = 1$) with sidebands appearing on the left of $\nu = 0$ for the two-photon transition ($k = 2$). For the triangular potential ($l = 1$), the ES exhibits a main symmetric double peak around $\nu = 1$ surrounded by sidebands associated with one-photon transitions. For two-photon transitions, the symmetry of the ES vanishes, and the spectrum expands with many spikes emerging (one peak being low and the other high, with a shift in the spacing of the two adjacent peaks). For the infinite well potential ($l \to \infty$), the atomic ES produces many peaks with amplitudes that decrease for the one-photon transitions and a decrease in the number of peaks for the two-photon transitions. From the results obtained, we note that a manipulation and control of the emission structure in the proposed model can be effectuated using a suitable combination of parameter values l and k.

We now introduce the fidelity that describes the degree of similarity of the quantum states; this measure is defined by [48, 49]

$$\xi(t) = \langle u(0)|\rho(t)|u(0)\rangle, \tag{9}$$

where $\rho(t) = |u(t)\rangle\langle u(t)|$, and $|u(t)\rangle = \exp(-i\hat{H}_T t/\hbar)|u(0)\rangle$.

To explore the effect of the main physical parameters of the model on the atomic fidelity, we plot $\xi(t)$ as a function of time with respect to various values of l and k. We observe that the fidelity starts from its maximum value and, as time elapses, it decreases gradually and exhibits oscillations featuring both collapse and revival phenomena. These oscillations depend significantly on the values of the exponent parameter l. The dynamical behavior of $\xi(t)$ changes completely for two-photon transitions. Interestingly,

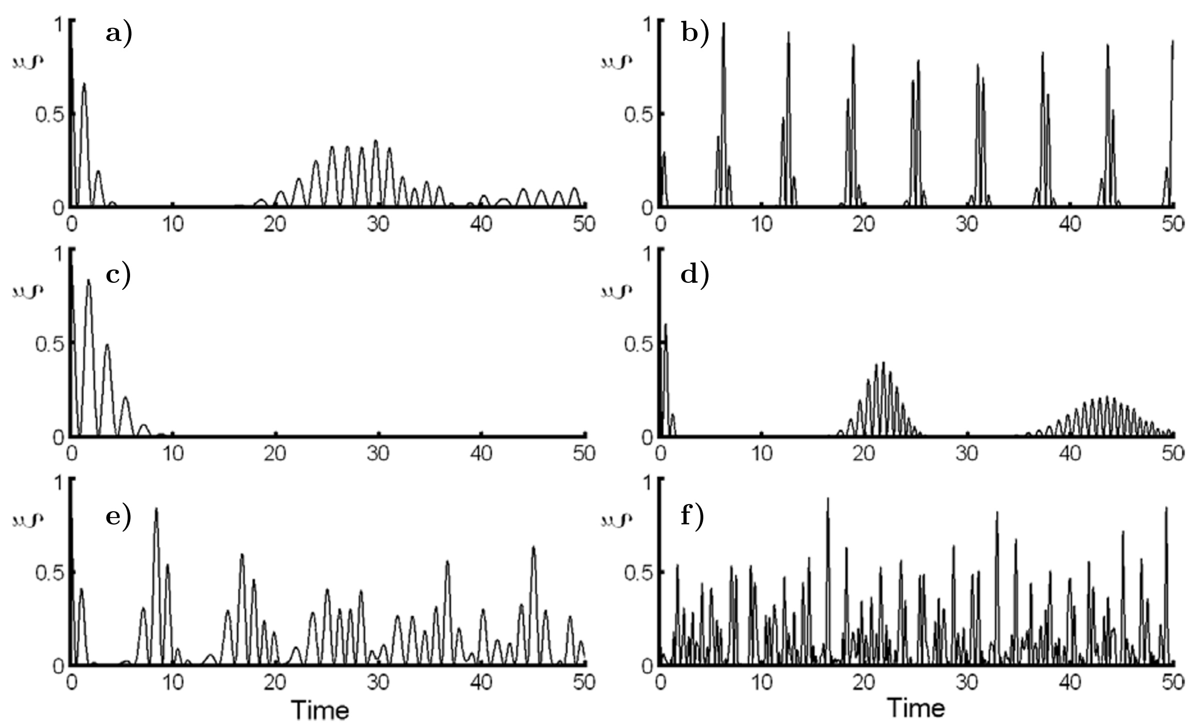

Fig. 2. Variation of the fidelity $\xi(t)$ as a function of time t for various parameter values l and k. Here, the harmonic oscillator potential for $k = 1$ (a) and $k = 2$ (b), the triangular potential for $k = 1$ (c) and $k = 2$ (d), and the infinite well potential for $k = 1$ (e) and $k = 2$ (f), respectively.

the oscillatory structure of the fidelity is affected by the photon transition number l and exhibits different local minimum and maximum values. Moreover, increasing the number of photon processes refreshes the fidelity for various values of the exponent parameter k, with the system attaining its pure state more often. This result indicates that the degree of similarity can be effectuated by a suitable choice of parameter values l and k.

4. Conclusions

We obtained the emission spectrum and state fidelity of our proposed physical model for various power-law potentials to demonstrate their high precision and for educational purposes in establishing the time development of the spectra and applications in quantum information technology. We investigated the atomic emission spectrum and atomic state fidelity for two-level atoms and field initially in the CS of PLP. The advantage of the scheme is its sensitivity to the photons-transition number and potential well. Based on the eigenvalues and eigenvector of the Hamiltonian, we evaluated the emission spectrum while the atomic state fidelity was evaluated based on the wave function or the atomic density matrix. The control of the emission spectrum and atomic state fidelity can be made by a suitable choice of the exponent parameter and photons-transition number.

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