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TO THE THOMSON CROSS SECTION OF LIGHT SCATTERED BY A MOVING PARTICLE

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The pertinence of his remarks would indeed be most strikingly vindicated by Pauli's continued work in the following years, resulting in the enunciation of the exclusion principle, which expresses a fundamental property of systems of identical particles for which, as for the quantum action itself, classical physics presents no analogy.

Niels Bohr, in: M. Fierz and V. F. Weisskopf (Eds.), Theoretical Physics in the Twentieth Century: a Memorial Volume to Wolfgang Pauli, Cambridge, USA (1960), p. 2.

Abstract

In this paper, within the framework of classical electrodynamics, we rederive in a simple way the formula for the cross section of light scattering by a moving particle. The results are of interest for the community of researchers working on laser-electron X-ray generators.

Keywords: Thomson scattering, cross-section, electrodynamics, relativity theory.

1. Introduction

The year 2020 will be the 120th Anniversary of Wolfgang E. Pauli, as well as the 75th anniversary of his winning the Nobel prize in physics. Wolfgang E. Pauli is the brightest representative of a brilliant galaxy of physicists of the early 20th Century, the creators of quantum theory. At this time, in essence, a new view of nature was formed, which subsequently affected the emerging new nonclassical human intuition [1, 2]. Pauli's works have played and continue to play a huge role in this process [3–5]. He was especially attracted to questions and ideas that, like a framework, link classical and quantum physics. Among them are the description of collisions of particles, the equations of electromagnetic field, and the

[†]Dr. Alexander Mikhailichenko passed away on August 14, 2018 in Ithaca, NY. We are very sad about this. More information is available on www.bangsfuneralhome.com/obituary/5294768.

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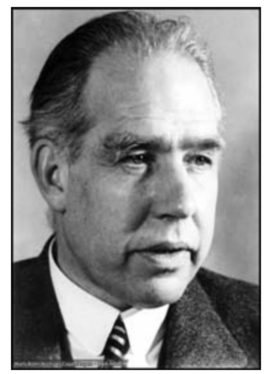
theory of relativity. A scientist, starting a new field of research, often looks for an adequate method or approach, referring to the works of predecessors.

It can be considered as a pure luck if similar questions were considered by Pauli. This happened to the community of researchers working since 1990 on the laser-electron X-ray generator (LEX), which is a new compact source of X-ray and gamma radiation for medical and other practical and scientific applications [6–19]. In laser-electron X-ray generators, X-rays are produced by collisions of relativistic electrons with laser photons in electron beam accelerators. The work of Pauli [20], where he introduced the concept of the effective cross section into relativistic physics, provides one of the efficient and widely-used approaches to simulate LEX's X-ray beams. It entered the textbooks [21] and continues to be cited today.

Wolfgang Ernst Pauli
(Institute for Advanced Study,
Princeton, New Jersey, USA)
1900-1958

The Nobel Prize in Physics 1945 was awarded to Wolfgang Pauli "for the discovery of the Exclusion Principle, also called the Pauli Principle."



Niels Henrik David Bohr
(Institute for Theoretical Physics
of the University of Copenhagen)
1885-1962

The Nobel Prize in Physics 1922 was awarded to Niels Henrik David Bohr "for his services in the investigation of the structure of atoms and of the radiation emanating from them."

2. Thomson Scattering Cross Section

As it is well known, a charged particle radiates in external fields with the total power determined by the formula [21]

$$I = \frac{2\,e^4\gamma^2}{3\,m^2c^3}\left\{\left(\vec{E} + \left[\vec{\beta}\vec{H}\right]\right)^2 - \left(\vec{\beta}\vec{E}\right)^2\right\}, \tag{1}$$

where e and m are the particle's charge and mass, $\vec{\beta} = \vec{v}/c$, $\vec{v}$ is the particle's velocity, $\gamma = 1/\sqrt{1-\beta^2}$ is a relativistic factor, $\beta = |\vec{\beta}|$, c is the speed of light, and $\vec{E}$ and $\vec{H}$ are vectors describing external electric and magnetic fields at the instant location of the particle. The time dependence of I in (1) is defined by the electron trajectory in an external electromagnetic field.

Below we consider the case where the external fields are the ones associated with a plane monochromatic laser wave. In this case, the vector $\vec{H} = \left[\vec{n}_L\vec{E}\right]$, $\left(\vec{n}_L\vec{E}\right) = 0$, and the total power (1) converts to

$$I = \frac{c\gamma^2\sigma_{T0}}{4\pi}\left|\vec{E}\right|^2\left[1-\left(\vec{\beta}\vec{n}_L\right)\right]^2, \tag{2}$$

where $\vec{n}_L$ is a unit vector in the wave-propagation direction, $\sigma_{T0} = 8\pi r_p^2/3$ is the Thomson cross-section of scattering of the laser wave by a particle at rest, and $r_p = e^2/(mc^2)$ is the particle's classical radius.

The plane monochromatic wave is a particular case of an undulator, which allows one to use the results describing the motion and radiation of particles in undulators; see [22] and references therein. The velocity in the expressions for power (1) and for trajectories (2) in the undulator can be represented as $\vec{v} = \overline{\vec{v}}_{\|} + \vec{v}_{\perp}$, where $\overline{\vec{v}}_{\|}$ is the average longitudinal component and $\vec{v}_{\perp}$ is the transverse component of the velocity.

The components of the plane monochromatic elliptically-polarized wave propagating along the x axis can be represented as follows:

$$E_y = E_{ym}\sin\varphi, \qquad E_z = E_{zm}\cos\varphi, \qquad E_x = 0, \tag{3}$$

where $\varphi = k_L x - \omega_L t + \varphi_0$ is the actual phase, φ_0 is the initial phase, $k_L = \omega_L/c$ is the wave vector, $\omega_L = 2\pi c/\lambda_L$ is the angular frequency, and λ_L is the wavelength. Usually the laser wave is represented by a wavelet with length $l_{\rm wp} = M\lambda_L$, where M is an integer number. Below we consider the case of a quasimonochromatic laser wave, i.e., $M \gg 1$.

The oscillation frequency of the particle moving at a collision angle $\theta_{\rm col}$ to the wave-propagation direction $\vec{n}_L$ is given by $\omega^* = \omega_L\left(1-\vec{n}_L\overline{\vec{\beta}}\right) = \omega_L\left(1-\beta\cos\theta_{\rm col}\right)$, and the frequency of emitted radiation in accordance with the Doppler effect is

$$\omega = \frac{\omega^*}{1-\vec{n}\overline{\vec{\beta}}} = \frac{\omega_L\left(1-\vec{n}_L\overline{\vec{\beta}}\right)}{1-\vec{n}\overline{\vec{\beta}}} = \omega_L\frac{(1-\beta\cos\theta_{\rm col})}{1-\vec{n}\overline{\vec{\beta}}}, \tag{4}$$

where $\vec{n}$ is the unit vector in the direction to an observer, and $\overline{\vec{\beta}} = \overline{\vec{v}}/c \simeq \overline{\vec{\beta}}_{\|} = \overline{\vec{v}}_{\|}/c$ [22].

In a weak field of the laser wave of type (3), this corresponds to the dipole approximation in the radiation process

$$|\vec{\beta}_{\perp}| = |\vec{v}_{\perp}/c| \ll 1/\gamma, \tag{5}$$

which is the same as the condition $eE\lambda_L \ll mc^2$ (in quantum language, it is the condition of the single-photon scattering process), and the radiation is emitted in the first harmonic only. Here, $E = |\vec{E}|$. Further on, we will proceed within the framework of approximation (5); this yields the relations $|\overline{\vec{\beta}}_{\|}| = |\vec{\beta}| = \beta = \text{const}$ and $\gamma = \text{const}$.

If the particle passes the way corresponding to many oscillations in a laser field ($M \gg 1$), then the emitted radiation in any direction will be monochromatic and distributed, according to (4), in a bandwidth $(\omega_{\min}, \omega_{\max})$, where

$$\omega_{\min} = \frac{\omega_L\left(1-\vec{n}_L\overline{\vec{\beta}}\right)}{1+|\overline{\vec{\beta}}|} \simeq \frac{\omega_L\left(1-\vec{n}_L\overline{\vec{\beta}}\right)}{1+\beta}, \qquad \omega_{\max} \simeq \frac{\omega_L\left(1-\vec{n}_L\overline{\vec{\beta}}\right)}{1-\beta}. \tag{6}$$

The spectral distribution of the energy flux emitted by a particle in a laser field under the dipole-radiation condition (5) is determined by the expression $I_\xi = If(\xi)$, where I states for the emitted wave power (2), $f(\xi)$ is the normalized spectral distribution of radiation over the dimensionless variable $\xi = \omega/\omega_{\max}$, $\int_{\xi_{\min}}^{\xi_{\max}} f(\xi)\, d\xi = 1$, $\xi_{\min} = \dfrac{1-\beta}{1+\beta}$, and $\xi_{\max} = 1$. The function $f(\xi)$ depends on the direction of the particle's oscillations with respect to the averaged velocity [22]. For relativistic particles ($\gamma \gg 1$, $\xi_{\min} \ll 1$) performing transverse and longitudinal harmonic oscillations, these functions are

$$f_{\perp}(\xi) = 3\,\xi(1 - 2\,\xi + 2\,\xi^2), \qquad f_{\|}(\xi) = 12\,\xi^2(1-\xi). \tag{7}$$

In some cases, the problem of radiation by a moving particle can be described in quantum language in terms of the number of radiated quanta. In this case, the spectral distribution of the photon flux of scattered laser radiation is determined by the expression $\dfrac{d\dot{N}_{\rm ph}}{d\omega} = \dfrac{I_\xi}{(\hbar\omega \cdot \omega_{\max})} = \dfrac{I_\xi}{(\xi\,\hbar\omega_{\max}^2)}$, which can be represented as

$$\frac{d\dot{N}_{\rm ph}}{d\xi} = \frac{I_\xi}{\xi\,\hbar\omega_{\max}}\,. \tag{8}$$

This yields for the total photon flux

$$\dot{N}_{\rm ph} = \frac{I}{\hbar\omega_{\max}} \int_0^1 \frac{f(\xi)}{\xi}\, d\xi. \tag{9}$$

According to (9), the total photon flux does not depend on the choice of functions (7) corresponding to oscillations of particles in transverse and longitudinal directions with respect to their velocities and can be calculated as follows:

$$\dot{N}_{\rm ph} = \frac{2\,I}{\hbar\omega_{\max}}\,. \tag{10}$$

The ratio of the photon flux of scattered radiation to the density of incoming laser flux is $S_L^{\rm ph} = \dfrac{cE_L^2}{4\,\pi\hbar\omega_L}$, i.e., the cross-section of scattering of laser photons by a moving particle is

$$\sigma = \sigma_{T0}\left(1 - \vec{n}_L\vec{\beta}\right) = \sigma_{T0}(1 - \beta\cos\theta_{\rm col}), \tag{11}$$

where $\theta_{\rm col}$ is the collision angle (angle between vectors $\vec{n}_L$ and $\overline{\vec{\beta}}_{\|}$). From (11) it follows that for collision angles $\theta_{\rm col} = \pm\pi$, as one can expect, the cross section becomes equal to $\sigma = (1 \pm \beta)\,\sigma_{T0}$. For collisions of particles at angle $\theta_{\rm col} = \pi/2$, the cross-section $\sigma = \sigma_{T0}$ does not depend on the velocity of motion in the transverse direction and becomes equal to the scattering cross section of laser photons emitted by the particle at rest.

Thus, we gave a brief derivation of the ratio of the Thomson scattering cross section on electron at rest and on the relativistic electron. In our case, we obviously did not address the Lorentz transformations of coordinates and also relativistic transformations of the electromagnetic field and the electron momentum. There is no contradiction with standard derivation of Pauli's formula (11) (see [21]). In fact, initial formula (1) contains free space electromagnetic fields ($\vec{E}$ and $\vec{H}$) that automatically satisfy the relativistic requirements. A similar problem for an arbitrary (non-electromagnetic) interaction of colliding particles was considered by Pauli [20] (quoted in [21, 23] where, in essence, the classical concept of the effective section is introduced in relativistic physics). The yield of elementary reactions in binary collisions of particles moving with arbitrary velocities is expressed through the ratio of the scattering cross section of one of them and the other one at rest. In particular, in the case of scattering of a photon by a moving electron, the relation thus obtained reads

$$K = \frac{\sigma}{\sigma_{T0}} = \sqrt{\left(\vec{n}_L - \vec{\beta}\right)^2 - \left[\vec{n}_L \vec{\beta}\right]^2}. \tag{12}$$

After some algebra, the above relation (12) can be transformed into (11). Thus, our consideration directly based on Maxwell's equations leads to Pauli's formula for the effective cross section of the electron–photon collision in relativistic mechanics.

We hope that the direct derivation of the invariant cross section we made has methodological interest.

Nowadays, the Pauli's formula is in use while one calculates processes in light sources based on backscattering of electron and ion beams; see, for example, [24–27]). The total number of scattered laser photons in this case can be calculated using the following expression:

$$N_{\text{ph}} = \sigma_{T0}\, c\, K \int n_L n_p \, dV \, dt, \tag{13}$$

where n_L and n_p are the densities of laser photons and particles, respectively, dV is an element of the volume where the interaction takes place, and dt is the time duration of the interaction.

The coefficient $K = 1 - \vec{n}_L \vec{\beta} = \sqrt{\left(\vec{n}_L - \vec{\beta}\right)^2 - \left[\vec{n}_L \vec{\beta}\right]^2}$ is called the kinematic factor of scattering, and the values $\dot{\nu} = \dfrac{\partial \nu}{\partial t}$ and $L = \dfrac{\dot{\nu}}{\sigma_{T0}}$ are called the photon flux and the luminosity of the light source, respectively.

To determine the spectral-angular, polarization, brightness, and other characteristics of light sources based on the radiation of particles in external fields, one should use an ordinary theory of electromagnetism; in these cases, the kinematic factor appears in the calculations automatically.

3. Conclusions

One can calculate the photon flux of Thomson scattering X-ray sources in view of formula (11), which follows from Paulis concept of collision cross section in relativistic mechanics. Whereas spectral and angular radiation distributions, polarization, and brightness should be described on the basis of ordinary electromagnetic field theory.

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