DOI 10.1007/s10946-019-09823-2
Journal of Russian Laser Research, Volume 40, Number 5, September, 2019

HARMONIC GENERATION BY RELATIVISTIC PLASMA RESONANCE

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Abstract

We present a theory on harmonic generation by relativistic plasma resonance mechanism in an inhomogeneous laser plasma. We find a transverse component of the resonance–enhanced electric field and electron velocities near the plasma critical density and calculate the nonlinear current as the source of the harmonic generation in vacuum. We obtain the power-law spectra of the radiation field and discuss their characteristics depending on the laser-plasma parameters.

Keywords: laser plasma, harmonic generation, relativistic effects, renormgroup symmetries, nonlinear differential equation.

1. Introduction

Harmonic generation in an inhomogeneous plasma is interesting as a method for obtaining high-frequency secondary radiation and as a laser-produced plasma diagnostic method. The harmonic generation of a low-intensity radiation propagating in an inhomogeneous plasma was investigated in [1, 2], in view of perturbation theory. The plasma resonance effect [3], which is the resonant enhancement of the potential electric field near the plasma critical density, where the laser frequency ω_0 coincides with the plasma frequency ω_L, was considered as the mechanism of generation. Within that framework, an exponential decrease in the harmonic amplitude with increase in the harmonic number was obtained. The authors of [4] went beyond the perturbation theory by taking into account the nonlinearity of electron motion, but neglecting the relativistic effects. By applying the renormalization group transformation approach [5], they calculated the amplitudes of harmonics of the electromagnetic wave incident on a inhomogeneous plasma. They also found that the intensity of harmonics decreases much more slowly with increasing harmonic number than was predicted by the weakly nonlinear theory. An adequate generalization of the nonlinear theory on the harmonic generation at high laser intensities, when relativistic effects near the plasma resonance becomes essential, is still lacking. In [6] it was shown that the resonance plasma field amplification can lead to an electric field strength up to the relativistic values even if the laser field intensity is lower than the relativistic one. Such a strong nonlinear resonance qualitatively changes the harmonic generation process, significantly reinforcing the higher harmonics.

Manuscript submitted by the authors in English on July 15, 2019.

In this study, we present the theory of harmonic generation by relativistic plasma resonance in an inhomogeneous plasma. A stationary analytical solution of the nonlinear equations governing the spatiotemporal structure of the transverse components of the electric field and electron velocity in the vicinity of the plasma resonance are calculated [3] via the renormgroup symmetry method. Application of the formalism of renormalization group transformations allowed us to take into account the nonlinearity of electron motion, including the relativistic effects, near plasma critical density. Then the harmonic amplitudes in vacuum are calculated.

2. Basic Equations and Their Solution

We consider a p-polarized electromagnetic wave described by the electric and magnetic field $\mathbf{E}, \mathbf{B}$ with frequency ω_0 that is incident on the inhomogeneous plasma with linear density profile at angle θ,

$$\begin{aligned}
&\mathbf{E} = \frac{1}{2}\{E_{0x}(x), E_{0y}(x), 0\} \exp\left(ik_y y - i\omega_0 t\right) \ + c.c.,\\
&\mathbf{B} = \frac{1}{2}\{0, 0, B_0(x)\} \exp\left(ik_y y - i\omega_0 t\right) \ + c.c.,\\
&k_y = (\omega_0/c)\sin\theta, \qquad n_0(x) = (1 + x/L)n_c, \qquad n_c = \frac{m{\omega_0}^2}{4\pi e^2}\,,
\end{aligned} \tag{1}$$

where L is the characteristic inhomogeneity scale length of the smooth $\omega_0 L/c \gg 1$ linear plasma density profile. To describe harmonic generation process, we use the following basic system of cold relativistic electron plasma collisionless hydrodynamics equations and Maxwell equations:

$$\begin{aligned}
&\partial_t \mathbf{p} + (\mathbf{v}\partial_{\mathbf{r}})\mathbf{p} = e\left(\mathbf{E} + \frac{1}{c}[\mathbf{v}\mathbf{B}]\right), \qquad \partial_t n_e + \operatorname{div}(n_e\mathbf{v}) = 0,\\
&\operatorname{rot}\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{B}, \qquad \operatorname{rot}\mathbf{B} = \frac{1}{c}\partial_t\mathbf{E} + \frac{4\pi}{c}en_e\mathbf{v}, \qquad \operatorname{div}\mathbf{E} = 4\pi(en_e + e_i n_i),\\
&\operatorname{div}\mathbf{B} = 0, \qquad \mathbf{p} \equiv m\mathbf{v}\gamma = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}\,.
\end{aligned} \tag{2}$$

Here m and e are the electron mass and charge, n_e, $\mathbf{v}$, and $\mathbf{p}$ are the density, velocity, and momentum of plasma electrons, respectively, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields in the plasma, and c is the speed of light in vacuum. Immobile ions of given density n_i correspond to the electron plasma approximation used in this work.

Through the expansion of fields and velocities (2) in series of the incident wave harmonics (1) and assuming a hierarchy of the electromagnetic field and electron velocities near plasma resonance region [4], we obtain the following equations for magnetic field harmonics with nonlinear current J as a source of harmonic generation:

$$\begin{aligned}
&\partial_{xx}R_n - \frac{\partial_x\varepsilon_n}{\varepsilon_n}\partial_x R_n + \left(\frac{n\omega_0}{c}\right)^2\left(\varepsilon_n - \sin^2\theta\right)R_n = -\frac{4\pi}{c}\left\{\frac{a}{4\pi}\operatorname{rot}\vec{J}_n\right\}_z, \quad n \geq 2,\\
&\vec{J}_n = \left\{v\partial_x P - \frac{i\omega_0}{n}v\partial_x(\gamma v) - \frac{\omega_0^2}{a}(\gamma - 1)v, u\partial_x P - \frac{i\omega_0}{n}v\partial_x(\gamma u) - \frac{\omega_0^2}{a}(\gamma - 1)u,\ 0\right\},\\
&\big\{v, u, P, Q, R\big\} = \sum_{-\infty}^{\infty}\big\{v, u, P, Q, R\big\}_n \exp[-in(\omega_0 t - k_y y)].
\end{aligned} \tag{3}$$

Here $a = -2e|B_1(0)|\sin\theta/m\omega_0^2 L$ is a dimensionless constant proportional to the magnetic field amplitude $|B_1(0)|$ at the plasma resonance point $x = 0$ in the linear approximation, $B_1(0) = |B_1(0)|\exp(i\arg B_1(0))$ is the complex amplitude of the first Fourier component of the magnetic field at the point $x = 0$ in the linear approximation $v = \mathrm{v_x/a}$, $u = \mathrm{v_y/a}$ are the normalized values of the electron velocity components, $\gamma = 1/\sqrt{1-(a^2/c^2)(v^2+u^2)}$, $P = eE_x/ma$, $Q = eE_y/ma$, $R = eB_z/ma$ are the normalized values of the electric (E_x, E_y) and (B_z) magnetic field components, respectively, $\varepsilon_n = 1 - (\omega_L^2)/(n^2\omega_0^2)$ is a plasma complex dielectric permittivity at the frequency $n\omega_0$, and $\omega_L \equiv \omega_L(x) = (4\pi|ee_i|n_i/m)^{1/2}$ is the Langmuir frequency. In the limit of a weakly inhomogeneous plasma in which the region of localization of the plasma resonance field is much smaller than the characteristic plasma inhomogeneity scale length L, we set $\omega_L = \omega_0$. From Eq. (3) it follows that to calculate the nth harmonic of the magnetic field in vacuum, it is necessary to know the structure of the electric field and the electron velocity in the plasma resonance region.

It was previously shown [6] that the longitudinal component of the electric field and electron velocity near relativistic plasma resonance are described by the following pair of first-order partial nonlinear differential equations

$$\partial_t v + av\partial_x v = P(1-\beta v^2)^{3/2}, \qquad \partial_t P + av\partial_x P = -\omega_0{}^2 v, \tag{4}$$

which has a solution in the form

$$\begin{gathered} P_0 = -\frac{A}{1+l^2}(l\cos\chi + \sin\chi), \qquad v_0 = -\frac{A}{1+l^2}(l\sin\chi - \cos\chi), \qquad x_0 = l + \frac{A}{1+l^2}(l\cos\chi + \sin\chi), \\ v_1 = \pm\frac{1}{b}\left[1 - \frac{1}{(1+b^2v_0^2/2)^2}\right]^{1/2}, \qquad \tau = \chi - \left(\sqrt{4+\frac{\beta}{1+l^2}}E(\varphi;k) - \frac{2}{\sqrt{4+\beta/(1+l^2)}}F(\varphi;k) - \varphi\right). \end{gathered} \tag{5}$$

Here $x_0 = x/\Delta$, $v_1 = (a/\omega_0\Delta)v$ and $P_0 = (a/\omega_0^2\Delta)P$ are normalized functions,

$$\varphi = -\arcsin\frac{l\cos\chi + \sin\chi}{\sqrt{1+l^2}}\ , \quad k = \sqrt{\frac{\beta/(1+l^2)}{4+\beta/(1+l^2)}}, \quad \beta = A^2b^2, \quad A = \frac{aL^2}{\Delta^2}, \quad b = \frac{\omega_0\Delta}{c},$$

and Δ is the plasma resonance width, wich is determined by either the electron thermal motion with velocity V_T or a low collision frequency ν between plasma particles,

$$\Delta = \max\left\{\nu L/\omega_0; (3V_T^2 L/\omega_0^2)^{1/3}\right\}. \tag{6}$$

Following the reasoning presented in [6], we define the equations for the transverse component of the electric field Q and electron velocities u as follows:

$$\partial_t u + av\partial_x u - Q = 0, \qquad \omega_0\partial_x Q + k_y\partial_t P = 0. \tag{7}$$

To find the solution of the system (7), we use the renormgroup symmetry method and follow the scheme outlined in [6]; omitting details, we give the solution

$$\begin{aligned} u_0 &= Ab\sin\theta\left[\frac{1}{2}\ln\left(\frac{b^2e^{2\varsigma}\sin^2\theta}{4}(1+l^2)\right)\cos\chi - \operatorname{arccot}(l)\sin\chi\right], \\ Q_0 &= Ab\sin\theta\left[-\frac{1}{2}\ln\left(\frac{b^2e^{2\varsigma}\sin^2\theta}{4}(1+l^2)\right)\sin\chi - \operatorname{arccot}(l)\cos\chi\right], \end{aligned} \tag{8}$$

where $u_0 = (a/\omega_0\Delta)u$, $Q_0 = (a/\omega_0^2\Delta)Q$, and $\varsigma = 0.5772156$ is the Euler constant. Note that the nonlinear dependence of the electric field P, Q and velocity v, u on the coordinate x and "time" $\tau = \omega_0 t$ is implicitly determined through the parametric variables χ and l, while the electric field P and velocity v play the role of parametric variables in the expression for τ. Dimensionless parameters A and b corresponds to contributions of the electron nonrelativistic and relativistic nonlinearities, respectievly [6].

Figure 1 shows the time dependences of the transverse components of the plasma electric field and electron velocity at the resonance point $x = 0$ for the incidence angle $\theta = 11°$, for different values of the dimensionless amplitude A, and fixed $b = 1$. As we can see, plasma-oscillation wave breaking occurs when the amplitude of the plasma wave exceeds a certain threshold value. This is evident from the figure for $A = 0.73$. The growth of the amplitude leads to the stationary plasma wave profile steepening up to the threshold value, when the derivative $\partial_x Q$ becomes infinite, and the hydrodynamic model fails. Knowing the structure of the electric field and the electron velocity of the plasma resonance, which are defined by the nonlinear current J as the radiation source and solving Eq. (3), we can calculate the amplitudes of the magnetic field harmonics in vacuum.

The solution of the inhomogeneous equation (3) is constructed in a standard way via fundamental solutions of the homogeneous equation. Assuming that the magnetic field has the form $B_n = C_n^- \exp\left[-i\dfrac{\omega_0}{c}(x+\infty)\cos\theta\right]$ at $x \to -\infty$, we write the magnetic field harmonic amplitude in vacuum with number n, $k_n(x) \equiv \dfrac{n\omega_0}{c}\sqrt{\varepsilon_n(x) - \sin^2\theta}$, as follows:

$$C_n^- = \frac{im_e\Delta^2\omega_0^3 \exp\left(in\,\mathrm{arg}R_1(0) - in\pi + i\int_{-\infty}^{0} k_n(x)\,\mathrm{d}x\right)}{4\pi ec(\cos^2\theta - 1/n^2)^{1/2}(1-1/n^2)^{1/2}\cos^{1/2}\theta}\left[\exp\left\{\frac{i4n^3L\omega_0}{3c}(\cos^2\theta - 1/n^2)^{3/2}\right\} I_n^- + iI_n^+\right], \quad (9)$$

$$I_n^{\pm} = \int\limits_{-\infty}^{\infty}\mathrm{d}l\int\limits_{0}^{2\pi}\mathrm{d}\chi\exp\left[in\tau(\chi,l) \pm inbx_0(\chi,l)\sqrt{\cos^2\theta - 1/n^2}\right]$$
$$\times\left\{\left[\partial_\chi\left(P_0 - \frac{i}{n}\gamma v_1\right)\partial_l\tau - \partial_l\left(P_0 - \frac{i}{n}\gamma v_1\right)\partial_\chi\tau - \left(\partial_l\tau\partial_\chi x_0 - \partial_l x_0\partial_\chi\tau\right)(\gamma-1)\right]v_1\sin\theta\right.$$
$$\pm\sqrt{\cos^2\theta - 1/n^2}\left[\left(u_0\partial_\chi P_0 - \frac{i}{n}v_1\partial_\chi(\gamma u_0)\right)\partial_l\tau - \left(u_0\partial_l P_0 - \frac{i}{n}v_1\partial_l(\gamma u_0)\right)\partial_\chi\tau\right.$$
$$\left.\left. - \left(\partial_l\tau\partial_\chi x_0 - \partial_l x_0\partial_\chi\tau\right)(\gamma-1)u_0\right]\right\}. \quad (10)$$

3. Radiation Field Spectra

To investigate the spectral composition of radiation, let us make the transition to the physical parameters of the laser–plasma system: laser radiation incidence angle θ, laser intensity q_0, W/cm^2, and inhomogeneity scale length $L[\lambda]$ (choosing laser wavelength $\lambda = 1.06\ \mu$m) of the plasma with temperature T, keV. Then the dimensionless parameter a is determined by the expression that connects the magnetic field amplitude in the plasma resonance point with the incident wave amplitude through the reflection coefficient R_1 in the linear theory [7],

$$a = \left|\frac{cB_0^2e^2}{\pi m^2}\frac{|\cos\theta|}{\omega_0^5L^3}(1-R_1^2)\right|^{1/2}. \quad (11)$$

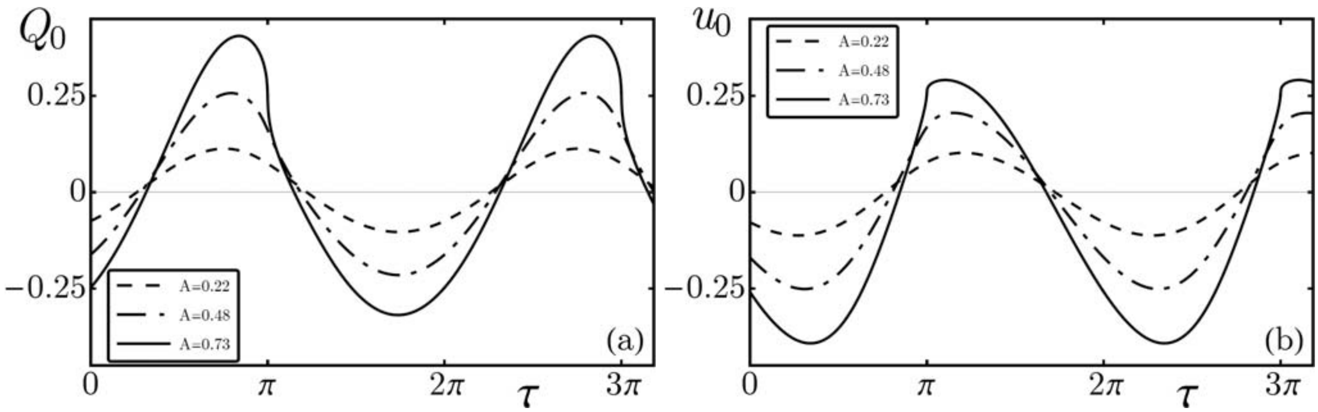

Fig. 1. Time dependences of the electric field (a) and electron velocity (b) transverse components at point $x = 0$ for $b = 1$ and different values of the parameter A.

Figure 2 shows the dependences of the harmonic intensities q_n on the number n at fixed incidence angle $\theta = 11^\circ$ and plasma temperature $T = 2$ keV for different laser intensities q_0, for two plasma inhomogeneity scales $L = 100\lambda$ (a) and $L = 20\lambda$ (b). The value $q_0 = 10^{17}$ W/cm^2 is close to the limiting one at which the wave breaking near plasma resonance occurs for given values θ, L, and T. Curves 1, 2, and 3 in Fig. 2 a correspond to the cases shown in Fig. 1. Comparison of the spectra shows that, with pump field amplitude growth, there are two effects in the spectrum structure changes that can be distinguished — flattening the spectral curve and modulating it, which reaches a maximum near wave breaking of the resonance plasma field. The predominance of an effect depends on the laser–plasma system parameters.

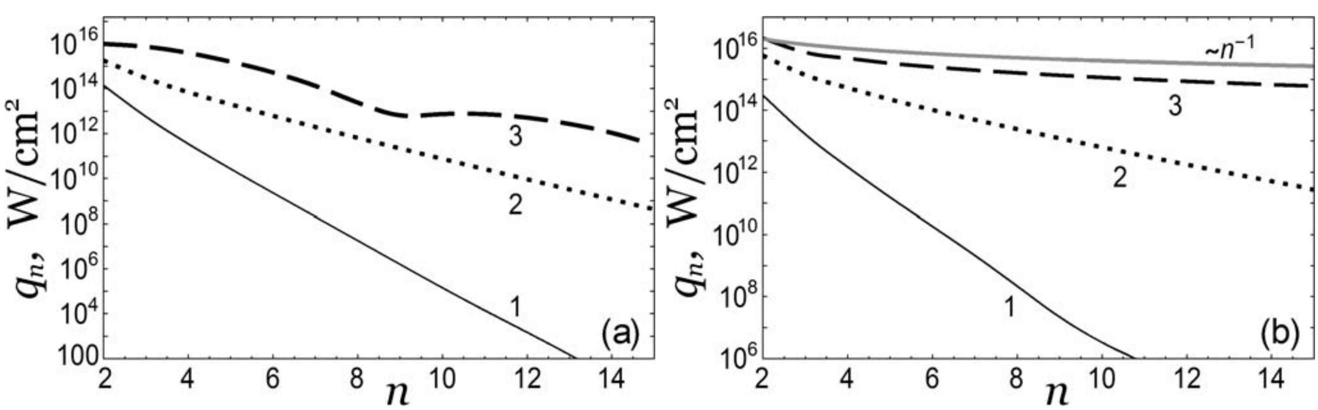

Fig. 2. Magnetic field spectra distributions calculated for different laser fluxes: $q_0 = 10^{16}$ W/cm^2 (1), $5 \cdot 10^{16}$ W/cm^2 (2), and 10^{17} W/cm^2 (3) for $T = 2$ keV at $L = 100\lambda$ (a) and $L = 20\lambda$ (b).

Comparing the spectral curves near wave breaking (curve 3) for different plasma inhomogeneity scales, we see that, in the case of a sharper gradient ($L = 20$), nonlinearity leads to a change in the slope of the curve, but, for a smoother gradient ($L = 100$), along with slight flattening, significant modulation are noticeable. We emphasize that these effects are direct consequences of plasma-oscillation modulation near the critical density, which was demonstrated in [6]. Figure 2 b also demonstrates that, in the case of sharper gradients near the plasma wave breaking threshold, the emission spectra are characterized by a power law that decreases with increasing number n but faster than $1/n$.

4. Summary

We constructed an analytical solution to the system of equations that describe the harmonic generation process through the relativistic plasma resonance mechanism. Using the renormgroup symmetry algorithm [6], we calculated the transverse components of the electric field and electron velocity in the vicinity of the critical plasma density. The nonlinear current as the source of the harmonic generation in vacuum and the harmonic amplitudes are found. The relativistic nonlinearity of the plasma wave leads to phase modulation of the electron oscillations, which leads to flattening of the spectral curve and its modulation. Also we showed that with fixed pump field intensity the efficiency of the higher harmonic generation increases with decreasing plasma inhomogeneity scale. This property is demonstrated for the plasma inhomogeneity scale $L = 20\lambda$ close to the limit laser intensity, at which wave breaking in the plasma resonance occurs when smooth power-law emission spectra can be generated.

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