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# HIDDEN CORRELATIONS AND INFORMATION-ENTROPIC INEQUALITIES IN SYSTEMS OF QUDITS†

**Igor Ya. Doskoch and Margarita A. Man'ko***

*Lebedev Physical Institute, Russian Academy of Sciences*
*Leninskii Prospect 53, Moscow 119991, Russia*

*Corresponding author e-mail: mankoma @lebedev.ru

**Abstract**

We present the results of our study of correlations in noncomposite systems like qudits ($N$-level atom or spin-$j$ system). We show that they correspond to correlations in composite systems like bipartite or multipartite systems. We establish the correspondence between correlations in noncomposite systems and composite ones using the map of indices labeling the random variables in classical probability theory or matrix elements of density matrices in quantum mechanics. We obtain new information-entropic inequalities for density matrices of noncomposite systems. Also we present Bayes' formula for noncomposite classical system with one random variable. Finally we discuss the possibility to use hidden correlations and entanglement of a single-qudit state.

**Keywords:** quantum suprematism, triangle geometry of qubits, probability distributions, quantum correlations, Bayes' formula.

## 1. Introduction

The aim of this work is to give a review of the problem of classical and quantum correlations and to show the invertible map of the problem of correlations available in multipartite systems onto the problem of correlations in systems (both classical and quantum) without the subsystems discussed in [1–14]. The problem of these correlations is important in connection with the development of quantum technologies related to quantum computing, quantum teleportation, quantum information, and quantum cryptography. These technologies are based on the researches in quantum mechanics, quantum optics, and quantum statistics.

The formalism of quantum mechanics and quantum optics is essentially different from the formalism of classical mechanics and classical electromagnetism, as well as classical statistics. The states of quantum systems are described by the vectors in Hilbert space [15] and by the wave functions [16] and density matrices [17, 18].

In quantum mechanics and quantum optics, especially in connection with emerging laser technology, a better understanding of photon states and photon statistics, as well as the phenomenon of strong quantum correlations like the entanglement, has appeared due to the study of specific properties of quantum-system states.

†Based on the talk presented by M. A. Man'ko at the 26th Central European Workshop on Quantum Optics {CEWQO} (Paderborn University, Germany, June 3–7, 2019) [https://cewqo2019.uni-paderborn.de/].

Manuscript submitted by the authors in English on August 20, 2019.

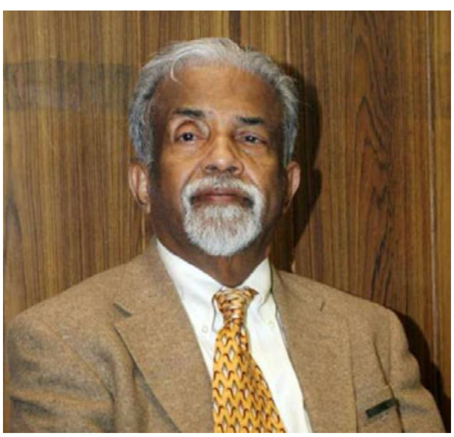
In this work, we concentrate on correlations in bipartite and multipartite systems and demonstrate that analogous correlations exist in the systems without subsystems like one qudit or $N$-level atom and spin-$j$ system. Also we discuss the tomographic-probability representation of quantum states [19–21] based on the experimental results of the photon-state tomography [22–25].

This paper is organized as follows.

The next section is Ad Memoriam of Prof. Roy Glauber and Prof. George Sudarshan, the founders of quantum optics and their collaboration with the Lebedev Physical Institute in Moscow. In Sec. 3, we consider examples of classical and quantum states. In Sec. 4, we discuss random variables and probabilities. In Sec. 5, we give a new formula for the probability distribution of one random variable. Finally, we present our conclusions and prospectives in Sec. 6.

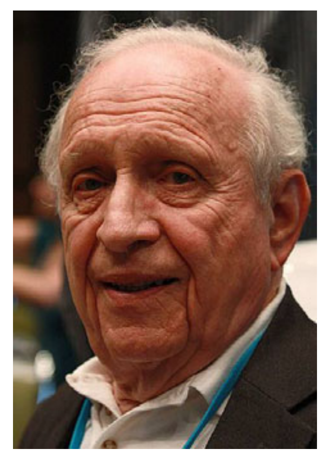
## 2. Ad Memoriam of Roy Glauber and George Sudarshan*

Last year we suffered great losses. Professor Roy J. Glauber and Professor E. C. George Sudarshan, founders of quantum optics, passed away in 2018. We are extremely sad about this.



**Ennackal Chandy George Sudarshan (Texas University) 1931-2018**

These distinguished scientists made famous discoveries in the foundations of quantum physics that provide the possibility today to raise quantum optics and quantum mechanics to a level of understanding such that quantum technologies can now be developed. The operation of such devices as lasers is based on understanding the coherence properties of radiation and realizing how to achieve the conditions for obtaining such properties.



**Roy Jay Glauber (Harvard University) 1925-2018**

In 1963, the notion of the coherent state of electromagnetic-field oscillations as well as the terminology "coherent state" of an arbitrary oscillator were introduced. Roy Glauber and George Sudarshan simultaneously published the papers [26, 27] where the properties of coherent states were discussed. These publications are cited in the majority of papers where quantum optics and quantum information technologies are discussed.

The general linear positive map of the density matrix to the other density matrices for finite-dimensional systems was presented in the form [28], which later on was generalized in [29] for arbitrary

*Ad Memoriam of Roy Glauber and George Sudarshan was also presented in the talks of M. A. Man'ko at the 16th International Conference on Squeezed States and Uncertainty Relations {ICSSUR} (Universidad Complutense de Madrid, Spain, June 17–21, 2019) [http://eventos.ucm.es/30364/detail/international-conference-on-squeezed-states-and-uncertainty-relations-2019.html] and the 18th International Symposium "Symmetries in Sciences" (Gasthof Hotel Lamm, Bregenz, Voralberg, Austria, August 4–9, 2019) [https://itp.uni-frankfurt.de/symmetries-in-science/] and will be published in *Quantum Reports* (2019) and the *Journal of Physics: Conference Series* (2020), respectively.

systems. Also we should point out that the new evolution equation for open quantum systems, which generalizes the Schrödinger equation for the wave function and the von Neumann equation for the density matrix considered in the case of unitary evolution to the case of nonunitary evolution, was obtained in [30, 31] and called the GKSL equation.

The pioneer scientific results obtained by these distinguished researchers play an important role in developing quantum optics. Studies of the properties of quantum states, especially the coherence and squeezing phenomena, correlations, unitary evolution, and the evolution of the electromagnetic fields in the presence of dissipation are based on the original results of Glauber and Sudarshan. All applications of quantum optics discussed in connection with the development of quantum technologies, motivated by the attempts to construct quantum computers and quantum information devices, are based on the theoretical notion and foundations of quantum mechanics associated with the results obtained by Glauber and Sudarshan.

A specific phase-space quasidistribution representation of quantum states, which employs the basis of coherent states, was introduced independently by Glauber and Sudarshan; it is called the Glauber–Sudarshan $P$-representation. This representation plays an important role in discussing the properties of quantum systems analogously to the Wigner quasidistribution function [32] and the Husimi–Kano quasidistribution [33, 34].

These famous contributions play an important role in the foundations of quantum optics and quantum information and in developing future quantum technologies.

Substantial developments in laser physics were made at the Lebedev Physical Institute in Moscow by Nikolay G. Basov, who won the Nobel Prize in 1964 together with Aleksandr M. Prokhorov and Charles H. Townes due to their revolutionary work in the invention of masers and lasers (as well as laser physics based on quantum mechanics and quantum optics). The official formulation reads that the Nobel Prize was given "for fundamental work in the field of quantum electronics, which has led to the construction of oscillators and amplifiers based on the maser-laser principle."

Roy J. Glauber received the Nobel Prize in 2005 "for his contribution to the quantum theory of optical coherence" along with John L. Hall and Theodor W. Hansch who received the Nobel Prize "for their contributions to the development of laser-based precision spectroscopy, including the optical frequency comb technique."

Professor Glauber exercised substantial influence on the development of quantum optics in European Scientific Centers; he used to invite young researchers to Harvard University for collaborations, among which we can list Fritz Haake from Germany, Paolo Tombesi from Italy, Vladimir Man'ko from the Soviet Union, and Stig Stenholm from Finland.

Since the entire scientific life of M. A. Man'ko was connected with the Lebedev Physical Institute, she was a witness and participant in the collaboration of Roy Glauber with Lebedev scientists. The results of these collaborations were published in such series as the Proceedings of the Lebedev Institute [35] and Journal of Experimental and Theoretical Physics [36]. Among the four Nobel-prize papers, there was a paper by G. Schrade, V. I. Man'ko, W. P. Schleich, and R. Glauber [37] where the collaboration of Harvard University, the Lebedev Institute (Moscow), and University of Ulm was mentioned.

As for George Sudarshan, the great scientist, we are happy to recall that he was in Moscow many times due to the International Workshops on Squeezed States, Group Theoretical Colloquium, and the International Conference on Squeezed states and Uncertainty Relations. Particular mention should be made of the University Federico II of Naples and especially Prof. Giuseppe Marmo who enabled many European scientists to collaborate with Prof. Sudarshan in Italy.

There exist other representations like the probability representation of quantum states [19, 38, 39] where states are identified with fair probability distributions. The state superposition is expressed in this representation using the addition rule for the probabilities. We are happy to let the readers of the Journal of Russian Laser Research know that famous results concerning the study of entanglement phenomenon, quantum tomography, and superposition principle in terms of density operators were published by Sudarshan with coauthors in [40–43]. The famous book *Foundations of Quantum Optics* [44] written by John Klauder and George Sudarshan and translated into Russian is an excellent textbook on foundation of quantum theory; it is used in all universities of the world, including universities in Russia, by students and professors to obtain basic knowledge in developing new quantum technologies in the future.


Foundations of Quantum Optics – Russian edition.

The results of Glauber and Sudarshan mentioned above provided the theoretical basis in studies of the evolution of open quantum systems, theory of quantum channels, and applications of these approaches in future quantum technologies.

Therefore, we are extremely pleased to be able to witness today these deep international connections and we show below a kaleidoscope of pictures taken recently and long enough time ago connected with participants of the CEWQO series (with Professor Jozsef Janszky, the Founder) and the ICSSUR series (with Professor Young Suh Kim, the Founder), as well as some other meetings.

CEWQO 2000, Balatonfüred, Hungary.

P. Tombesi, M. A. Man'ko, R. J. Glauber, Y. S. Kim, and D. Han at ICSSUR 1992, Moscow.

J. Janszky and M. A. Man'ko at Wigner Centennial Conference 2002, Pecs, Hungary.


CEWQO 2006, Vienna, Austria.

G. Sudarshan with his wife Bhamathi, Y. S. Kim, and M. Man'ko at Quantumlike Models 1994, Erice, Sicily, Italy.

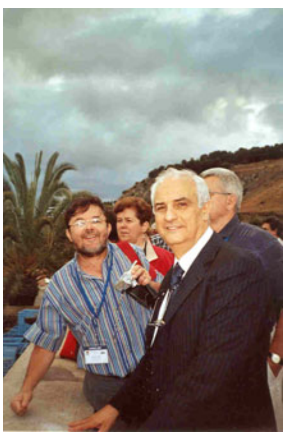
E. C. G. Sudarshan at CEWQO 2007, Palermo, Italy.

J. Janszky, J. Klauder, and M. A. Man'ko at ICSSUR 2003, Puebla, Mexico.


Y. S. Kim, J. Janszky, and V. I. Man'ko at ICSSUR 2003, Puebla, Mexico.

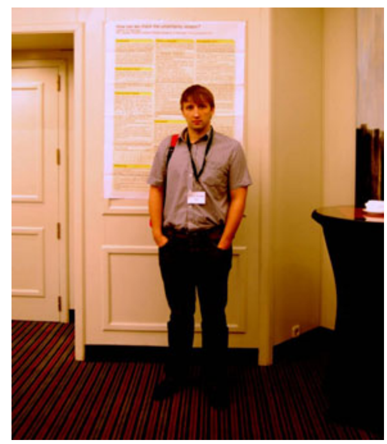
CEWQO 2008, Beograde, Serbia.

CEWQO 2009, Turku, Finland.



A. Isar and A. Messina at CEWQO 2007, Palermo, Italy.

S. Stenholm at CEWQO 2009, Turku, Finland.



V. Chernega at CEWQO 2011, Madrid.

CEWQO 2017, Lyngby, Denmark.


CEWQO 2018, Palma de Mallorca, Spain.

M. A. Man'ko, A. Zeilinger, and K.-A. Suominen
at CEWQO 2009, Turku, Finland.

E. C. G. Sudarshan and M. Paris
at CEWQO 2007, Palermo, Italy.

Springer exhibition at CEWQO 2007, Palermo, Italy.


CEWQO 2019, Paderborn, Germany.

E. C. G. Sudarshan at CEWQO 2007, Palermo, Italy.

M. Man'ko and R. Glauber at ICSSUR 1992, Moscow.

S. Wallentowitz, M. A. Man'ko, and A. Wünsche at Social Dinner at CEWQO 2004, Trieste, Italy.


A. Solomon at CEWQO 2007, Palermo, Italy.


O. V. Man'ko at CEWQO 2008, Beograde, Serbia.

D. Davidović, V. A. Andreev, V. V. Dodonov, V. S. Gorelik, and M. A. Man'ko at CEWQO 2008, Beograde, Serbia.

A. Zeilinger at CEWQO 2009, Turku, Finland.

A. Messina, T. Marian, and A. Isar with students at CEWQO 2017, Lyngby, Denmark.

## 3. Classical and Quantum States and Hidden Correlations

To clarify the idea of introducing the notion of "hidden correlations," first we consider some relations between the positive numbers on a numerical example. Namely, let us have four nonnegative numbers $\frac{1}{10}$, $\frac{2}{10}$, $\frac{4}{10}$, $\frac{3}{10}$. These numbers satisfy the "normalization condition" $\frac{1}{10}+\frac{2}{10}+\frac{4}{10}+\frac{3}{10}=1.$

These numbers can be used to produce other pairs of numbers like $\frac{1}{10}+\frac{2}{10}=\frac{3}{10}$, $\frac{4}{10}+\frac{3}{10}=\frac{7}{10}$

and $\dfrac{1}{10}+\dfrac{4}{10}=\dfrac{5}{10}$, $\dfrac{2}{10}+\dfrac{3}{10}=\dfrac{5}{10}$, which are also normalized, i.e., $\dfrac{3}{10}+\dfrac{7}{10}=1$ and $\dfrac{5}{10}+\dfrac{5}{10}=1$.

From these four numbers, it is possible to create pairs with the same normalization condition, i.e., to produce the numbers $\dfrac{1}{10}\left[\dfrac{1}{10}+\dfrac{4}{10}\right]^{-1}=\dfrac{1}{5}$ and $\dfrac{4}{10}\left[\dfrac{1}{10}+\dfrac{4}{10}\right]^{-1}=\dfrac{4}{5}$.

There exist also other analogous possibilities. Now we clarify how these procedures are related to probabilistic properties of a system with two subsystems or with two random variables.

Assume that we have two nonideal classical coins I and II in such a game as coin flipping, coin tossing, or heads (UP, $\oplus$) or tails (DOWN, $\ominus$), which is the practice of throwing a coin in the air and checking which side is showing when it lands, in order to choose between two alternatives $P_k$ or $(1-P_k)$; $k=1,2$.

There are four possible results of such coin tossing realizing four possibilities $\begin{array}{cccc} I & II & III & IV \\ \oplus\oplus & \oplus\ominus & \ominus\oplus & \ominus\ominus \end{array}$.

We denote the probabilities of these four results using the notation for four numbers, e.g., the mentioned ones $\dfrac{1}{10}$, $\dfrac{2}{10}$, $\dfrac{4}{10}$, $\dfrac{3}{10}$, namely, $P(1,1)=\dfrac{1}{10}$, $P(1,2)=\dfrac{2}{10}$, $P(2,1)=\dfrac{4}{10}$, and $P(2,2)=\dfrac{3}{10}$.

In this picture, the probabilities to have the head (UP, $\oplus$) or the tail (DOWN, $\ominus$)
for the first coin: $\mathcal{P}(1)=P(1,1)+P(1,2)=\dfrac{3}{10}$ and $\mathcal{P}(2)=P(2,1)+P(2,2)=\dfrac{7}{10}$;
for the second coin: $\Pi(1)=P(1,1)+P(2,1)=\dfrac{5}{10}$ and $\Pi(2)=P(1,2)+P(2,2)=\dfrac{5}{10}$.

After introducing an analogous notation for the numbers $P(j,k)$; $j,k=1,2$, we see that addition of the numbers received the sense of probabilities characterizing the situation with two random variables – positions of the first and second coins considered as a composite system containing these two coins as subsystems. The second procedure providing two pairs of nonnegative numbers like $\dfrac{1}{5}$ and $\dfrac{4}{5}$ has also a probabilistic interpretation. In view of the notation introduced, we have

$$\frac{1}{5}=\frac{P(1,1)}{P(1,1)+P(2,1)} \qquad \text{and} \qquad \frac{4}{5}=\frac{P(2,1)}{P(1,1)+P(2,1)}. \tag{1}$$

We denote the first and second numbers as $P(1\mid 1)=\dfrac{1}{5}$ and $P(2\mid 1)=\dfrac{4}{5}$. Thus, $\dfrac{1}{5}$ is the conditional probability $P(1\mid 1)$ for the first coin to have position "head" (UP, $\oplus$) provided the second coin has also position "head" (UP, $\oplus$). Also, $\dfrac{4}{5}$ is the conditional probability $P(2\mid 1)$ for the first coin to have position "tail" (DOWN, $\ominus$) provided the second coin has position "head" (UP, $\oplus$).

Now we give another probabilistic interpretation to the same four numbers by introducing a different notation for them, namely,

$$\frac{1}{10}=P(1), \qquad \frac{2}{10}=P(2),$$
$$\frac{4}{10}=P(3), \qquad \frac{3}{10}=P(4).$$

We can consider these numbers as probabilities given by the probability distributions of four pockets on the wheel of a casino roulette (Fig. 1) assuming that the roulette is nonideal (this means that the pockets are not absolutely equivalent).

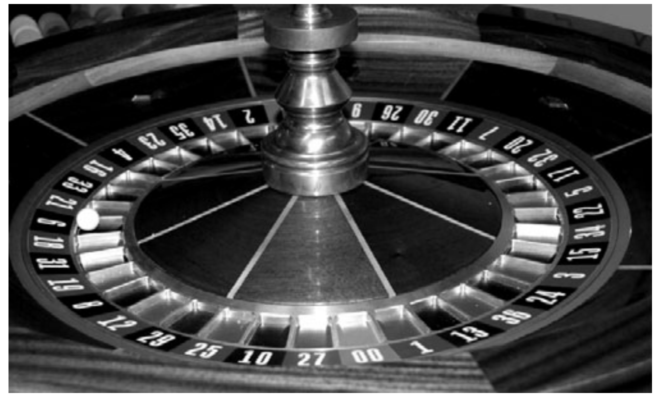


**Fig. 1.** The casino roulette.

The probability distribution $P(n)$; $n = 1, 2, 3, 4$, is the distribution of only one random variable.

On the other hand, the probability distribution function takes exactly the same values that the probability distribution of two random variables considered for the positions of the two coins. Also, in the case of two random variables, one has the notion of correlations of the two-subsystem degrees of freedom.

There is no correlation if $P(j,k) = \mathcal{P}(j)\Pi(k)$. In this case, the joint probability distribution $P(j,k)$ is the product of marginal probability distributions $\mathcal{P}(j)$ and $\Pi(k)$ describing statistics of both coins, which behave completely without the dependence of their positions on each of the other in the tossing coin game.

One can see that our four numbers do not satisfy this condition for the chosen example $\frac{1}{10} \neq \frac{7}{10} \cdot \frac{1}{2}$. Correlations mean that in two-coin tossing there is some statistical dependence of the second-coin position on the first-coin position. This dependence is characterized by mutual information, which is the difference of Shannon entropies

$$\begin{aligned} I &= P(1,1)\ln P(1,1) + P(1,2)\ln P(1,2) + P(2,1)\ln P(2,1) + P(2,2)\ln P(2,2) \\ &\quad -\mathcal{P}(1)\ln\mathcal{P}(1) - \mathcal{P}(2)\ln\mathcal{P}(2) - \Pi(1)\ln\Pi(1) - \Pi(2)\ln\Pi(2) \geq 0. \end{aligned} \tag{2}$$

This inequality is called the subadditivity condition.

If there is no correlation, the mutual information is equal to zero.

In our numerical example,

$$I = \frac{1}{10}\ln\frac{1}{10} + \frac{2}{10}\ln\frac{2}{10} + \frac{4}{10}\ln\frac{4}{10} + \frac{3}{10}\ln\frac{3}{10} - 2\frac{5}{10}\ln\frac{5}{10} - \frac{3}{10}\ln\frac{3}{10} - \frac{7}{10}\ln\frac{7}{10} > 0. \tag{3}$$

This means that there are correlations in statistical behavior of two coins in our example, which we checked assuming the absence of the property $P(j,k) = \mathcal{P}(j)\Pi(k)$. We see that all the properties we discussed are based on the properties of chosen nonnegative numbers. In view of this fact, we address the following problem.

For the casino roulette, we have exactly the same four numbers $\frac{1}{10}$, $\frac{2}{10}$, $\frac{4}{10}$, and $\frac{3}{10}$, which are probabilities. The only difference from the case of two coins is our different notation for these numbers $P(1)$, $P(2)$, $P(3)$, and $P(4)$ instead of $P(1,1)$, $P(1,2)$, $P(2,1)$, and $P(2,2)$; nevertheless, the properties of these numbers are the same. We can produce the same calculations using the map of indices (or arguments of probabilities $P(n)$ and $P(j,k)$) of the form

$$1 \Leftrightarrow 1,1,\ 2 \Leftrightarrow 1,2,\ 3 \Leftrightarrow 2,1,\ 4 \Leftrightarrow 2,2. \tag{4}$$

We see that there exist mutual information and formal correlations associated with the positive value of information $I$, which is determined only by the four nonnegative numbers. We call these correlations "hidden correlations" available in a noncomposite system (the casino roulette).

In the next sections, we discuss these aspects for Bayes' formula and for quantum systems.

First, we demonstrate hidden correlations in quantum states of the four-level atom or spin-3/2 system. The state of the atom is determined by the 4×4 density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \tag{5}$$

The idea of hidden correlations means that we can interpret this matrix $\rho$ as the density matrix of two artificial qubit states; for this, we rewrite it in the form

$$\rho = \begin{pmatrix} \rho_{11,11} & \rho_{11,12} & \rho_{11,21} & \rho_{11,22} \\ \rho_{12,11} & \rho_{12,12} & \rho_{12,21} & \rho_{12,22} \\ \rho_{21,11} & \rho_{21,12} & \rho_{21,21} & \rho_{12,22} \\ \rho_{22,11} & \rho_{22,12} & \rho_{22,21} & \rho_{22,22} \end{pmatrix}. \tag{6}$$

This matrix can be considered as the density matrix of a composite system of two qubits.

The density matrix of the first artificial qubit is given by the partial tracing over indices of the second qubit, i.e., $\rho_{jj'}(1) = \sum_{k=1}^{2} \rho_{jk,j'k}$; it reads

$$\rho(1) = \begin{pmatrix} \rho_{11,11} + \rho_{12,12} & \rho_{11,21} + \rho_{12,22} \\ \rho_{21,11} + \rho_{22,12} & \rho_{21,21} + \rho_{22,22} \end{pmatrix}. \tag{7}$$

Using the initial notation, the form of this matrix is

$$\rho(1) = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix}. \tag{8}$$

Analogously, the density matrix of the second artificial qubit is given by the partial tracing of the initial matrix over indices of the first qubit, i.e., $\rho_{kk'}(2) = \sum_{j=1}^{2} \rho_{jk,jk'}$; explicitly it reads

$$\rho(2) = \begin{pmatrix} \rho_{11,11} + \rho_{21,21} & \rho_{11,12} + \rho_{21,22} \\ \rho_{12,11} + \rho_{22,21} & \rho_{12,12} + \rho_{22,22} \end{pmatrix}. \tag{9}$$

Using the initial notation, the form of this matrix is

$$\rho(2) = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix}. \tag{10}$$

We see that there is the rule how to construct the density matrices, if the initial matrix $\rho = \rho^{\dagger}$ has the block form

$$\rho = \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}, \tag{11}$$

where blocks $R_1$, $R_2$, $R_3$, and $R_4$ are 2×2 matrices, and the recipe is the general one. Namely, we have the 2×2 matrix $\rho(1)$ given by the rule

$$\rho(1) = \begin{pmatrix} \mathrm{Tr}\, R_1 & \mathrm{Tr}\, R_2 \\ \mathrm{Tr}\, R_3 & \mathrm{Tr}\, R_4 \end{pmatrix}, \tag{12}$$

and the 2×2 matrix $\rho(2)$ reads $\rho(2) = R_1 + R_4$. By construction, matrices $\rho(1)$ and $\rho(2)$ are Hermitian,

$$\rho(1) = \rho^\dagger(1), \qquad \rho(2) = \rho^\dagger(2), \qquad \mathrm{Tr}\, R_1 = \mathrm{Tr}\, R_2 = 1, \tag{13}$$

and eigenvalues of these matrices are nonnegative.

Correlations of two qubits in the case where the 4×4 density matrix $\rho_{jk,j'k'}$ is the density matrix of composite system, are characterized by the mutual quantum information

$$I_q = -\,\mathrm{Tr}\left[\rho(1)\ln\rho(1)\right] - \mathrm{Tr}\left[\rho(2)\ln\rho(2)\right] + \mathrm{Tr}\,(\rho\ln\rho) \geq 0;$$

in the case of no correlations, i.e., $\rho = \rho(1)\otimes\rho(2)$, the mutual quantum information $I_q = 0$.

On the other hand, even if the density matrix $\rho$ is the density matrix of the noncomposite system (like four-level atom or spin-3/2 system), all numerical relations we employed for the case of two-qubit system are valid. For example, the mutual quantum information characterizes the presence of correlations of artificial qubits. Also for noncomposite system (four-level atom or spin-3/2 system), one can introduce the notion of strong quantum correlations, which are associated with an analog of entangled state of composite system like the two-qubit system. For example, if the density matrix of the four-level atom or spin-3/2 system $\rho_{mm'}$ has the form

$$\rho = \begin{pmatrix} \rho_{3/2,3/2} & \rho_{3/2,1/2} & \rho_{3/2,-1/2} & \rho_{3/2,-3/2} \\ \rho_{1/2,3/2} & \rho_{1/2,1/2} & \rho_{1/2,-1/2} & \rho_{1/2,-3/2} \\ \rho_{-1/2,3/2} & \rho_{-1/2,1/2} & \rho_{-1/2,-1/2} & \rho_{-1/2,-3/2} \\ \rho_{-3/2,3/2} & \rho_{-3/2,1/2} & \rho_{-3/2,-1/2} & \rho_{-3/2,-3/2} \end{pmatrix}, \tag{14}$$

with

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \tag{15}$$

being the matrix elements, it cannot be presented in a separable form, i.e.,

$$\rho \neq \sum_k p_k \rho_1^{(k)} \otimes \rho_2^{(k)}, \qquad 1 \geq p_k \geq 0, \qquad \sum_k p_k = 1, \tag{16}$$

where $\rho_1^{(k)}$ and $\rho_2^{(k)}$ are 2×2 density matrices due to the Peres–Horodecki criterion. The matrix $\rho$ formally corresponds to the entangled state (Bell state) of two artificial qubits. On the other hand, it is a pure superposition state of the spin-3/2 particle with spin projections +3/2 and −3/2; it reads

$$|\,\psi\rangle = \frac{1}{\sqrt{2}}\left(\,|\,3/2, 3/2\rangle + |\,3/2, -3/2\rangle\,\right). \tag{17}$$

For two spin-1/2 particles, this state is associated with the pure superposition state

$$|\,\psi\rangle = \frac{1}{\sqrt{2}}\big(\,|\,1/2,1/2\rangle\otimes\,|\,1/2,1/2\rangle+\,|\,1/2,-1/2\rangle\otimes\,|\,1/2,-1/2\rangle\,\big). \tag{18}$$

The one-to-one correspondence of the state of two spin-1/2 particles and the state of one spin-3/2 particle is described by the map of spin-3/2 particle projections onto a pair of projections of two spin-1/2 particles onto the $z$ axis, namely,

$$3/2 \Leftrightarrow 1/2,1/2, \quad 1/2 \Leftrightarrow 1/2,-1/2, \quad -1/2 \Leftrightarrow -1/2,1/2, \quad -3/2 \Leftrightarrow -1/2,-1/2. \tag{19}$$

Using this map of indices, one can formally map all correlation properties of the two-qubit system onto "hidden-correlation" properties of one ququart system.

An analogous consideration can be presented for the $N\times N$ density matrix of any qudit system (noncomposite system) state if $N = n_1m_1$. We can interpret its correlation properties as correlation properties of two qudit systems. We can study the entanglement properties and introduce the entropies and quantum information for a qudit with $N\times N$ density matrix based on entropies of artificial qudits with $n_1\times n_1$ and $m_1\times m_1$ density matrices of artificial qudit states determined by the initial qudit $N\times N$ density matrix, respectively.

We formulate the rule as a conjecture for a given $N\times N$ density matrix of a qudit state to obtain two density matrices $\rho(1)$ and $\rho(2)$ in the case of equality $N = nm$. We represent the $N\times N$ matrix in block form

$$\rho = \begin{pmatrix} R_{11} & R_{12} & \cdots & R_{1m} \\ R_{21} & R_{22} & \cdots & R_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ R_{m1} & R_{m2} & \cdots & R_{mm} \end{pmatrix}, \tag{20}$$

where $R_{jk}$; $j,k = 1,2,\ldots,m$ are $n\times n$ matrices. The $m\times m$ matrix $\rho(1)$ of the form

$$\rho(1) = \begin{pmatrix} \operatorname{Tr} R_{11} & \operatorname{Tr} R_{12} & \cdots & \operatorname{Tr} R_{1m} \\ \operatorname{Tr} R_{21} & \operatorname{Tr} R_{22} & \cdots & \operatorname{Tr} R_{2m} \\ \cdots & \cdots & \cdots & \cdots \\ \operatorname{Tr} R_{m1} & \operatorname{Tr} R_{m2} & \cdots & \operatorname{Tr} R_{mm} \end{pmatrix} \tag{21}$$

is Hermitian, $\rho^\dagger(1) = \rho(1)$, $\operatorname{Tr}\rho(1) = 1$, and its eigenvalues are nonnegative.

The other $n\times n$ matrix $\rho(2) = R_{11} + R_{22} + \cdots + R_{mm}$ is also Hermitian, $\rho^\dagger(2) = \rho(2)$, $\operatorname{Tr}\rho(2) = 1$, and its eigenvalues are nonnegative. Thus, for a given $N\times N$ density matrix of an arbitrary qudit state such that $N = nm$, we have two artificial qudits with density matrices $\rho(1)$ and $\rho(2)$, which we obtain in view of the representation of the initial $N\times N$ density matrix in block form. A rigorous proof of these matrix properties is presented in the work of G. Marmo et al. [45]. This property follows from the bijective map of indices labeling the matrix elements of the initial matrix $\rho_{jj'}$; $j,j' = 1,2,\ldots,N$ onto a pair of indices $j \Leftrightarrow (\alpha,\beta)$, $j' \Leftrightarrow (\alpha',\beta')$, corresponding to labeling the density matrix of composite system of two qudits.

Then we apply the tool of partial tracing

$$\rho_{\alpha\beta,\alpha'\beta'} \mapsto \rho_{\alpha\alpha'}(1) = \sum_\beta \rho_{\alpha\beta,\alpha'\beta}, \qquad \rho_{\alpha\beta,\alpha'\beta'} \mapsto \rho_{\beta\beta'}(2) = \sum_\alpha \rho_{\alpha\beta,\alpha\beta'}, \tag{22}$$

which provides the density matrices of each of the qudits. Since numerically the density matrix has the same form for both composite and noncomposite systems, the obtained numerical matrices $\rho(1)$ and $\rho(2)$ have the same properties. For a composite system of two qudits, the partial tracing procedure is equivalent to constructing the matrix $\rho$ in block form and obtaining the matrices $\rho(1)$ and $\rho(2)$ from these blocks. Thus, all correlation properties available for composite systems containing two qubits are formally available for artificial qubit states corresponding to density matrices $\rho(1)$ and $\rho(2)$, which we called "hidden correlations."

If the number $N = nml$, the partial tracing can be done first using the above described procedure where we introduce the number $\widetilde{m} = nl$ and consider the formulated tool for $N = n\widetilde{m}$. After this, we can apply the elaborated tool for the $\widetilde{m}\times\widetilde{m}$ density matrix; this means that we can describe "hidden correlations" for three artificial qudits associated with the $N\times N$ density matrix if $N = nml$.

An example of such situation is the 8×8 density matrix $\rho$ of spin-7/2 (or eight-level atom) system; it can be considered as the density matrix of three artificial spin-1/2 (or three two-level atom) systems.

For such density matrix, one has the strong subadditivity condition

$$-\operatorname{Tr}\rho\ln\rho - \operatorname{Tr}\rho(\widetilde{2})\ln\rho(\widetilde{2}) \leq -\operatorname{Tr}\rho(1,2)\ln\rho(1,2) - \operatorname{Tr}\rho(2,3)\ln\rho(2,3); \tag{23}$$

here, if the matrix $\rho$ corresponds to the composite-system state of three qubits, the 2×2 matrix $\rho(\widetilde{2})$ is the density matrix of the second qubit, the 4×4 matrix $\rho(1,2)$ is the density matrix of the first and second qubits, and the 4×4 matrix $\rho(2,3)$ is the density matrix of the second and third qubits.

If the matrix $\rho$ describes one qudit (spin-7/2 or eight-level atom system), the same numerical matrices in this inequality are the 2×2 matrix $\rho(\widetilde{2})$ and the 4×4 matrices $\rho(1,2)$ and $\rho(2,3)$ describing the density matrices of artificial qubits. For the qudit state of the noncomposite system, the presented inequality is a new relation that was not considered in the literature. Analogous new inequalities can be considered for other noncomposite systems like qudits with $N = n_1 n_2 \cdots n_s$.

## 4. Random Variables and Probabilities

After considering specific examples of classical random variables and quantum qubit and ququart systems, we present a general consideration of noncomposite system states with an arbitrary number $N$. In probability theory, the notion of random variables and probability distributions was discussed using rigorous approaches presented in many books, for example, in [46–49].

We employ here the following empiric approach. We define the relation of random variables to sets of integer numbers following [1, 6, 8, 9, 11, 14].

Given a set of $N$ different events, these events are associated with integers $j = 1, 2, \ldots, N$. We call the relative frequencies $P(j)$ of realization of these random events in a series of experiments "the probabilities of the events" where $0 \leq P(j) \leq 1$. The function $P(j)$ is the probability distribution; it is normalized, $\sum_{j=1}^{N} P(j) = 1$. The properties of the events are characterized by some functions $f(j)$, which we call observables. In this approach, random variables are mapped onto the integers $j = 1, 2, \ldots, N$. The physical meaning of the events can be different; for example, in the casino roulette, the event is the appearance of some integer number $j$ chosen from a set of integer numbers located between 1 and $N$. The event may be also considered as positions "UP" and "DOWN" of coins; in this case, the integer number $j$ is mapped onto a pair $(a, b)$ of integer numbers labeling the position of each coin.

In both cases, the relative frequencies of the events can be associated with the integer $j$, but the interpretation of this random variable is different. In the case of the casino roulette, we speak of one random variable, and, in the case of coins, we have random variables associated with labeling positions of two coins by other two integer numbers $(a, b)$. An analogous approach to random events can be employed in quantum mechanics.

We extend the above approach to classical probabilities using in this case the identification of random events with the integers $1 \leq j, j' \leq N$ labeling the matrix elements of the density matrix $\rho_{jj'}$ determining the states, e.g., of qudit with spin $s$, where $N = 2s + 1$, or of the $N$-level atom. The physical observables are given by the Hermitian matrices $f_{jj'}$, where the indices of rows and columns are identified with the random variables $1 \leq j, j' \leq N$. It is important that we can interpret the above described association of integers $j$ analogously to the case of a classical casino roulette and the case of classical coins considering numerically the same density matrices $\rho_{jj'}$ either as the density matrices of noncomposite (nondivisible) systems (an analog of the casino roulette) or as the density matrices of bipartite systems (an analog of the states of coins).

In the next section, we consider Bayes' formula, in view of the approach under discussion, using it for one random variable and applying the map of integer numbers $1, 2, \ldots, N$ onto pairs of random numbers.

## 5. Bayes' Formula for the Probability Distribution of One Random Variable

In this section, we discuss the application of Bayes' formula available for probability distributions of several random variables to the case of the probability distribution of one random variable. Bayes' formula is one of the basic results of the probability theory for classical random variables [46–49].

First, we recall Bayes' formula and the notion of conditional probability distribution for the statistics of two random variables. Given the function $1 \geq P(j, k) \geq 0$, where $j = 1, 2, \ldots, n_1$, $k = 1, 2, \ldots, n_2$, and $n_1 n_2 = N$, with the normalization condition $\sum_{j=1}^{n_1} \sum_{k=1}^{n_2} P(j, k) = 1$. This function is identified with the probability distribution of two random variables $j$ and $k$.

The marginal probability distributions

$$P_1(j) = \sum_{k=1}^{n_2} P(j, k) \qquad \text{and} \qquad P_2(k) = \sum_{j=1}^{n_1} P(j, k)$$

determine the statistical properties of each random variable. The conditional probability distribution of the first random variable $j$ for given $k$ is presented by the formula $P(j \mid k) = \dfrac{P(j, k)}{P_2(k)}$, which means that $P(j, k) = P_2(k) P(j \mid k)$.

For the case of joint probability distributions of two random variables describing the statistics of the bipartite system, these relations correspond to Bayes' formula connecting marginal probability distributions and conditional probability distributions of these random variables. We present an example of application of the above formulas for a particular case $N = 4$. In view of the example, we are in the position to formulate the rule for introducing Bayes' formula for the probability distribution $\mathcal{P}(n)$; $N = n_1 n_2$ of one random variable. We apply the map of integers $n$ onto pairs of integers $j$ and $k$, such that $j = 1, 2, \ldots, n_1$ and $k = 1, 2, \ldots, n_2$. Then, for marginal probability distributions and conditional probability distributions, we use the known expression for joint probability distribution of two variables and define these distributions, in view of the invertible map of integers $1, 2, \ldots, N \leftrightarrow (1,1), (2,1), \ldots, (n_1,1)(1,2), (2,2), \ldots, (n_1,2), \ldots, (n_1,n_2)$, where $N = n_1 n_2$.

We determine the functions $y(x_1,x_2)$, $x_1(y)$, and $x_2(y)$, where $1 \leq x_1 \leq X_1$, $1 \leq x_2 \leq X_2$, and $1 \leq y \leq N = X_1 X_2$, as

$$y(x_1,x_2) = x_1 + (x_2-1)X_1, \tag{24}$$

$$x_1(y) = y \mod X_1, \quad 1 \leq y \leq N, \tag{25}$$

$$x_2(y) - 1 = \frac{y - x_1(y)}{X_1} \mod X_2, \quad 1 \leq y \leq N. \tag{26}$$

We use these functions for representing the probability distribution of one random variable as a joint probability distribution of two random variables. To do this, we introduce in (24)–(26) the following notation: $y \equiv n$, $x_1 \equiv j$, $x_2 \equiv k$, $X_1 \equiv n_1$, $X_2 \equiv n_2$, $N = n_1 n_2 = X_1 X_2$, $n = 1, 2, \ldots, N$, $P(j,k) \equiv y(x_1,x_2)$, and $f(y) = \mathcal{P}(n)$. In the case of $N = 4$ and $n_1 = n_2 = 2$, the map introduced just provides the relations $\mathcal{P}(1) = P(1,1)$, $\mathcal{P}(2) = P(2,1)$, $\mathcal{P}(3) = P(1,2)$, and $\mathcal{P}(4) = P(2,2)$ discussed above. Nevertheless, the functions introduced describe the invertible map of the probability distribution of one random variable $\mathcal{P}(n)$; $n = 1, 2, \ldots, N$ onto the joint probability distribution $P(j,k)$ of two random variables $j = 1, 2, \ldots, n_1$ and $k = 1, 2, \ldots, n_2$, with $N = n_1 n_2$, for arbitrary integers $n_1$ and $n_2$. In our new notation, $n = n(j,k)$, $j = j(n)$, and $k = k(n)$. Taking into account this discussion, we introduce Bayes' formula for the probability distribution $\mathcal{P}(n)$ of one random variable; it reads

$$P\big(j(n) \mid k(n)\big) = \frac{\mathcal{P}\big(n(j,k)\big)}{\sum_{j=1}^{n} \mathcal{P}\big(n(j,k)\big)}, n = n_1 n_2. \tag{27}$$

where the functions $n(j,k)$, $j(n)$, and $k(n)$ are constructed by Zhanat Zeilov [50]. The relation of the joint probability distribution $P(j,k)$ to the marginal probability distributions corresponds to the presence of correlations in the system with two random variables. Since we introduced an analog of two random variables and their marginal and conditional probability distributions, the relation between these distributions reflects correlations, which we called hidden correlations for systems without subsystems. Such correlations exist for both classical and quantum systems.

## 6. Conclusions

We demonstrated that quantum systems without subsystems (one qudit) have the density matrices of their states with properties analogous to the properties of the density matrices of composite system states. This means that correlations in composite system states formally have analogs also for the density matrices of noncomposite system states.

For these noncomposite system states, we considered the notion of mutual information characterizing "hidden correlations" and discussed information-entropic inequalities available for single qudit states associated with artificial qudits determined by the initial density matrix.

For qudit states, the density operator $\hat{\rho}(q)$ is bijectively mapped onto the density operator $\hat{\rho}(j,k)$, where two artificial qudits are introduced. For example, the density operator of four-level atom can be mapped bijectively onto the density operator of two qubits. In view of the map constructed, after obtaining an analog of Bayes' formula for the classical system with one random variable, we are in the position to write a new information-entropic inequality for single qudit states [5]. For example, we show that the ququart-state density 4×4-matrix mapped onto the density matrix of two artificial qubits demonstrates hidden correlations expressed through the violation of the formal Bell's inequalities, being given as some relations of matrix elements of the ququart-state density matrix.

Since the density matrix of an arbitrary qudit state can be mapped onto a set of probability distributions of classical-like random variables [5,51–57] or a single specific probability distribution of one random variable, the new information-entropic inequalities for matrix elements of the qudit density matrix can be derived. These inequalities can be checked experimentally using superconducting circuit devices based on employing Josephson junctions.

Some new information-entropic inequalities for the states of systems in thermodynamic equilibrium are obtained in [58]. The fundamental results of probability theory [46,47,49] related to entropic inequalities can be extended to the case of probabilities describing quantum states [22–25,59,60] by taking into account specific quantum correlations in quantum-system states, and this will be done in a future publication.

We obtained an analog of Bayes' formula (27) for systems without subsystems.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