

# TRIANGLE GEOMETRY FOR QUTRIT STATES IN THE PROBABILITY REPRESENTATION

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## Abstract

We express the matrix elements of the density matrix of the qutrit state in terms of probabilities associated with artificial qubit states. We show that the quantum statistics of qubit states and observables is formally equivalent to the statistics of classical systems with three random vector variables and three classical probability distributions obeying special constraints found in this study. The Bloch spheres geometry of qubit states is mapped onto triangle geometry of qubits. We investigate the triada of Malevich's squares describing the qubit states in quantum suprematism picture and the inequalities for the areas of the squares for qutrit (spin-1 system). We expressed quantum channels for qutrit states in terms of a linear transform of the probabilities determining the qutrit-state density matrix.

**Keywords:** qudit, Malevich's squares, probability representation, density matrix.

## 1. Introduction

The pure states of qutrit are described by a vector in the three-dimensional Hilbert space [1]. The mixed states of qutrit are described by the three-dimensional density matrix [2]. The qutrit states can be realized as the states of a spin-1 particle or as the states of the three-level atom. The density matrix of the spin state in the spin tomographic probability representation [3,4] is determined by a fair probability distribution of spin projections on arbitrary directions in the space called the spin tomogram. The von Neumann entropy [5] of the qutrit state was shown [6] to satisfy the entropic inequality, which is the subadditivity condition analogous to the subadditivity condition for bipartite systems of two qubits. Recently [7,8], the triangle geometry of qubit states, in which the density matrix of the spin-1/2 particle was associated with the triada of Malevich's squares, was investigated. The areas of the Malevich's squares are determined by three tomographic probabilities of spin projections  $m = 1/2$  onto three perpendicular directions in the space.

The aim of this work is to construct the triada of Malevich's squares associated with the density matrix of qutrit states using the approach connecting the qutrit states with the states of two artificial

qubits found in [6] and extra artificial qubit associated with the permutation of the axes  $x \longleftrightarrow z$  in the three-dimensional space. We review the probability description of qubit states [7–9] and derive compact formulas for spin tomograms of these states. We use the relation of qutrit states to the states of artificial qubits to express the density matrix elements of the qutrit state in terms of probabilities of the spin-1/2 projection.

This paper is organized as follows.

In Sec. 2, we review the quantum suprematism picture of spin-1/2 particle states suggested in [7, 8]. In Sec. 3, we discuss the statistical properties of the quantum spin-1/2 observable. In Sec. 4, we consider the qutrit-state density matrix and express its matrix elements in terms of probabilities of spin-1/2 projections related to three artificial qubit states connected with the given indivisible qutrit system. In Sec. 5, we discuss the triangle geometry of the qutrit state and study the inequalities for the tomographic probabilities determining the state density matrix. We present our conclusions and prospectives in Sec. 6.

## 2. Qubits in the Quantum Suprematism Picture

The density matrix of qubit states is the Hermitian  $2 \times 2$  matrix  $\rho$  satisfying the conditions  $\rho^\dagger = \rho$ ,  $\text{Tr} \rho = 1$ , and  $\rho \geq 0$ . This means that the density matrix has two eigenvalues, which are nonnegative numbers  $\lambda_1$  and  $\lambda_2$ , with  $\lambda_1 + \lambda_2 = 1$ . We consider the matrix  $\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$ . The eigenvalues  $\lambda_1$  and  $\lambda_2$  of the density matrix  $\rho$  satisfy the equation

$$(\rho_{11} - \lambda)(\rho_{22} - \lambda) - \rho_{12}\rho_{21} = 0. \quad (1)$$

It was shown in [9] that the matrix elements of the density matrix  $\rho$  can be expressed within the framework of the probability representation of qubit states in terms of three probabilities  $0 \leq p_1, p_2, p_3 \leq 1$ , namely,

$$\rho = \begin{pmatrix} p_3 & p_1 - ip_2 - (1/2) + (i/2) \\ p_1 + ip_2 - (1/2) - (i/2) & 1 - p_3 \end{pmatrix}. \quad (2)$$

In this expression, nonnegative probabilities  $p_1$ ,  $p_2$ , and  $p_3$  are the probabilities of spin-1/2 projections  $m = 1/2$  onto three perpendicular directions in the space, namely,  $p_1$  is the probability to have the spin projection along the  $x$  direction,  $p_2$  is the probability to have the spin projection along the  $y$  direction, and  $p_3$  is the probability to have the spin projection along the  $z$  direction. The eigenvalues of the density matrix (2) read

$$\lambda_1 = \frac{1}{2} + \left[ \sum_{j=1}^3 \left( p_j - \frac{1}{2} \right)^2 \right]^{1/2}, \quad \lambda_2 = \frac{1}{2} - \left[ \sum_{j=1}^3 \left( p_j - \frac{1}{2} \right)^2 \right]^{1/2}. \quad (3)$$

The nonnegativity of the density matrix provides the inequality [9] for three probabilities  $p_j$ , namely,

$$(p_1 - 1/2)^2 + (p_2 - 1/2)^2 + (p_3 - 1/2)^2 \leq 1/4. \quad (4)$$

This inequality is the nonnegativity condition for the density matrix of the qubit state; it reflects the presence of quantum correlations of the single-spin states.

There exists the geometrical interpretation of the introduced parameters of the spin-state density matrix. The probabilities  $p_1$ ,  $p_2$ , and  $p_3$  can be associated with a triangle on the plane [7,8]. The lengths  $L_n$  ( $n = 1, 2, 3$ ) of the triangle sides are expressed in terms of the probabilities as follows:

$$\begin{aligned} L_1 &= (2 + 2p_2^2 - 4p_2 - 2p_3 + 2p_3^2 + 2p_2p_3)^{1/2}, \\ L_2 &= (2 + 2p_3^2 - 4p_3 - 2p_1 + 2p_1^2 + 2p_3p_1)^{1/2}, \\ L_3 &= (2 + 2p_1^2 - 4p_1 - 2p_2 + 2p_2^2 + 2p_1p_2)^{1/2}. \end{aligned} \quad (5)$$

The probabilities  $p_1$ ,  $p_2$ , and  $p_3$  satisfy the inequality

$$L_n + L_{n-1} > L_{n+1}, \quad n = 1, 2, 3. \quad (6)$$

Three squares with these sides and the areas  $S_n = L_n^2$  were introduced in [7,8]; they were called the triada of Malevich's squares. The area of triangle with the sides  $L_n$  reads

$$S_{\text{tr}} = (1/4) [(L_1 + L_2 + L_3)(L_1 + L_2 - L_3)(L_2 + L_3 - L_1)(L_3 + L_1 - L_2)]^{1/2}. \quad (7)$$

Usually, the density matrix (2) is associated with a point in the Bloch ball. In the triangle geometry picture under discussion, the density matrix is represented by the triada of Malevich's squares. This means that we construct the invertible map of any point in the Bloch ball onto the triangle with sides  $L_n$  and the triada of Malevich's squares. The obvious inequalities for the triangle sides give the inequalities for the probabilities  $p_1$ ,  $p_2$ , and  $p_3$  (6). These inequalities are compatible with the condition (4). The three squares introduced in [7,8] and called the triada of Malevich's squares provide the quantum suprematism picture of the qubit states.\* It is worth noting that Zeilinger, emphasizing in [10] the importance in physics to make experiments as simple as possible and with the smallest efforts, compared such approach with the creation of Malevich's black square in the art.

The sum of areas of three Malevich's squares expressed in terms of the probabilities  $p_1$ ,  $p_2$ , and  $p_3$  reads

$$S = 2 [3(1 - p_1 - p_2 - p_3) + 2p_1^2 + 2p_2^2 + 2p_3^2 + p_1p_2 + p_2p_3 + p_3p_1]. \quad (8)$$

The sum satisfies the inequality

$$3/2 \leq S \leq 9/2. \quad (9)$$

For classical system of three coins, an analogous suprematism picture of Malevich's squares provides for this sum the domain  $3/2 \leq S \leq 6$ . The difference between numbers  $9/2$  and  $6$  reflects the difference of classical and quantum correlations in the two systems – qubit and three coins, though the states in both cases are determined by three probabilities  $p_1$ ,  $p_2$ , and  $p_3$ .

### 3. Statistical Properties of Quantum Observable

In this section, we discuss the properties of means of an observable  $A$  given by the Hermitian matrix  $A_{jk} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ . The mean values of the observable in the state with the density matrix (2) read

$$\langle A \rangle = \text{Tr } A\rho = p_3A_{11} + (1 - p_3)A_{22} + A_{12}(p_1 + ip_2 - (1 + i)/2) + A_{21}(p_1 - ip_2 - (1 - i)/2). \quad (10)$$

\*We thank Dr. Tommaso Calarco for informing us about available discussions of Malevich's square picture related to quantum states of a single atom (private communication).

This relation can be interpreted using the picture of three classical random observables, which are described by three probability distributions.

In fact, there are three probability vectors

$$\vec{P}_1 = \begin{pmatrix} p_1 \\ 1 - p_1 \end{pmatrix}, \quad \vec{P}_2 = \begin{pmatrix} p_2 \\ 1 - p_2 \end{pmatrix}, \quad \vec{P}_3 = \begin{pmatrix} p_3 \\ 1 - p_3 \end{pmatrix}.$$

For a spin-1/2 system, probabilities  $(1 - p_1)$ ,  $(1 - p_2)$ , and  $(1 - p_3)$  are the probabilities to have the spin-projection  $m = -1/2$  along the axes  $x$ ,  $y$ , and  $z$ , respectively. The matrix elements of the matrix  $A_{jk}$  ( $j, k = 1, 2$ ) can be considered as linear functions of classical random variables, which take the real values

$$X_1 = \frac{A_{12} + A_{21}}{2}, \quad Y_1 = \frac{i(A_{12} - A_{21})}{2}, \quad X_2 = -\frac{A_{12} + A_{21}}{2}, \quad Y_2 = -\frac{i(A_{12} - A_{21})}{2} \tag{11}$$

and

$$Z_1 = A_{11}, \quad Z_2 = A_{22}. \tag{12}$$

The inverse relations are

$$A_{12} = X_1 - iY_1, \quad A_{11} = Z_1, \quad A_{22} = Z_2, \quad A_{21} = X_1 + iY_1. \tag{13}$$

Introducing the vector notation for the classical variables

$$\vec{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}, \quad \vec{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \tag{14}$$

we obtain the expression for the mean value of quantum observable  $\langle A \rangle$  in terms of the mean values of classical observables  $\vec{X}$ ,  $\vec{Y}$ , and  $\vec{Z}$  of the form

$$\langle A \rangle = \vec{P}_1 \vec{X} + \vec{P}_2 \vec{Y} + \vec{P}_3 \vec{Z}. \tag{15}$$

Thus, the quantum relation for the mean value of the spin observable  $A$  in the state with the density matrix  $\rho$  given by (2) is presented as the sum of three classical means of random variables  $\vec{X}$ ,  $\vec{Y}$ , and  $\vec{Z}$ ,

$$\langle A \rangle = p_1 X_1 + (1 - p_1) X_2 + p_2 Y_1 + (1 - p_2) Y_2 + p_3 Z_1 + (1 - p_3) Z_2. \tag{16}$$

These observations provide a possibility to construct the model of quantum observable  $A$  using the classical observables  $\vec{X}$ ,  $\vec{Y}$ , and  $\vec{Z}$ .

In fact, for given arbitrary three real two-vectors  $\vec{X}$ ,  $\vec{Y}$ , and  $\vec{Z}$  such that  $X_1 + X_2 = Y_1 + Y_2 = 0$ , we construct the Hermitian matrix  $A_{jk}$  ( $j, k = 1, 2$ ) with matrix elements (13). Since the density matrix (2) is expressed in terms of classical probability vectors  $\vec{P}_1$ ,  $\vec{P}_2$ , and  $\vec{P}_3$ , the measurable quantum observable  $A$  has the mean value determined by classical observables  $\vec{X}$ ,  $\vec{Y}$ ,  $\vec{Z}$  and classical probability distributions.

Quantumness of the model is formulated as inequality (4) reflecting the condition for classical probabilities  $p_1$ ,  $p_2$ , and  $p_3$ , and the definition of the second moment of quantum observable  $\langle A^2 \rangle$  in terms of classical random variables  $\vec{X}$ ,  $\vec{Y}$ , and  $\vec{Z}$ ,

$$\begin{aligned} \text{Tr } \rho A^2 &= p_3 Z_1^2 - (1 - p_3) Z_2^2 + X_1^2 + Y_1^2 + 2(Z_1 + Z_2) [X_1(p_1 - 1/2) + Y_1(p_2 - 1/2)] \\ &= (Z_1 + Z_2) [\vec{X} \vec{P}_1 + \vec{Y} \vec{P}_2] + (X_1^2 + Y_1^2) + p_3(Z_1^2 - Z_2^2) + Z_2^2. \end{aligned} \tag{17}$$

The constructed relations (13)–(16) and formulas (16) and (17) for the quantum mean and dispersion of any observable  $A$ , expressed in term of classical random variables and classical probabilities, demonstrate that quantum mechanics of qubits can be formulated using only standard ingredients of classical probability theory. We conjecture that quantum mechanics of any qudit system can also be formulated using only classical random variables and classical probability distributions. The difference from classical statistical mechanics is expressed by specific inequalities for classical probabilities distributions, reflecting hidden correlations in quantum systems analogous to (4) for qubits.

### 4. Qutrit in the Probability Representation

The tomographic probability distribution for the spin-1 system for a minimum number of probabilities can be described by eight parameters, which are spin projections  $m = +1, 0$  onto four directions; these probabilities are discussed in [9]. In this section, we develop another approach to associate the density matrix of the qutrit state with probabilities determining the states of artificial qubits.

We follow the approach applied to get a new entropic subadditivity condition for the qutrit state suggested in [6]. The density matrix of the spin-1 system is given by the matrix  $\rho$ , such that  $\rho^\dagger = \rho$ ,  $\text{Tr } \rho = 1$ , and  $\rho \geq 0$ ; it reads

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} \\ \rho_{21} & \rho_{22} & \rho_{23} \\ \rho_{31} & \rho_{32} & \rho_{33} \end{pmatrix}. \tag{18}$$

Applying the tool to consider the matrix  $\rho$  as the  $3 \times 3$  block matrix in the  $4 \times 4$  density matrix of two qubits with zero fourth column and zero fourth row, we obtain two qubit-state density matrices of the artificial qubits using the partial tracing procedure. The  $2 \times 2$  matrices are

$$\rho(1) = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} \\ \rho_{31} & \rho_{33} \end{pmatrix}, \quad \rho(2) = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \tag{19}$$

For these two qubit-state density matrices, we have the expressions in the probability representation in terms of probabilities  $p_{1,2,3}^{(k)}$ ,  $k = 1, 2$ , of the form

$$\rho(k) = \begin{pmatrix} p_3^{(k)} & p_1^{(k)} - ip_2^{(k)} - (1/2) + (i/2) \\ p_1^{(k)} + ip_2^{(k)} - (1/2) - (i/2) & 1 - p_3^{(k)} \end{pmatrix}, \quad k = 1, 2. \tag{20}$$

This means that a part of the matrix elements of the density matrix  $\rho$  is expressed in terms of the probabilities  $p_j^{(k)}$ ,  $k = 1, 2$ ,  $j = 1, 2, 3$ , namely,

$$\rho_{11} = p_3^{(2)} - (1 - p_3^{(1)}), \quad \rho_{22} = 1 - p_3^{(2)}, \quad \rho_{33} = 1 - p_3^{(1)}. \tag{21}$$

For off-diagonal matrix elements, we have

$$\rho_{12} = p_1^{(2)} - ip_2^{(2)} - (1/2) + (i/2), \quad \rho_{21} = \rho_{12}^*, \tag{22}$$

$$\rho_{13} = p_1^{(1)} - ip_2^{(1)} - (1/2) + (i/2), \quad \rho_{31} = \rho_{13}^*. \tag{23}$$

To obtain an explicit expression for the matrix element  $\rho_{23}$  in terms of probabilities, we consider the density matrix of the state where we use the permutation of axes  $x \leftrightarrow z$ ; this means that we use another qutrit state. For a three-level atom, we use the permutation of the ground state level and maximum excited energy level; in such a case, we have the extra qubit with the density matrix

$$\rho^{(3)} = \begin{pmatrix} \rho_{33} + \rho_{11} & \rho_{32} \\ \rho_{23} & \rho_{22} \end{pmatrix}. \tag{24}$$

The probabilities  $p_j^{(3)}$  for this artificial qubit state read

$$p_3^{(3)} = \rho_{11} + \rho_{33} = p_3^{(2)}, \quad p_1^{(3)} - ip_2^{(3)} - (1 - i)/2 = \rho_{32}, \quad \rho_{23} = \rho_{32}^*. \tag{25}$$

Thus, we provide the final expression of the qutrit density matrix  $\rho$  in terms of eight parameters – probabilities  $p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_1^{(3)}$ , and  $p_2^{(3)}$ . The density matrix  $\rho$  is

$$\rho = \begin{pmatrix} p_3^{(2)} + p_3^{(1)} - 1 & p_1^{(2)} - ip_2^{(2)} - (1 - i)/2 & p_1^{(1)} + ip_2^{(1)} - (1 + i)/2 \\ p_1^{(2)} + ip_2^{(2)} - (1 + i)/2 & 1 - p_3^{(2)} & p_1^{(3)} + ip_2^{(3)} - (1 + i)/2 \\ p_1^{(1)} - ip_2^{(1)} - (1 - i)/2 & p_1^{(3)} - ip_2^{(3)} - (1 - i)/2 & 1 - p_3^{(1)} \end{pmatrix}. \tag{26}$$

The parameters  $p_j^{(k)}$ ,  $k, j = 1, 2, 3$  must satisfy the inequalities

$$\sum_{j=1}^3 (p_j^{(k)} - 1/2)^2 \leq 1/4. \tag{27}$$

In addition to these inequalities, one has the cubic inequality  $\det \rho \geq 0$  and the quadratic inequality like

$$(1 - p_3^{(2)})(1 - p_3^{(1)}) - |p_1^{(3)} + ip_2^{(3)} - (1 + i)/2|^2 \geq 0. \tag{28}$$

To check all the inequalities, one needs to provide the probabilities of spin-projections  $m = +1/2$  onto three perpendicular directions for the three artificial qubits. For the two qubits, the directions are given by the axes  $x, y$ , and  $z$ , and for the third qubit the direction corresponds to the permutation of the first and the third directions,  $x \leftrightarrow z$ .

The density matrix  $\rho$  can be rewritten in the form

$$\rho = \begin{pmatrix} p_3^{(2)} + p_3^{(1)} - 1 & p^{(2)*} - \gamma^* & p^{(1)} - \gamma \\ p^{(2)} - \gamma & 1 - p_3^{(2)} & p^{(3)} - \gamma \\ p^{(1)*} - \gamma^* & p^{(3)*} - \gamma^* & 1 - p_3^{(1)} \end{pmatrix}, \tag{29}$$

where the complex numbers  $p^{(k)}$  are  $p^{(k)} = p_1^{(k)} + ip_2^{(k)}$ ,  $k = 1, 2, 3$ , and  $\gamma = (1 + i)/2$ . Then we express the purity of the qutrit state  $\mu = \text{Tr} \rho^2$  in terms of three classical probabilities  $p_j^{(k)}$ ,  $j, k = 1, 2, 3$ ,

$$\mu = (p_3^{(2)} + p_3^{(1)} - 1)^2 + (1 - p_3^{(2)})^2 + (1 - p_3^{(1)})^2 + 2[|p^{(1)} - \gamma|^2 + |p^{(2)} - \gamma|^2 + |p^{(3)} - \gamma|^2]. \tag{30}$$

The nonnegativity condition of the density matrix  $\det \rho \geq 0$  yields the inequality for the probabilities  $p_j^{(k)}$ , which looks like the inequality for the cubic polynomial,

$$\begin{aligned} & (p_3^{(2)} + p_3^{(1)} - 1)(1 - p_3^{(2)})(1 - p_3^{(1)}) + (p^{(2)} - \gamma)(p^{(1)} - \gamma)(p^{(3)*} - \gamma^*) \\ & + (p^{(2)*} - \gamma^*)(p^{(1)*} - \gamma^*)(p^{(3)} - \gamma) - |p^{(1)*} - \gamma^*|^2(1 - p_3^{(2)}) - |p^{(2)} - \gamma|^2(1 - p_3^{(1)}) \\ & - |p^{(3)} - \gamma|^2(p_3^{(2)} + p_3^{(1)} - 1) \geq 0. \end{aligned} \tag{31}$$

The obtained inequalities (27), (28), and (31) are quantum characteristics of the qutrit state expressed in terms of classical probabilities  $p_j^{(k)}$ . One can extend the model of qubit state based on the properties of classical random variables  $\vec{X}, \vec{Y},$  and  $\vec{Z}$  (14) to the case of the qutrit state. We consider random classical variables  $\vec{X}^{(k)}, \vec{Y}^{(k)},$  and  $\vec{Z}^{(k)}, k = 1, 2, 3$  with the probability distributions given by the vectors  $\vec{P}_1^{(k)}, \vec{P}_2^{(k)},$  and  $\vec{P}_3^{(k)}$ .

If inequalities (27), (28), and (31) are not valid, the system properties correspond to the behavior of sets of classical “coins.” Namely, quantum correlations are described by inequalities (27), (28), and (31). Thus, we obtain new inequalities for qutrit states, which are entropic inequalities for the probability vectors  $\vec{P}_j^{(k)}, j, k = 1, 2, 3.$  For example, the inequality for relative entropy

$$\sum_{j=1}^2 p_j^{(k)} \ln (p_j^{(k)} / p_j^{(k')}) \geq 0, \quad k, k' = 1, 2, 3, \tag{32}$$

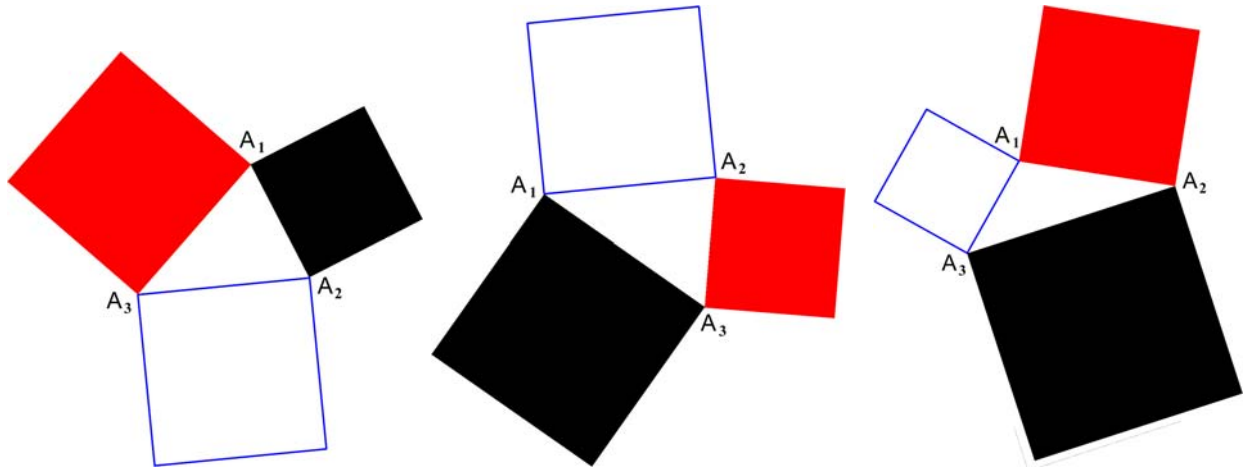
is valid for two arbitrary probability distributions. Since the probabilities are expressed in terms of the density matrix elements of the qutrit state, one has new entropic inequalities for the qutrit-state density matrix; it is just inequality (32), which provides the entropic inequality for the matrix elements of the qutrit-state density matrix.

For example, one has the new relative-entropy inequality for the matrix elements of the qutrit-state density matrix

$$\frac{1}{2} (\rho_{12} + \rho_{21} + 1) \ln \left[ \frac{\rho_{12} + \rho_{21} + 1}{\rho_{13} + \rho_{31} + 1} \right] + \frac{1}{2} [i(\rho_{12} - \rho_{21}) - 1] \ln \left[ \frac{i(\rho_{12} - \rho_{21}) - 1}{i(\rho_{13} - \rho_{31}) - 1} \right] \geq 0. \tag{33}$$

An arbitrary permutation of indices 1, 2, 3 in (33) yields another entropic inequality for the matrix elements of the qutrit-state density matrix.

Now we discuss the geometric picture of the qutrit state using the quantum suprematism approach. Each qubit state is visualized in terms of the triada of Malevich’s squares. The qutrit state, as we have shown, is mapped onto three qubit states, which are described by probabilities  $p_j^{(k)}, j, k = 1, 2, 3.$  Among these nine probabilities, eight are independent, but  $p_3^{(3)} = p_3^{(2)}$ . Thus, the state can be described by three triadas of Malevich’s squares; see Fig. 1. Three sets of the Malevich’s squares are determined by the probabilities  $p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_1^{(3)}, p_2^{(3)},$  and  $p_3^{(3)},$  where  $p_3^{(2)} = p_3^{(3)}$ . The sums of the areas of the triadas of Malevich’s squares are given by (8). For each of the three triadas of Malevich’s squares, one has the inequality for the sums of the areas given by (9). The inequality reflects the presence of quantum correlations between the artificial qubits in the single-qutrit state. During the time evolution of qutrit states, the inequalities for the areas of Malevich’s squares are respected.



**Fig. 1.** Three triadas of Malevich's squares describing the qutrit state and corresponding to three artificial qubit states, respectively.

### 5. Quantum Channels for Qutrit States

The linear maps of the qutrit-state density matrix (26) can be expressed as a linear transform of the eight-dimensional vector  $\vec{\Pi}$  with the components  $p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_1^{(3)},$  and  $p_2^{(3)}$ . These components are the probabilities for three artificial spin-1/2 systems and three spin projections  $m = 1/2$  onto three perpendicular directions in the space. In fact, we have also the probabilities  $p_3^{(3)} = p_3^{(2)}$ . The unitary transform of the density matrix  $\rho$

$$\rho \longrightarrow \rho_u = u\rho u^\dagger, \tag{34}$$

where  $uu^\dagger = 1$ , provides the linear transform of the eight-vector  $\vec{\Pi}$ . One can get this transform (quantum channel) in an explicit form.

In fact, the nine-dimensional vector  $\vec{\rho}$  with components  $(\rho_{11}, \rho_{12}, \rho_{13}, \rho_{21}, \rho_{22}, \rho_{23}, \rho_{31}, \rho_{32}, \rho_{33})$ , which can be denoted as  $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8, \rho_9)$ , after the unitary transform converts to the vector  $\vec{\rho}_u = u \otimes u^* \vec{\rho}$ . Then we arrive at

$$\Pi'_k = \sum_{j=1}^8 \bar{U}_{kj} \Pi_j + \Gamma_k, \tag{35}$$

where the  $9 \times 9$  unitary matrix  $U = u \otimes u^*$  determines the  $8 \times 8$  matrix  $\bar{U}_{kj}$  and the eight-vector  $\vec{\Gamma}$ .

Since the components of vectors  $\vec{\Pi}$  and  $\vec{\Pi}'$  are expressed in terms of the probabilities  $p_j^{(k)}$  and  $p_j^{(k)'}$ , the channel under discussion provides an explicit transform of the probabilities determining the qutrit density matrices. For a unital channel of the form  $\rho \longrightarrow \rho_U = \sum_k p_k u_k \rho u_k^\dagger$  ( $0 \geq p_k \geq 0, \sum_k p_k = 1$ ), the transform of the vector  $\vec{\rho} \longrightarrow \vec{\rho}_U$  reads

$$\vec{\rho}_U = \left( \sum_k p_k u_k \otimes u_k^* \right) \vec{\rho}. \tag{36}$$

It provides the transform of the eight-vector  $\vec{\Pi} \longrightarrow \vec{\Pi}_U$  of the form

$$\vec{\Pi}_U = \left( \sum_k p_k \bar{U}_k \right) \vec{\Pi} + \vec{\Gamma}, \tag{37}$$



where the eight-vectors  $\vec{\Pi}$  and  $\vec{\Pi}_U$  are expressed in terms of eight probabilities determining the qutrit-state density matrix  $\rho$ . An analogous relation describes the generic completely positive map of the qutrit-state density matrix

$$\rho \longrightarrow \rho_{\text{pos}} = \sum_k V_k \rho V_k^\dagger, \quad (38)$$

where  $V_k$  are arbitrary  $3 \times 3$  matrices satisfying the relation  $\sum_k V_k^\dagger V_k = 1$ .

For example, the channel providing the transform  $\rho_{kk} \longrightarrow \rho'_{kk} = \rho_{kk}$  and  $\rho_{kj} \longrightarrow \rho'_{kj} = 0$  ( $k \neq j$ ) for the qutrit-state density matrix determines the transform of the probabilities  $p_1^{(k)'} = 1/2$ ,  $p_2^{(k)'} = 1/2$ ,  $p_3^{(k)'} = p_3^{(k)}$ ,  $k = 1, 2, 3$ . This means that such a channel transforms the states of three artificial qubits determining the initial density matrix of the qutrit state into the state with maximum entropy  $S = \ln 2$  for the probability distributions of spin-projections  $m = \pm 1/2$  along the axes  $x$  and  $y$ . The positive map can be determined by the combination of the described maps with the transforms  $\rho(1) \longrightarrow \rho^{\text{tr}}(1)$  and  $\rho(2) \longrightarrow \rho^{\text{tr}}(2)$  of the two artificial qubit-state density matrices, as well as an analogous transposition of the third artificial qubit-state density matrix.

## 6. Conclusions

To conclude, we point out the main results of this work.

We presented the matrix elements of the qutrit-state density matrix as linear combinations of nine classical probabilities  $p_1^{(1)}, p_2^{(1)}, p_3^{(1)}, p_1^{(2)}, p_2^{(2)}, p_3^{(2)}, p_1^{(3)}, p_2^{(3)}, p_3^{(3)}$ . We interpreted the probabilities  $p_j^{(k)}$ ,  $j, k = 1, 2, 3$  as the probabilities to have “spin-1/2 projections”  $m = 1/2$  in three perpendicular directions of three artificial qubits. This means that such quantum system as qutrit has states whose density matrices are given in the classical formulation by eight independent parameters – eight probabilities corresponding to the states of eight classical coins.

We found new inequalities for the introduced classical probabilities, including new entropic inequalities for the qutrit-state density matrix elements. The new inequalities provide the condition of quantumness of qutrit. The states of eight classical coins are described by the same probabilities, but these probabilities should not satisfy these constraints. The new relations for the qutrit-state density matrices obtained can be checked in experiments with superconducting circuits [11] based on Josephson junctions, which have been discussed in connection with the nonstationary (dynamical) Casimir effect in [12–15]; see also recent publications [16–22]. The dynamical Casimir effect was discovered in [23] and discussed in [24–27].

We presented the observables of the spin-1/2 system in the form of three classical random variables  $\vec{X}$ ,  $\vec{Y}$ , and  $\vec{Z}$ , which are described by classical probability vectors  $\vec{P}_1$ ,  $\vec{P}_2$ , and  $\vec{P}_3$ . These three classical variables are organized in the form of the Hermitian matrix. The quantumness of the construction is reflected by the introduced inequalities for the probability distributions. We extended an analogous construction for qutrit states and observables. We discussed quantum channels for qutrit states in the probability representation. Some aspects of the quantum channel properties in the tomographic-probability picture are presented in [28, 29]. We considered the triangle geometry of qutrit states and described the states by the three triadas of Malevich’s squares in the quantum suprematism approach (suprematism in the art is reviewed in [30]). The consideration can be extended to arbitrary systems of qudit states. We will study this problem in a future publication.

## Acknowledgments

The formulation of the problem of quantum suprematism and the results of Sec. 2 are due to V. I. Man'ko, who is supported by the Russian Science Foundation under Project No. 16-11-00084; the work was partially performed at the Moscow Institute of Physics and Technology.

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