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CONDITIONAL INFORMATION AND HIDDEN CORRELATIONS IN SINGLE-QUDIT STATES

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Abstract

We discuss the notions of mutual information and conditional information for noncomposite systems, classical and quantum; both the mutual information and the conditional information are associated with the presence of hidden correlations in the state of a single qudit. We consider analogs of the entanglement phenomena in the systems without subsystems related to strong hidden quantum correlations.

Keywords: hidden correlations, entanglement, single qudit, four-level atom, probability, entropy, information.

1. Introduction

The main goal of this work is to show that systems without subsystems and multipartite systems, both classical and quantum, have identical correlation properties; the difference between the systems is related to interpretation of the correlations. We call the correlations in systems without subsystems the hidden correlations.

The probability distribution $P(k) \geq 0$ $(k = 1, 2, \ldots, N)$ for one random variable is the normalized function $\sum_{k=1}^{N} P(k) = 1$, which characterizes the state of a classical finite system [1–3]. If the classical system contains two subsystems, the joint probability distribution $\mathcal{P}(j, k) \geq 0$, where $j = 1, 2, \ldots, N_1$, $k = 1, 2, \ldots, N_2$, and $N = N_1N_2$, characterizes the statistical properties of the system with two random variables. The normalization condition for the joint probability distribution of two random variables $\sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \mathcal{P}(j, k) = 1$ is the condition for N nonnegative numbers $\mathcal{P}(j, k)$ organized in the form of a table analogous to a rectangular matrix with the jth row and the kth column.

In the case of a finite tripartite classical system, the statistical properties of the system with three random variables are described by a joint probability distribution $\Pi(j, k, l)$, where $j = 1, 2, \ldots, N_1$, $k = 1, 2, \ldots, N_2$, and $\ell = 1, 2, \ldots, N_3$, and the probability distribution satisfies the normalization condition $\sum_{j=1}^{N_1} \sum_{k=1}^{N_2} \sum_{\ell=1}^{N_3} \Pi(j, k, \ell) = 1$; we assume that $N_1N_2N_3 = N$.

For quantum systems, the states are described by the density matrices [4–6] or by the tomographic probability distributions (see, for example, [7]).

For the single-qudit state or the spin state, the density matrix $\rho_{mm'}$, where $m, m' = -j, -j+1, \ldots, j-1, j$, is the Hermitian nonnegative matrix $\rho = \rho^\dagger$ with unit trace $\operatorname{Tr}\rho = 1$ and nonnegative eigenvalues. For a bipartite system containing two qudits, the density matrix $\rho_{m_1m_2,\,m_1'm_2'}$, where $m_1, m_1' = -j_1, -j_1+$

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$1,\ldots,j_1-1,j_1$ and $m_2,m_2'=-j_2,-j_2+1,\ldots,j_2-1,j_2$, has nonnegative eigenvalues, the property of hermiticity, and unit trace. Analogously, for the system of three qudits, the density matrix of the system state $\rho_{m_1m_2m_3,\,m_1'm_2'm_3'}$, where $m_a,m_a'=-j_a,-j_a+1,\ldots,j_a-1,j_a$ $(a=1,2,3)$, is the nonnegative Hermitian matrix with unit trace.

If one makes an appropriate map of indices in the density matrices, the matrix elements in these matrices can be labeled by integers as follows: $\rho_{jj'}$ (one qudit), $\rho_{jk,\,j'k'}$ (two qudits), and $\rho_{jk\ell,\,j'k'\ell'}$ (three qudits), where $j,j'=1,2,\ldots,N_1$, $k,k'=1,2,\ldots,N_2$, and $\ell,\ell'=1,2,\ldots,N_3$, The diagonal elements of the density matrices provide the probability distributions $\rho_{jj}=P(j)$, $\rho_{jk,\,jk}=\mathcal{P}(j,k)$, and $\rho_{jk\ell,\,jk\ell}=\Pi(j,k,\ell)$.

As we have discussed in [8–24], one can consider probability distributions of composite and noncomposite systems as a set of N nonnegative numbers; in the case of $N=N_1N_2$ or $N=N_1N_2N_3$, where N_1, N_2, and N_3 are integers, one can obtain new entropy–information relations for noncomposite systems. Analogously, for quantum indivisible systems, one can obtain the relations for entropy and information analogous to the relations known for multiqudit systems.

The aim of this study is to extend the notions of mutual information and conditional information available for composite systems to the single-qudit state in order to employ them as characteristics of hidden correlations in the system, including the correlations associated with the entanglement phenomena in single-qudit states. It is worth noting that strong quantum correlations in single-qudit states have been studied in [25]. Also we apply the introduced notion to quantum thermodynamics of N-level atoms [26]. Recently, new aspects of quantum thermodynamics were discussed, for example, in [27–29]; see also [30, 31].*

2. Correlations and Entropies

In this section, we review known properties of correlations in bipartite and tripartite classical systems and their relations with the mutual information and conditional information, respectively.

For two classical random variables, the joint probability distribution $\mathcal{P}(j,k)$, where $j=1,2,\ldots,N_1$ and $k=1,2,\ldots,N_2$, determines the marginal probability distributions

$$\mathcal{P}_1(j)=\sum_{k=1}^{N_2}\mathcal{P}(j,k),\qquad \mathcal{P}_2(k)=\sum_{j=1}^{N_1}\mathcal{P}(j,k) \tag{1}$$

describing the statistical properties of the first and second subsystems, respectively.

By definition, we have three Shannon entropies associated with three probability distributions $\mathcal{P}(j,k)$, $\mathcal{P}_1(j)$ and $\mathcal{P}_2(k)$; they read

$$H(1,2)=-\sum_{j=1}^{N_1}\sum_{k=1}^{N_2}\mathcal{P}(j,k)\ln\mathcal{P}(j,k),\quad H(1)=-\sum_{j=1}^{N_1}\mathcal{P}_1(j)\ln\mathcal{P}_1(j),\quad H(2)=-\sum_{k=1}^{N_2}\mathcal{P}_2(k)\ln\mathcal{P}_2(k). \tag{2}$$

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If there is no correlations between the degrees of freedom of two subsystems, the joint probability distribution $\mathcal{P}(j,k)$ has the factorized form in terms of the marginal probability distributions $\mathcal{P}_1(j)$ and $\mathcal{P}_2(k)$,

$$\mathcal{P}(j,k) = \mathcal{P}_1(j)\mathcal{P}_2(k). \tag{3}$$

Such expression means that the entropy of the system $H(1,2)$ is the sum of entropies of two subsystems, i.e., the following equality holds:

$$H(1,2) = H(1) + H(2). \tag{4}$$

If there are correlations between the degrees of freedom of two subsystems, one has the inequality for the entropies

$$0 \leq I = H(1) + H(2) - H(1,2), \tag{5}$$

which is the nonnegativity condition for the mutual information I. Thus, the value of mutual information is a characteristic of the correlations in the system consisting of two subsystems or the system with two random variables.

Analogously, for tripartite classical system, the joint probability distribution $\Pi(j,k,\ell)$, $j = 1,2,\ldots,N_1$, $k = 1,2,\ldots,N_2$, and $\ell = k = 1,2,\ldots,N_3$, describing the statistical properties and correlations in the system with three random variables determines the marginal probability distributions

$$\mathcal{P}_{12}^{(\Pi)}(j,k) = \sum_{\ell=1}^{N_3} \Pi(j,k,\ell), \quad \mathcal{P}_{23}^{(\Pi)}(k,\ell) = \sum_{j=1}^{N_1} \Pi(j,k,\ell), \quad P_2^{(\Pi)}(k) = \sum_{j=1}^{N_1}\sum_{\ell=1}^{N_3} \Pi(j,k,\ell). \tag{6}$$

These marginal probability distributions for three different subsystems of tripartite system are characterized by three Shannon entropies

$$H^{(\Pi)}(1,2) = -\sum_{j=1}^{N_1}\sum_{k=1}^{N_2} \mathcal{P}_{12}^{(\Pi)}(j,k) \ln \mathcal{P}_{12}^{(\Pi)}(j,k), \qquad H^{(\Pi)}(2,3) = -\sum_{k=1}^{N_2}\sum_{\ell=1}^{N_3} \mathcal{P}_{23}^{(\Pi)}(k,\ell) \ln \mathcal{P}_{23}^{(\Pi)}(k,\ell),$$
$$H^{(\Pi)}(2) = -\sum_{k=1}^{N_2} P_2^{(\Pi)}(k) \ln P_2^{(\Pi)}(k). \tag{7}$$

If there is no correlations between the degrees of freedom in the system with three random variables, the joint probability distribution $\Pi(j,k,\ell)$ has the factorized form in terms of marginal probability distributions, i.e.,

$$\Pi(j,k,\ell) = \left(\sum_{k'=1}^{N_2} \mathcal{P}_{12}^{(\Pi)}(j,k')\right) P_2^{(\Pi)}(k) \left(\sum_{k''=1}^{N_2} \mathcal{P}_{2,3}^{(\Pi)}(k'',\ell)\right). \tag{8}$$

In the case of tripartite classical system with three random variables, for the conditional information I_C we have the nonnegativity condition

$$0 \leq I_C = H^{(\Pi)}(1,2) + H^{(\Pi)}(2,3) - H^{(\Pi)}(2) - H(1,2,3), \tag{9}$$

where the entropy of the tripartite-system state reads

$$H(1,2,3) = -\sum_{j=1}^{N_1}\sum_{k=1}^{N_2}\sum_{\ell=1}^{N_3} \Pi(j,k,\ell) \ln \Pi(j,k,\ell). \tag{10}$$

For the systems without correlations determined by the joint probability distribution (8), the conditional information is equal to zero.

Thus, the value of conditional information I_C is a characteristics of the correlations in the tripartite system. The properties of the set of nonnegative numbers $\mathcal{P}(j,k)$ and $\Pi(j,k,\ell)$, from the mathematical point of view, are characterized by the values of numbers I and I_C, which show the difference of two possibilities. The first one is either to represent the number $\mathcal{P}(j,k)$ in the product form (3) or to represent the numbers $\Pi(j,k,\ell)$ in the product form (8) or as $\Pi(j,k,\ell) = P_1(j)P_2(k)P_3(\ell)$. Information I and conditional information I_C show how much the sets of numbers $\mathcal{P}(j,k)$ and $\Pi(j,k,\ell)$ differ from products of the corresponding marginal probability distributions. In this formulation, we do not interpret the sets of numbers as probability distributions but simply consider the sets as tables of given nonnegative numbers $\mathcal{P}(j,k)$ and $\Pi(j,k,\ell)$ satisfying the normalization conditions. Thus, the numbers I and I_C can be associated not only with the joint probability distributions of bipartite and tripartite systems but also with abstract sets of nonnegative numbers.

3. Quantum States of Bipartite and Tripartite Systems

For quantum states of any system, one has the density matrix ρ which determines the von Neumann entropy,

$$S = -\mathrm{Tr}\,\rho \ln \rho. \tag{11}$$

For bipartite system with two qudits, the density matrix $\rho(1,2)$ determines the von Neumann entropy,

$$S(1,2) = -\mathrm{Tr}\,\rho(1,2) \ln \rho(1,2). \tag{12}$$

The density matrices of the subsystem states of the first and second qudits are given as

$$\rho(1) = \mathrm{Tr}_2\rho(1,2), \qquad \rho(2) = \mathrm{Tr}_1\rho(1,2), \tag{13}$$

and the von Neumann entropies of these states

$$S(1) = -\mathrm{Tr}\,\rho(1) \ln \rho(1), \qquad S(2) = -\mathrm{Tr}\,\rho(2) \ln \rho(2) \tag{14}$$

satisfy the nonnegativity condition. The mutual quantum information I_q is given by the relationship

$$0 \le I_q = S(1) + S(2) - S(1,2). \tag{15}$$

For the system without quantum correlations between the subsystem degrees of freedom, one has $I_q = 0$ and, in this case, the density matrix has the factorized form

$$\rho(1,2) = \rho(1) \otimes \rho(2). \tag{16}$$

We consider the system of three qudits with the density matrix $\rho(1,2,3)$ and three density matrices of subsystems $\rho(1,2)$, $\rho(2,3)$, and $\rho(2)$, respectively; the three density matrices are obtained from the density matrix $\rho(1,2,3)$ using partial traces, namely,

$$\rho(1,2) = \mathrm{Tr}_3\,\rho(1,2,3), \qquad \rho(2,3) = \mathrm{Tr}_1\,\rho(1,2,3), \qquad \rho(2) = \mathrm{Tr}_1\,\rho(1,2). \tag{17}$$

The conditional quantum information I_{Cq} is defined in terms of von Neumann entropies of the subsystem states; it reads

$$I_{Cq} = S(1,2) + S(2,3) - S(1,2,3) - S(2) \tag{18}$$

and satisfies the nonnegativity condition $I_{Cq} \geq 0$. For the system states without correlations such that $\rho(1,2,3) = \rho(1)\otimes\rho(2)\otimes\rho(3)$, the conditional quantum information $I_{Cq} = 0$. Thus, the value of conditional quantum information characterizes the degree of quantum correlations in the tripartite system.

Now we express the von Neumann entropies (12) and (13) in terms of the matrix elements of the density matrices $\rho_{jk,\,j'k'}(1,2) = \langle jk \mid \widehat{\rho}(1,2) \mid j'k'\rangle$, $\rho_{jj'}(1) = \langle j \mid \widehat{\rho}(1) \mid j'\rangle$, and $\rho_{kk'}(2) = \langle k \mid \widehat{\rho}(2) \mid k'\rangle$ of the corresponding density operators $\widehat{\rho}(1,2)$, $\widehat{\rho}(1)$, and $\widehat{\rho}(2)$ of the qudit states. We have

$$\rho_{jj'}(1) = \sum_{k=1}^{N_2} \rho_{jk,\,j'k}(1,2), \qquad \rho_{kk'}(2) = \sum_{j=1}^{N_1} \rho_{jk,\,jk'}(1,2). \tag{19}$$

The von Neumann entropies (14) of the qudit states read

$$S(1) = -\sum_{j=1}^{N_1} \big[\rho(1)\ln\rho(1)\big]_{jj}\,, \qquad S(2) = -\sum_{k=1}^{N_2} \big[\rho(2)\ln\rho(2)\big]_{kk}\,, \tag{20}$$

and the von Neumann entropy of the bipartite system states is

$$S(1,2) = -\sum_{j=1}^{N_1}\sum_{k=1}^{N_2} \big[\rho(1,2)\ln\rho(1,2)\big]_{jk,\,jk}\,. \tag{21}$$

Analogous expressions can be easily obtained for the von Neumann entropies of tripartite system states.

The idea of our approach to introduce the notion of conditional information and the notion of mutual information for single qudit states is related to the employment of bijective maps of indices determining the matrix elements of density matrices. We describe the map in the next section.

4. The Functions Detecting the Hidden Correlations

We introduce the sets of functions describing the bijective map of integers $y = 1,2,\ldots,N$, where $N = X_1X_2$, onto the pairs of integers $y \leftrightarrow x_1, x_2$, where $x_1 = 1,2,\ldots,X_1$ and $x_2 = 1,2,\ldots,X_2$. Following [32, 33], we obtain

$$\begin{aligned} &y(x_1,x_2) = x_1 + (x_2-1)X_1, \qquad 1 \leq x_1 \leq X_1, \qquad 1 \leq x_2 \leq X_2,\\ &x_1(y) = y \bmod X_1, \qquad x_2(y) - 1 = \frac{y - x_1(y)}{X_1} \bmod X_2, \qquad 1 \leq y \leq N. \end{aligned} \tag{22}$$

It is worth noting that analogous functions were discussed in [34, 35]. In the case where the integer N is the product of three integers $N = X_1X_2X_3$, we introduce the functions $y(x_1,x_2,x_3)$, $x_1(y)$, $x_2(y)$, and $x_3(y)$ given by the expressions [32]

$$\begin{aligned} &y = y(x_1,x_2,x_3) = x_1 + (x_2-1)X_1 + (x_3-1)X_1X_2, \qquad 1 \leq x_i \leq X_i, \qquad i \in [1,3],\\ &x_1(y) = y \bmod X_1, \qquad x_2(y) - 1 = \frac{y - x_1(y)}{X_1} \bmod X_2,\\ &x_3(y) - 1 = \frac{y - x_1(y) - (x_2(y)-1)X_1}{X_1X_2} \bmod X_3. \end{aligned} \tag{23}$$

The introduced functions provide the bijective map of integers onto pairs of integers and triples of integers. These maps give the possibilities to interpret the probability distributions of one random variable $P(y)$, $y = 1, 2, \ldots, N$ as the probability distributions of two random variables $\mathcal{P}(x_1, x_2) \equiv P\big(y(x_1, x_2)\big)$ if $N = N_1 N_2$.

If $N = N_1 N_2 N_3$, the functions introduced provide the possibility to interpret the probability distributions of one random variable $P(y)$ as joint probability distributions $\Pi(x_1, x_2, x_3) \equiv P\big(y(x_1, x_2, x_3)\big)$ of three random variables. In view of such interpretation, we introduce the notion of mutual information and the notion of conditional information for single qudit states.

First, we introduce the notion of artificial subsystems in the case of classical system described by the probability distribution $P(y)$, $y = 1, 2, \ldots, N$ and $N = N_1 N_2$. We construct two marginal probability distributions $\mathcal{P}_1(x_1)$ and $\mathcal{P}_2(x_2)$,

$$\mathcal{P}_1(x_1) = \sum_{x_2=1}^{N_2} P\big(y(x_1, x_2)\big), \qquad \mathcal{P}_2(x_2) = \sum_{x_1=1}^{N_1} P\big(y(x_1, x_2)\big). \tag{24}$$

The von Neumann entropies $H_a(1)$ and $H_a(2)$ for artificial subsystem states read

$$H_a(1) = -\sum_{x_1=1}^{N_1} \mathcal{P}_1(x_1) \ln \mathcal{P}_1(x_1), \quad H_a(2) = -\sum_{x_2=1}^{N_2} \mathcal{P}_2(x_2) \ln \mathcal{P}_2(x_2). \tag{25}$$

The mutual information I_a for the classical system of one random variable and the two artificial subsystems introduced is

$$I_a = H_a(1) + H_a(2) + \sum_{x_1=1}^{N_1} \sum_{x_2=1}^{N_2} P\big(y(x_1, x_2)\big) \ln P\big(y(x_1, x_2)\big) \geq 0. \tag{26}$$

The value of information I_a characterizes the degree of hidden correlations in the system with one random variable, and the hidden correlations demonstrate the difference of the probability distribution $P\big(y(x_1, x_2)\big)$ and the product of marginals: $\Delta(x_1, x_2) = \mathcal{P}_1(x_1)\mathcal{P}_2(x_2) - P\big(y(x_1, x_2)\big)$. In the absence of hidden correlations, this difference is equal to zero, and the mutual information $I_a = 0$ also.

For the classical system with one random variable and $N = N_1 N_2 N_3$, we introduce an analogous construction of three artificial subsystems and define the marginal probability distributions,

$$\begin{gathered} \mathcal{P}_{12}^{(a)}(x_1, x_2) = \sum_{x_3=1}^{N_3} P\big(y(x_1, x_2, x_3)\big), \qquad \mathcal{P}_{23}^{(a)}(x_2, x_3) = \sum_{x_1=1}^{N_1} P\big(y(x_1, x_2, x_3)\big), \\ \mathcal{P}_{2}^{(a)}(x_2) = \sum_{x_1=1}^{N_1} \sum_{x_3=1}^{N_3} P\big(y(x_1, x_2, x_3)\big). \end{gathered} \tag{27}$$

The above probability distributions determine the artificial subsystem entropies which, in turn, determine the nonnegative conditional information; it reads

$$\begin{aligned} 0 \leq I_{ac} = &-\sum_{x_1=1}^{N_1} \sum_{x_2=1}^{N_2} \mathcal{P}_{12}^{(a)}(x_1, x_2) \ln \mathcal{P}_{12}^{(a)}(x_1, x_2) - \sum_{x_1=1}^{N_2} \sum_{x_3=1}^{N_3} \mathcal{P}_{23}^{(a)}(x_2, x_3) \ln \mathcal{P}_{23}^{(a)}(x_2, x_3) \\ &+ \sum_{x_2=1}^{N_2} \mathcal{P}_{2}^{(a)}(x_2) \ln \mathcal{P}_{2}^{(a)}(x_2) + \sum_{x_1=1}^{N_1} \sum_{x_2=1}^{N_2} \sum_{x_1=3}^{N_3} P\big(y(x_1, x_2, x_3)\big) \ln P\big(y(x_1, x_2, x_3)\big). \end{aligned} \tag{28}$$

The conditional information is equal to zero if there is no hidden correlations in the system.

Analogously, we can introduce artificial subsystems for an arbitrary classical system with one random variable for the case of $N = N_1 N_2 \cdots N_n$. For this, we use an appropriate partition of the integer N.

For the quantum system, i.e., a single qudit with the density matrix $\rho_{yy'}$, where $y, y' = 1, 2, \ldots N$, we introduce the notion of mutual quantum information (for $N = N_1 N_2$) and the notion of conditional quantum information (for $N = N_1 N_2 N_3$) applying an analogous partition tool. This means that we can interpret the density matrix $\rho_{yy'}$ as the density matrix of the bipartite system determining it, namely, $\rho_{x_1x_2,\, x_1'x_2'} \equiv \rho_{y(x_1x_2),\, y'(x_1'x_2')}$. In view of this definition, we introduce the density matrix of the first artificial subsystem (the first qudit) as $\rho^{(a)}_{x_1x_1'}(1) = \sum_{x_2=1}^{N_2} \rho_{y(x_1x_2),\, y'(x_1'x_2)}$, and the density matrix of the second artificial subsystem (the second qudit) as $\rho^{(a)}_{x_2x_2'}(2) = \sum_{x_1=1}^{N_1} \rho_{y(x_1x_2),\, y'(x_1x_2')}$. The construction of the density matrices provides the possibility to introduce the notion of nonnegative mutual quantum information defined through von Neumann entropies, i.e.,

$$0 \leq I_q^{(a)} = -\sum_{x_1=1}^{N_1} \left(\rho^{(a)}(1) \ln \rho^{(a)}(1)\right)_{x_1x_1} - \sum_{x_2=1}^{N_2} \left(\rho^{(a)}(2) \ln \rho^{(a)}(2)\right)_{x_2x_2} + \sum_{x_1=1}^{N_1} \sum_{x_2=1}^{N_2} \left(\rho \ln \rho\right)_{y(x_1,\,x_2),\, y'(x_1,\,x_2)}. \tag{29}$$

The value of mutual information characterizes the difference of the density matrix $\rho_{x_1x_2,\, x_1'x_2'}(1,2)$ and the product of two density matrices $\rho^{(a)}_{x_1x_2}(1)$ and $\rho^{(a)}_{x_1'x_2'}(2)$, namely,

$$\left(\Delta\rho\right)_{x_1x_2,\, x_1'x_2'} = \rho_{x_1x_2,\, x_1'x_2'}(1,2) - \left(\rho^{(a)}(1) \otimes \rho^{(a)}(2)\right)_{x_1x_2,\, x_1'x_2'}. \tag{30}$$

If there is no hidden correlations in the system of two artificial qudits, this difference is equal to zero and the mutual quantum information $I_q^{(a)} = 0$ also.

For a single-qudit state with spin $s = (N-1)/2$, where the integer $N = N_1 N_2 N_3$, the density matrix $\rho_{yy'}$ with indices $y, y' = 1, 2, \ldots, N$ can be interpreted as the density matrix of the tripartite system with three artificial qudits using the definition $\rho_{x_1x_2x_3,\, x_1'x_2'x_3'} \equiv \rho_{y(x_1,x_2,x_3),\, y'(x_1',x_2',x_3')}$, where indices x_1, x_2, x_3, x_1', x_2', x_3' take the values $x_1, x_1' = 1, 2, \ldots, N_1$, $x_2, x_2' = 1, 2, \ldots, N_2$, and $x_3, x_3' = 1, 2, \ldots, N_3$. Such interpretation provides the possibility to introduce the density matrices of three artificial subsystems with density matrices of their states obtained by the partial traces,

$$\rho^{(a)}_{x_1x_2,\, x_1'x_2'}(1,2) = \sum_{x_3=1}^{N_3} \rho_{x_1x_2x_3,\, x_1'x_2'x_3}, \qquad \rho^{(a)}_{x_2x_3,\, x_2'x_3'}(2,3) = \sum_{x_1=1}^{N_1} \rho_{x_1x_2x_3,\, x_1x_2'x_3'}, \qquad \rho^{(a)}_{x_2x_2'}(2) = \sum_{x_1=1}^{N_1} \rho_{x_1x_2,\, x_1x_2'}(1,2). \tag{31}$$

After obtaining these matrices, we introduce the conditional quantum information for the single qudit state using the definition

$$\begin{aligned} 0 \leq I_{Cq}^{(a)} &= -\mathrm{Tr}\left\{\rho^{(a)}(1,2) \ln \rho^{(a)}(1,2)\right\} - \mathrm{Tr}\left\{\rho^{(a)}(2,3) \ln \rho^{(a)}(2,3)\right\} \\ &\quad +\mathrm{Tr}\left\{\rho^{(a)}(2) \ln \rho^{(a)}(2)\right\} + \mathrm{Tr}\left\{\rho \ln \rho\right\}, \end{aligned} \tag{32}$$

where the conditional quantum entropy for the single-qudit state is an analog of the entropy given by Eq. (18) for the three-qudit state. The difference of this entropy from zero characterizes the difference of the density matrix of the single-qudit state from the product of the matrices $\rho^{(a)}(1) \otimes \rho^{(a)}(2) \otimes \rho^{(a)}(3)$. The value $I_{Cq}^{(a)}$ is a measure of hidden quantum correlations in single-qudit states; if it is equal to zero, the density matrix of the single-qudit state can be presented in the product form $\rho^{(a)}(1)\otimes\rho^{(a)}(2)\otimes\rho^{(a)}(3) = \rho$.

The notion of entanglement in the states of a single qudit can be defined introducing an analog of the notion of entanglement in bipartite system. We consider the state of a single qudit with the density matrix $\rho_{yy'}$, where $y, y' = 1, 2, \ldots, N$, as the separable state if it can be expressed in the form

$$\rho_{y(x_1,\,x_2),\,y'(x_1',\,x_2')} = \sum_k p_k \left(\rho^{(a)(k)}(1) \otimes \rho^{(a)(k)}(2)\right)_{x_1x_2,\,x_1'x_2'}, \tag{33}$$

with p_k being any probability distribution. If the matrix $\rho_{yy'}$ cannot be presented in such a form, we call the state of a single qudit the entangled state. The entanglement phenomenon in the single qudit state reflects the presence of hidden quantum correlations. The mathematical aspects of the entanglement in the single qudit state are identical to the mathematical aspects of the bipartite system states of two qudits where quantum correlations are correlations of degrees of freedom of two subsystems. The definition of artificial multiqudit entanglement in a single-qudit state is the straightforward generalization of the above bipartite-state entanglement definition.

5. Example of the Four-Level Atom

In this section, within the framework of the approach with functions detecting hidden correlations, we consider the example of a single-qudit state realized by the four-level atom or qudit with spin $j = 3/2$. The density matrix of this qudit $\rho_{mm'} = \langle 3/2, m \mid \widehat{\rho} \mid 3/2, m'\rangle$, where $m, m' = -3/2, -1/2, 1/2, 3/2$, can be denoted as $\rho_{yy'}$ $(y, y' = 1, 2, 3, 4)$ using the map $-3/2 \leftrightarrow 1$, $-1/2 \leftrightarrow 2$, $1/2 \leftrightarrow 3$, $3/2 \leftrightarrow 4$. The numbers used to define the functions detecting hidden correlations in the system of two artificial qubits are as follows: $N = 4$, $X_1 = 2$, and $X_2 = 2$. Then we have for (22) the functions $y(x_1, x_2)$, $x_1(y)$, and $x_2(y)$, namely,

$$\begin{gathered} y(x_1, x_2) = x_1 + (x_2 - 1)X_1, \qquad x_1 = 1, 2, \qquad x_2 = 1, 2, \qquad X_1 = 2, \\ x_1(y) = y \bmod 2, \qquad x_2(y) = \frac{y - x_1(y)}{2} \bmod 2 \ + 1, \qquad y = 1, 2, 3, 4. \end{gathered} \tag{34}$$

These functions are described explicitly by numbers

$$\begin{gathered} y(1,1) = 1, \quad y(2,1) = 2, \quad y(1,2) = 3, \quad y(2,2) = 4, \quad x_1(1) = 1, \quad x_1(2) = 2, \\ x_1(3) = 1, \quad x_1(4) = 2, \quad x_2(1) = 1, \quad x_2(2) = 1, \quad x_2(3) = 2, \quad x_2(4) = 2. \end{gathered}$$

Thus, the density matrix of the qudit with $j = 3/2$ can be written as the density matrix of two two-level atoms (two artificial qubits), i.e.,

$$\rho_{yy'} \equiv \rho_{y(x_1,\,x_2),\,y'(x_1',\,x_2')} \equiv \rho_{x_1x_2,\,x_1'x_2'}. \tag{35}$$

For example, the state of the four-level atom with the density matrix $\rho_{yy'} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ can be considered as the state with the density matrix $\rho_{x_1,x_2,x_1',x_2'} = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$, for which the density matrices of two artificial qubits $\rho_{x_1,x_1'}$ and $\rho_{x_2,x_2'}$ are $\rho_{x_1,x_1'} = \rho_{x_2,x_2'} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

Due to the *ppt*-criterion [36, 37], the above state of two artificial qubits is known to be an entangled state. Thus, the pure state of the four-level atom has the hidden correlations associated with the behavior of artificial qubit degrees of freedom. Hidden correlations also exist for single qudits with N given by prime numbers. A tool to understand this property is to introduce $\widetilde{N} = N + k$, where $\widetilde{N} = N_1 N_2$, and to extend the density matrix $\rho_{yy'}$ adding an appropriate number of zero columns and rows. Thus, for the new density matrix $\rho_{\widetilde{y}\widetilde{y}'}$ with $\widetilde{y}, \widetilde{y}' = 1, 2, \ldots, N + k$, we repeat our construction by introducing artificial qudits in a new (extended) Hilbert space. Thus, the five-level atom can be considered as a bipartite system of an artificial qubit and an artificial qutrit.

6. Quantum Thermodynamics and Hidden Correlations

Recently, some problems of quantum thermodynamics were discussed in [27,28]; the problems are connected with different relationships between the density matrices of thermal-equilibrium states $\rho(\hat{H}, T) = \exp(-\hat{H}/T) \big/ \mathrm{Tr}\exp(-\hat{H}/T)$, where $\hat{H}$ is the system Hamiltonian and T is a parameter, which can be considered as the temperature, and the other density matrices (see, for example, [26]). The density matrix of the thermal-equilibrium state contains the partition function $Z(\hat{H}, T) = \mathrm{Tr}\exp(-\hat{H}/T)$; this function determines the von Neumann entropy $S(\hat{H}, T) = -\mathrm{Tr}\left(\rho(\hat{H}, T)\ln\rho(\hat{H}, T)\right)$ and free energy $F(\hat{H}, T) = E(\hat{H}, T) - TS(\hat{H}, T)$, where energy $E(\hat{H}, T) = \mathrm{Tr}\left(\hat{H}\rho(\hat{H}, T)\right)$ of the thermal-equilibrium state.

In [26], we obtain the new inequality for dimensionless energy and entropy associated with any other state with a finite-dimensional density matrix ρ; it reads

$$E(\rho, \hat{H}) + S(\rho) \leq \ln Z(\hat{H}, T = -1), \tag{36}$$

where $E(\rho, \hat{H}) = \mathrm{Tr}(\hat{H}\rho)$, $S(\rho) = -\mathrm{Tr}(\rho\ln\rho)$, and the value of T in the partition function is equal to -1. The equality in this relation takes place only if the density matrix ρ coincides with the density matrix of the thermal-equilibrium state [24]; the other new inequalities were obtained in [29].

Our goal in this section is to discuss the notion of hidden correlations in quantum thermodynamics.

For a single-qudit state with the density matrix $\rho(\beta) = \exp(-\beta/H) \big/ \mathrm{Tr}\exp(-\beta H)$, where β is a parameter and H is any Hermitian matrix (for example, a Hamiltonian matrix), one can introduce the notion of artificial qudits and corresponding hidden correlations.

In the case of a four-level atom, the matrix $\rho(\beta)$ has the matrix elements $\rho(\beta)_{yy'}$, which can be presented in the form

$$\rho(\beta)_{yy'} = \begin{pmatrix} \rho_{11}(\beta) & \rho_{12}(\beta) & \rho_{13}(\beta) & \rho_{14}(\beta) \\ \rho_{21}(\beta) & \rho_{22}(\beta) & \rho_{23}(\beta) & \rho_{24}(\beta) \\ \rho_{31}(\beta) & \rho_{32}(\beta) & \rho_{33}(\beta) & \rho_{34}(\beta) \\ \rho_{41}(\beta) & \rho_{42}(\beta) & \rho_{43}(\beta) & \rho_{44}(\beta) \end{pmatrix}, \tag{37}$$

and the normalization condition $\sum_{y=1}^{4} \rho(\beta)_{yy} = 1$ holds.

The same matrix (37) can be written as the density matrix of two artificial qubits employing Eq. (35); this provides the possibility to construct the density matrices of two artificial qubits,

$$\rho(\beta)^{(1)}_{x_1x_1'} = \begin{pmatrix} \rho_{11}(\beta) + \rho_{22}(\beta) & \rho_{13}(\beta) + \rho_{24}(\beta) \\ \rho_{31}(\beta) + \rho_{42}(\beta) & \rho_{33}(\beta) + \rho_{44}(\beta) \end{pmatrix}, \quad \rho(\beta)^{(2)}_{x_2x_2'} = \begin{pmatrix} \rho_{11}(\beta) + \rho_{33}(\beta) & \rho_{12}(\beta) + \rho_{34}(\beta) \\ \rho_{21}(\beta) + \rho_{43}(\beta) & \rho_{22}(\beta) + \rho_{44}(\beta) \end{pmatrix}.$$

The mutual quantum information for the four-level atom state is introduced for a thermal-equilibrium-like state $\rho(\beta) = \exp(-\beta H) \big/ \mathrm{Tr}\exp(-\beta H)$ as follows:

$$I_q(\beta, H) = -\mathrm{Tr}\,\rho^{(1)}(\beta)\ln\rho^{(1)}(\beta) - \mathrm{Tr}\,\rho^{(2)}(\beta)\ln\rho^{(2)}(\beta) + \mathrm{Tr}\,\rho(\beta)\ln\rho(\beta).$$

If the Hermitian matrix H coincides with the Hamiltonian $\hat{H}$ and the parameter $T = \beta^{-1}$ coincides with the temperature, one has the mutual quantum information $I_q(\beta = T^{-1}, \hat{H})$ characterizing the hidden correlations of two artificial qubits in the thermal-equilibrium state of the four-level atom.

The discussed characteristics can be studied in experiments with superconducting qudits realized in devices based on Josephson junctions [38–40]. The problem of using hidden correlations in single qudits in quantum technologies mentioned in [41, 42] needs extra study.

7. Concluding Remarks

To conclude, we point out the main results of this discussion.

We suggested a systematic approach to introducing the notion of hidden correlations and their characteristics in indivisible systems (systems without subsystems). Thus, for single-qudit states, we applied the concrete map of the density-matrix indices described by specific functions detecting the hidden correlations. These functions provide the partitions of the sets of natural numbers. In view of these partitions, we present the density matrices of single qudit states in the form of density matrices of multiqudit states. Due to these forms of the density matrices, we introduced the notion of entanglement for the single-qudit state. Also we obtained the formulas for mutual information and for conditional information for single-qudit states in terms of the functions detecting the hidden correlations in single-qudit states.

We considered an example of the four-level atom and described artificial qubits and their density matrices determined by the density matrix of this indivisible system. In addition, we discussed the thermal-equilibrium-like states of the four-level atoms and properties of artificial qubit states related to these states.

The prospectives of using the hidden correlations in indivisible systems in quantum technologies will be studied in future publications.

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References

1. A. N. Kolmogorov, *Foundations of the Theory of Probability*, 2nd ed., Chelsea Publishing Company, New York (1956).
2. A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam (1982).
3. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press, Cambridge, UK (2000).
4. L. D. Landau, *Z. Phys.*, **45**, 430 (1927).
5. J. von Neumann, *Göttingenische Nachrichten*, **11** (Nov. 1927), S. 245–272.
6. J. von Neumann, *Mathematische Grundlagen der Quantummechanik*, Springer, Berlin (1932).
7. M. Asorey, A. Ibort, G. Marmo, and F. Ventriglia, *Phys. Scr.*, **90**, 074031 (2015).
8. M. A. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **34**, 203 (2013).
9. M. A. Man'ko, *Phys. Scr.*, **T153**, 014045 (2013).
10. M. A. Man'ko and V. I. Man'ko, *J. Phys.: Conf. Ser.*, **442**, 012008 (2013).
11. M. A. Man'ko, V. I. Man'ko, G. Marmo, et al., *Nuovo Cimento C*, **36**, 163 (2013).
12. M. A. Man'ko and V. I. Man'ko, *J. Phys.: Conf. Ser.*, **538**, 012016 (2014).
13. M. A. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **35**, 509 (2014).
14. M. A. Man'ko and V. I. Man'ko, *Int. J. Quantum Inform.*, **12**, 156006 (2014).
15. P. Adam, V. A. Andreev, J. Janszky, et al., *Phys. Scr.*, **T160**, 014001 (2014).
16. M. A. Man'ko, V. I. Man'ko, G. Marmo, et al., *J. Russ. Laser Res.*, **35**, 79 (2014).
17. M. A. Man'ko and V. I. Man'ko, *Phys. Scr.*, **T160**, 014030 (2014).
18. M. A. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **35**, 298 (2014).
19. M. A. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **35**, 509 (2014)
20. M. A. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **35**, 582 (2014).
21. M. A. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **36**, 301 (2015).
22. M. A. Man'ko and V. I. Man'ko, *Entropy*, **17**, 2876 (2015).
23. M. A. Man'ko and V. I. Man'ko, *J. Phys.: Conf. Ser.*, **698**, 012004 (2016).
24. M. A. Man'ko, V. I. Man'ko, and G. Marmo, *Nuovo Cimento C*, **38**, 167 (2016).
25. A. A. Klyachko, M. A. Can, S. Binicioğlu, and A. S. Shumovsky, *Phys. Rev. Lett.*, **101**, 020403 (2008).
26. A. Figueroa, J. Lopez, O. Castaños, et al., *J. Phys. A: Math. Theor.*, **48**, 065301 (2015).
27. M. Horodecki and J. Oppenheim, *Nature Commun.*, **4**, 2059 (2013).
28. J. Goold, M. Huber, A. Riera, et al., *J. Phys. A: Math. Theor.*, **49**, 143001 (2016).
29. J. A. López-Saldívar, O. Castaños, M. A. Man'ko, and V. I. Man'ko, "New entropic inequalities for qubit and unimodal Gaussian states," *Eur. Phys. Lett.* (2017, submitted).

30. P. Facchi, G. Garnero, and M. Ligabo, “Quantum fluctuation relations,” arXiv:1705.06096 [quant-ph].
31. P. Facchi and G. Garnero, “Quantum thermodynamics and canonical typicality,” arXiv:1705.02270 [quant-ph].
32. V. I. Man’ko and Zh. Seilov, *J. Russ. Laser Res.*, **38**, 50 (2017).
33. V. N. Chernega, O. V. Man’ko, V. I. Man’ko, and Zh. Seilov, “New information and entropic inequalities for Clebsch–Gordan coefficients,” arXiv:1606.00854v4 [quant-ph]; *Theor. Math. Phys.* (2017, in press).
34. A. D. Pasquale, “Bipartite entanglement of large quantum systems,” arXiv:1206.6749v2 [quant-ph].
35. A. D. Pasquale, P. Facchi, V. G. Giovannetti, et al., *J. Phys. A: Math. Theor.*, **45**, 015308 (2012).
36. A. Peres, *Phys. Rev. Lett.*, **77**, 1413 (1996).
37. M. Horodecki, P. Horodecki, and R. Horodecki, *Phys. Lett. A*, **223**, 1 (1996).
38. A. K. Fedorov, E. O. Kiktenko, O. V. Man’ko, and V. I. Man’ko, *Phys. Rev. A*, **91**, 042312 (2015).
39. E. O. Kiktenko, A. K. Fedorov, A. A. Strakhov, and V. I. Man’ko, *Phys. Lett. A*, **379**, 1409 (2015).
40. E. Glushkov, A. Glushkova, and V. I. Man’ko, *J. Russ. Laser Res.*, **36**, 448 (2015).
41. I. Ya. Doskoch and M. A. Man’ko, *J. Russ. Laser Res.*, **36**, 503 (2015).
42. M. A. Man’ko and V. I. Man’ko, *J. Russ. Laser Res.*, **37**, 1 (2016).