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ENTANGLEMENT AND PANCHARATNAM PHASE OF A FOUR-LEVEL ATOM IN COHERENT STATES WITHIN GENERALIZED HEISENBERG ALGEBRA

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Abstract

We consider a four-level atom (FLA) interacting with a field mode that is initially in a coherent state associated with a generalized Heisenberg algebra (CSGHA). The dynamical behavior of quantum entropy, the Pancharatnam phase, and the Mandel parameter are investigated. The statistical and nonclassical properties of the field in regard to its CSGHA are discussed through the evolution of the Mandel parameter, and the effects of the initial atomic state position and time-dependent coupling given in terms of atomic speed and acceleration are examined. The results show that the CSGHA strength and time-dependent coupling based on the atomic speed and acceleration have the potential to affect the time evolution of the entanglement, the Pancharatnam phase, and the Mandel parameter.

Keywords: entanglement, coherent states, Heisenberg algebras, Mandel parameter, Pancharatnam phase.

1. Introduction

The advanced concept of coherent states (CSs) is widely used in quantum estimation theory (QET). CS was introduced by Schrödinger [1] within the context of the harmonic oscillator (HO) model, intriguingly using them once to find the quantum states. With either the quantum or classical formulation, any physical system can be interpreted in terms of CSs. Later, the notion of CS grew to be primary in quantum optics as eigenstates of the annihilation operator by Glauber [2]. In further developments, deformed CSs associated with quantum groups have been constructed by exploiting its mathematical description as a deformed Lie algebra. The resulting nonlinear states appear as a natural extension of the concept of CSs [3,4]. Currently, these deformed states have garnered interest through their practical application in many areas of quantum information [5–7]. These states have some nonclassical properties, which include sub-Poissonian photon distribution [8], photon antibunching [9], and squeezing [10,11]. Moreover, it has been realized that, in experiments, real laser beams are expressible as Poissonian or sub-Poissonian distributions [12,13].

The nonclassical properties such as specific photon distributions in the f-coherent states were discussed in [14]. Also, generalized coherent states have been constructed as a broader generalization of

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q-coherent states [15]. These can be associated with different statistical properties in quantum optics such as photon distribution functions, Wigner functions, and Q-functions.

Controlling quantum correlations between subsystems is one of the main goals of quantum technologies [16–18]. The most thoroughly explored quantum correlation is quantum entanglement [19]. In recent years, different devices have been proposed and realized experimentally to generate quantum entanglement, such as beam splitters [20–22], nanoresonators [23,24], and nuclear-magnetic-resonance (NMR) systems [25].

The geometric phase is a basic intrinsic feature in quantum mechanics that has been investigated by almost two generations of physicists [26–28]. Berry proved that the solution of the quantum object (wave function) retains the memory of its evolution in its complex phase argument (known as the geometric phase factor), which, when observed from the perspective of its dynamical contribution, depends only on the geometry of the path traversed by the system [29, 30]. It is robust against environmental perturbations and imperfections in control. This explains why it has gained attention in the implementation of fault-tolerant quantum computation. More recently, we investigated the quantum phase and field purification for quantum systems in the coherent state of a generalized Heisenberg algebra (GHACS) [31]. The geometric phase and field purification have been found to be very sensitive to the number of photon transitions and the initial atomic state setting. Nevertheless, these results were obtained neglecting the effect of atomic speed and acceleration. Hence the main goal of this article is to develop the model by investigating the behavior of an electromagnetic field in a GHACS interacting with a four-level atom (FLA). We explore the relationship between quantum quantifiers, including nonlocal correlation, physical properties, and geometric phase for constant and time-dependent coupling based on the atomic speed and acceleration.

2. Four-Level Atom and Field in GHA Coherent States

We introduce a model for the field mode interacting with a FLA with energy levels denoted by $|j\rangle$, $j = 1, 2, 3, 4$ where the lower level is $|4\rangle$, the two intermediate levels are $|2\rangle$, $|3\rangle$, and $|1\rangle$ is the upper level. The interaction Hamiltonian reads

$$H_I(t) = \sum_{j=1}^{3} g_j(t) \left\{ \hat{a} \, |j\rangle \, \langle j+1| + \hat{a}\dagger \, |j+1\rangle \, \langle j| \right\}, \tag{1}$$

where $\hat{a}$ ($\hat{a}^{\dagger}$) is the annihilation (creation) operator, and $g_j(t)$ is the time-dependent coupling parameterized by atomic speed v and acceleration a and given in the form $g(t) = \epsilon \sin(at^2 + vt + c)$ with coupling constant ϵ. The time dependence of the coupling is negligible if $a = v = 0$ and $c = \pi/2$ [32, 33]. We consider the symmetric case where the coupling between the field mode and atomic levels $g_1(t) = g_2(t) = g_3(t)$.

As is well known, the superposition principle is crucial in quantum information processing (QIP). Hence we assume that the initial state of the FLA is a superposition between the two upper levels $|1\rangle$ and $|2\rangle$. The field is initially set in the GHACS $|\beta\rangle$, hence the combined atomic–field (A-F) system is then $|\varpi(0)\rangle = \cos(\vartheta) \, |\beta, 1\rangle + \sin(\vartheta) \exp(i\varphi) \, |\beta, 2\rangle$, where ϑ is the initial atomic-state position and φ is the relative phase between the two upper levels.

The GHACS is the CS associated with the anharmonic oscillators and is expressed in the form [31,34]

$$|\beta\rangle = \frac{1}{\mathcal{N}\left(|\beta|^2\right)} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{G(n)}} |n\rangle, \tag{2}$$

with

$$G(n)=\prod_{k=1}^{n}(k^2+\mu k),\qquad G(0)=1. \tag{3}$$

The normalization factor is then given as

$$\mathcal{N}\left(|\beta|^2\right)=\sqrt{\Gamma(\mu+1)|\beta|^{-\mu}\mathcal{I}_\mu(2|\beta|)}, \tag{4}$$

where $\mathcal{I}_\mu(\beta)$ is the modified Bessel function of the first kind. The final form of the state vector at any time is

$$|\varpi(t)\rangle=\exp\left\{-i\int_0^t H_I(T)dT\right\}|\varpi(0)\rangle. \tag{5}$$

We can now use the wave function (5) to calculate the Pancharatnam phase, atom–field entanglement (A-FE) measured by the von Neumann entropy, and the Mandel parameter.

3. Pancharatnam Phase and A-F Entanglement

The Mandel parameter is a good measure of the nonclassical and statistical properties. It is defined in terms of the average photon number of reduced field (RF) states $\langle N\rangle$ as [35, 36]

$$Q_M=\frac{\langle N^2\rangle}{\langle N\rangle}-\langle N\rangle-1, \tag{6}$$

and is an indicator of whether the photon distribution of the RF is sub-Poissonian ($-1\leq Q_M\leq 0$), inferring the presence of nonclassical states, Poissonian ($Q_M<0$) for semiclassical states, and super-Poissonian ($Q_M>0$) for a classical system.

To quantify the A-FE, we use the von Neumann entropy or quantum entropy given by [37–39]

$$S_A=-\mathrm{Tr}\left\{\rho^A\ln\rho^A\right\}=-\sum_{j=1}^{4}\xi_j\ln\xi_j, \tag{7}$$

where ρ^A denotes the atomic density matrix obtained by taking the trace over the atomic basis (i.e., $\rho_A=\mathrm{Tr}_F\left(|\varpi(t)\rangle\langle\varpi(t)|\right)$) and ξ_j is the eigenvalues of the atomic density matrix ρ^A. Quantum entropy varies from zero for separable states to ln(4) for maximum entangled states.

The evolution of the quantum system is defined as noncyclic if the initial and final states are distinct. Hence the final and initial wave functions are not related by a simple multiplication with a complex number. We suppose that the initial state $|\varpi(0)\rangle$ evolves to $|\varpi(t)\rangle$ after a certain time t. If the scalar product $L(t)=\langle\varpi(0)|\varpi(t)\rangle$ can be formulated by a real number F, where $L(t)=Fe^{i\phi}$, then the noncyclic phase is the angle ϕ.

The phase between the two states for such an evolution is nontrivial. The noncyclic phase generalizes the cyclic geometric phase and can be regarded as a special case of the former in which $F=1$.

The Pancharatnam phase contains the dynamical phase and the geometric phase, which is prescribed as the phase acquired during an arbitrary evolution of the wave function from the initial vector $|\varpi(0)\rangle$ to $|\varpi(t)\rangle$ [27, 40],

$$\phi_P(t)=\arg(\langle\varpi(0)|\varpi(t)\rangle). \tag{8}$$

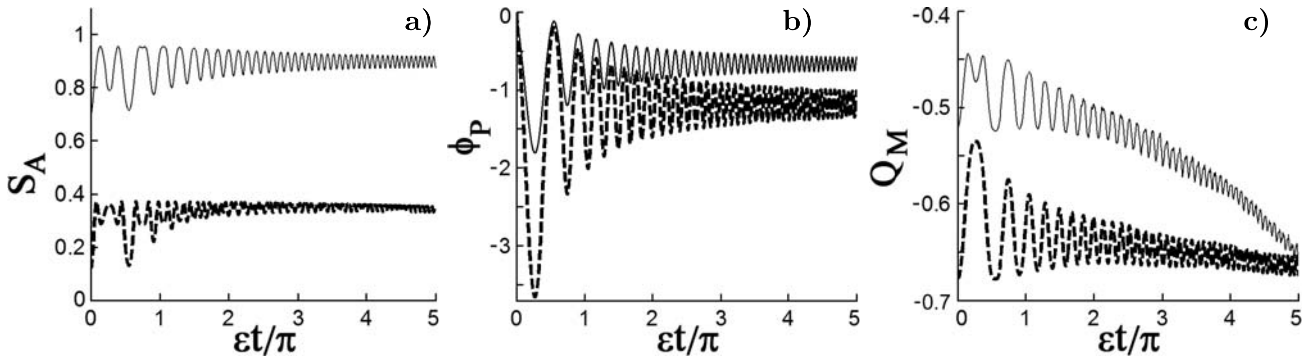

Fig. 1. Effects of the GHACS strength β on the time evolution of the von Neumann entropy S_A (a), the Pancharatnam phase ϕ_P (b), and the Mandel parameter Q_M (c) of a FLA interacting with a field initially in the GHACS for $\mu = 1.5$ and $\vartheta = \varphi = 0$ for nonzero atomic speed and acceleration (specifically, $a = v = 1$ and $c = 0$). Here, the solid curve corresponds to $|\beta| = 1$ and the dashed curve to $|\beta| = 6$.

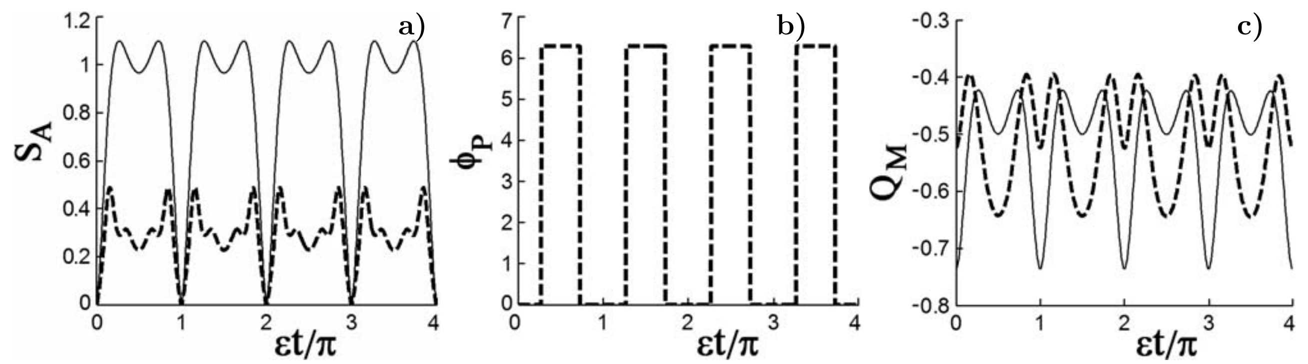

Fig. 2. Effect of the atomic speed (i.e., $v = 1$ and $a = c = 0$) on the time evolution. The figure descriptions are the same as for Fig. 1.

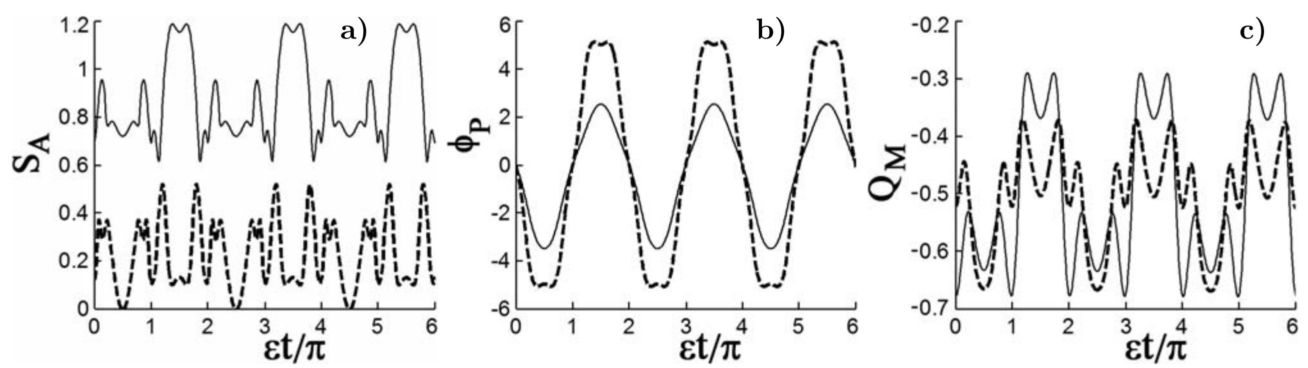

Fig. 3. The same as Fig. 2 but for $\vartheta = \pi/2$ and $\varphi = \pi/4$.

4. Numerical Results and Discursion

In Figs. 1–4, we present our main results by exhibiting the influence of the physical parameters on the Pancharatnam phase ϕ_P, the von Neumann entropy S_A, and the Mandel parameter Q_M. A reasonable comparison between the results enables us to understand the contribution of the atomic speed, acceleration, and atomic state position on the evolution of the different physical quantities.

In Fig. 1, the respective variations of ϕ_P, S_A and Q_M are plotted against the scaled time ϵt setting nonzero values for the atomic speed and acceleration (i.e., $a = v = 1$ and $c = 0$). The QE oscillates between a minimum and a maximum value although the maximum A-FE value is not reached (i.e., ln(4)). Entanglement increases when the FLA is in a superposition state. Interestingly, the A-FE is more enhanced when $\theta = \pi/4$ than in the upper-state case, $\theta = 0$. In addition, the dynamical behavior of ϕ_P involves oscillations where the widths of the amplitude of oscillation decrease as time evolves. As we see in Fig. 1 c, the Mandel parameter is always negative, and hence the state exhibits sub-Poissonian statistics. According to Fig. 1, there is a clear correspondence between the appearance of the oscillations in S_A, ϕ_P, and Q_M for a single photon transition. Also, all the quantities present a richer structure for the superposition state compared with the upper state, where the ϕ_P and S_A are enhanced whereas the field mixes less and becomes more quantum mechanical.

In Fig. 2, we present the effect of the GHACS strength on the dynamical properties of the quantities for nonzero atomic speed and negligible acceleration. We observe that an increasing CS strength leads to a significantly enhanced efficiency for the A-FE. In this regard, the maximum A-FE is obtained at $\epsilon t = (2m + 1)\pi/2 \;\; m = 1, 2, 3, \ldots$ At these points, the A-FE is at a minimum in the strong CS-strength regime and corresponds to high values for ϕ_P. Also, the field is close to Poissonian statistics. Hence the results in Fig. 2 show clearly that the maximum A-FE can be obtained by considering a nonzero atomic speed and weak CS strength regime. With the figure, we examined the influence of the atomic-state angle on the behavior of the quantities for the strong CS strength regime. Strong A-FE is obtained when the Pancharatnam phase is zero.

In Fig. 3, we see the effect of the GHACS strength on the evolution of S_A, ϕ_P, and Q_M when the FLA starts the interaction from a superposition between the two upper levels. The main observation from Fig. 3 in comparison with Fig. 2 is that a Pancharatnam phase appears for both strong and weak CS-strength regimes.

In Fig. 4, we compare the dynamical behavior when the time-dependent coupling (i.e., $g(t) = \epsilon$) is negligible. The initial atomic position is the excited state (solid curve) and the superposition state (dashed curve). A comparison between these two settings verifies that the dynamics are changed by the atomic-state position. Note that the A-FE and the Mandel parameter exhibit nonperiodic behavior. Additionally, the superposition-state parameter θ has a clear effect on the dynamic behavior of the entanglement and the Pancharatnam phase. The rectangular peaks of ϕ_P change to sharp nonperiodic peaks with increasing A-FE when the initial atomic state is changed from the excited state to the superposition state.

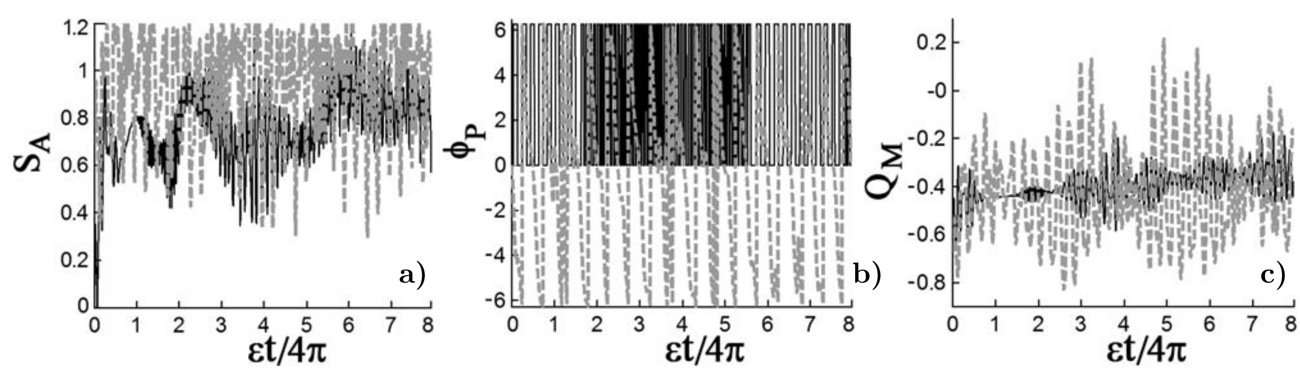

Fig. 4. Effect of the initial state position ϑ and relative phase φ on the dynamical behavior. The figure descriptions are the same as for Fig. 1, except that the solid curve represents $\vartheta = \varphi = 0$ and the grey dashed curve represents $\vartheta = \pi/2$ and $\varphi = \pi/4$.

5. Conclusions

We investigated the dynamical behavior of the Pancharatnam phase and entanglement of an input field in terms of the interaction between a FLA and an optical field within the GHACS formalism. The statistical properties of the field were also discussed using the evolution of the Mandel parameter. We found that the time-dependent coupling in terms of the atomic speed and acceleration have a central role in the dynamics of the Pancharatnam phase, Mandel parameter, and entanglement. Our results clarify the link between the dynamical behavior of the Mandel parameter and the A-FE for the strong and weak coherent-state strength regimes. We showed that the statistical properties of these states based on GHA exhibit the Poissonian and sub-Poissonian distributions, and that the Mandel Q-parameter decreases as the coherent-state strength increases. Additionally, we observed that high entanglement can be obtained through a suitable choice of the initial atomic-state position and the weak coherent-state strength regime.

References

1. E. Schrödinger, *Naturwissenschaften*, **14**, 664 (1926).
2. R. J. Glauber, *Phys. Rev.,* **131**, 2766 (1963).
3. B. Jurco, *Lett. Math. Phys.*, **21**, 51 (1991).
4. V. V. Dodonov, *J. Opt. B: Quantum Semiclass. Opt.*, **4**, R1 (2002).
5. D. Ellinas, "On coherent states and q-deformed algebras, arXiv hepth/9309072 (1993).
6. S. Chaturvedi, A. K. Kappoor, R. Sandhya, and V. Srinivasan, *Phys. Rev. A*, **43**, 4555 (1991).
7. S. H. Chiu, R. W. Gray, and C. A. Nelson, *Phys. Lett. A*, **164**, 237 (1992).
8. M. C. Teich and B. E. A. Salch, *J. Opt. Soc. Am. B*, **2**, 275 (1985).
9. H. J. Kimble, M. Dagenais, and L. Mandel, *Phys. Rev. Lett.*, **39**, 691 (1977).
10. C. K. Hong and L. Mandel, *Phys. Rev. Lett.*, **54**, 323 (1985).
11. L. A. Wu, H. J. kimble, J. L. Hall, and H. Wu, *Phys. Rev. Lett.*, **57**, 2520 (1985).
12. R. Short and L. Mandel, *Phys. Rev. Lett.*, **51**, 384 (1983).
13. J. Perina, Z. Hradil, and B. Jurco, *Quantum Optics and Fundamentals of Physics*, Kluwer Academic Publishers, Dordrechet (1994).
14. V. I. Man'ko, G. Marmo, E. C. G. Sudarshan, and F. Zaccaria, *Phys. Scr.*, **55**, 528 (1997).
15. R. Lopez-Peña, V. I. Man'ko, G. Marmo, et al., *J. Russ. Laser Res.*, **21**, 305 (2000).
16. A. Kay, *Phys. Rev. A*, **92**, 062329 (2015).
17. E. A. Sete, A. A. Svidzinsky, H. Eleuch, et al., *J. Mod. Opt.*, **57**, 1311 (2010).
18. K. Berrada, S. Abdel-Khalek, and A. Eid, *J. Russ. Laser Res.*, **37**, 45 (2016).
19. M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge University Press (2000).
20. S. M. Tan, D. F.Walls, and M. J. Collett, *Phys. Rev. Lett.*, **66**, 252 (1991).
21. G. Toth, C. Simon, and J. I. Cirac, *Phys. Rev. A*, **68**, 062310 (2003).
22. K. Berrada, S. Abdel-Khalek, H. Eleuch, and Y. Hassouni, *Quantum Inform. Process.*, **12**, 69 (2013).
23. T. P. Purdy, P.-L. Yu, R. W. Peterson, et al., *Phys. Rev. X,* **3**, 031012 (2013).
24. E. A. Sete, H. Eleuch, and C. H. R. Ooi, *J. Opt. Soc. Am. B*, **31**, 2821 (2014).
25. N. Gershenfeld and I. L. Chuang, *Science*, **275**, 350 (1997).
26. M. V. Berry, *Proc. R. Soc. London A,* **329**, 45 (1984).
27. S. Pancharatnam, *Proc. Indian Acad. Sci. A,* **44**, 247 (1956).

28. F. Wilczek and A. Zee, *Phys. Rev. Lett.*, **52**, 2111 (1984).
29. B. Simon, *Phys. Rev. Lett.*, **51**, 2167 (1983).
30. P. Facchi and S. Pascazio, *Acta Phys. Slovaca*, **49**, 677 (1999).
31. E. Edfawy and S. Abdel-Khalek, *J. Russ. Laser Res.*, **37**, 345 (2016).
32. M. S. Abdalla, A. S. F. Obada, and S. Abdel-Khalek, *Eur. Phys. J. Plus*, **128**, 1 (2013).
33. S. Abdel-Khalek, *Open Syst. Inform. Dyn.*, **22**, 1550015 (2015).
34. Y. Hassouni, E. M. F. Curado, and M. A. Rego-Monteiro, *Phys. Rev. A*, **71**, 022104 (2005).
35. L. Mandel, *Opt. Lett.*, **4**, 205 (1979).
36. K. Berrada, S. Abdel-Khalek, and A. Eid, *J. Russ. Laser Res.*, **37**, 45 (2016).
37. J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press (1955)
38. S. J. Phoenix, and P. L. Knight, *Ann. Phys. (N.Y.),* **186**, 381 (1988).
39. S. Abdel-Khalek and N. H. Abdel-Wahab, *Int. J. Theor. Phys*, **50**, 562 (2011).
40. Q. V. Lawande, S. V. Lawande, and A. Joshi, *Phys. Lett. A*, **251**, 164 (1999).