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THE DYNAMICS OF QUANTUM CORRELATIONS IN MIXED CLASSICAL ENVIRONMENTS

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Abstract

We present a comparative study of the dynamics of entanglement and quantum discord in a bipartite system in the presence of mixed classical noises. In particular, the joint effects of three different types of classical noises, namely, random telegraphic noise (RTN), Ornstein–Uhlenbeck noise (OU), and static noise, are studied by combining them in two different ways. In each case, one marginal system is coupled with random telegraphic noise, and the other marginal system is coupled with either OU or static noise. We make a comparison between the behaviors of both correlations in the two setups. In the weak coupling regime, the qualitative behavior of entanglement is unaffected by switching the coupling of only one marginal system from OU to static noise, and vice versa. However, the behavior of quantum discord strongly depends on whether it is coupled with OU or static noise. On the other hand, in the strong coupling regime, the static noise is more fatal to the survival of both correlations as compared to the other two noises.

Keywords: entanglement, quantum discord, decoherence.

1. Introduction

Composite quantum systems are more resourceful for information processing than its classical counterparts due to the existence of nonclassical correlations between its constituent marginal systems [1]. The well-known and widely investigated phenomena among such nonclassical correlations is entanglement. It has been recognized as the first candidate of nonclassical correlations and was, initially, used to distinguish between classical and quantum realms. It is considered as a vital resource for quantum information science [2]. Remarkable progress through strenuous efforts for the detection, quantification, application, and its dynamics in several practically realizable systems have been made in the last few decades [3–16].

In addition to entanglement, a quantum state ρ is endowed with classical as well as other kinds of quantum correlations. Understanding and quantification of these correlations may be, in some respects, useful for practical realization of various quantum information tasks. Generally, these correlations are different in nature. Each of them behaves independently and is usually not captured by the measures of each other. In addition to entanglement, quantum discord is one of such other correlations [17–19]. It is

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considered a more general measure of quantum correlations, which also captures those that do not come into the domain of quantum entanglement. Its benefits, as a potential resource for different quantum information tasks, have already been exploited in certain quantum computation models [20] encoding information onto a quantum state [21] and quantum state merging [22, 23].

Pertaining to its thrilling role in quantum information theory, the studies of its dynamics have been recently extended to continuous variable systems of Gaussian and non-Gaussian states [24–26]. Another study on the dynamics of quantum discord in non-inertial frame reveals that quantum discord for a two-mode squeezed state asymptotically goes to zero in the limit of infinite acceleration [27]. Some further details about the behavior of quantum discord in open quantum systems can be found in [28–30].

A quantum system very easily interacts with other nearby quantum systems of large dimensions. Such quantum systems are ubiquitous and are generally known as environment. The interaction between a quantum system and an environment is very fatal to the survival of correlations existing between different marginal systems that constituting the principal one. The interaction between a system and an environment can be modeled in a classical or quantum-mechanical picture of the environment. Dealing with the environment through the classical approach is convenient and often more accurate in cases where the degrees of freedom of the environment become considerably large, because quantum-mechanical treatment in such situations is usually approximated. The effects of different environments in their classical pictures on the dynamics of quantum correlations in several quantum systems have been studied, and important results have been obtained [31–34].

In this paper, we investigate the dynamics of bipartite quantum correlations by making a comparative study under the influence of different mixed classical noises that have no direct interaction with each other. In particular, we consider the effects of static noise, random telegraphic noise (RTN), and the Ornstein–Uhlenbeck noise (OU) on the behaviors of entanglement and quantum discord under two different setups. In each case, one marginal system of the bipartite system is coupled to RTN, and the other is either coupled to OU or to static noise in such a way that the environments are isolated from each other. We make a comparative analysis between the two setups by limiting the coupling of one marginal system to the weak coupling regime, and the other is varied from the weak to the strong coupling regime. Such a study for the behavior of entanglement in the presence of quantum environments in accelerated frames is made in [35].

Our study reveals that even in the absence of direct interaction between the environments in mixed setups, the influence of one classical environment on the dynamics of quantum correlations is strongly affected by the coupling of another type of classical environment that directly interacts with a different marginal system.

2. The Physical Model

To carry out the present work in a more composite way, we begin this section by presenting a review of the basic concepts of the required mathematical machinery. As mentioned above, the focus of our study is the dynamics of different types of quantum correlations that initially exist between two noninteracting identical qubits and evolves under the action of various local classical environments. Therefore, we first write the Hamiltonian that describes the evolution of the system. If I represents the identity matrix acting on the Hilbert space of a qubit, then the Hamiltonian in its most general form can be expressed as follows:

$$H(t) = H_A \otimes I + I \otimes H_B, \tag{1}$$

where $H_{A(B)}$ is a single-qubit Hamiltonian describing its dynamics in the presence of noise and can be explicitly written as

$$H_{A(B)} = \varepsilon I_{A(B)} + g\chi_{A(B)}(t)\sigma^x_{A(B)}. \tag{2}$$

In Eq. (2), ε represents the energy of an isolated qubit, g defines the strength of the coupling of the qubit with its local environment, $\chi_{A(B)}(t)$ stands for a stochastic variable, which depends on the nature of the coupled noise, and $\sigma^x_{A(B)}$ is the spin-flip Pauli matrix. The two qubits of our system are identical in the sense that they are characterized by the same energy ε.

The three different kinds of classical noises incorporated in this study whose local coupling with the qubits that we consider are random telegraphic noise (RTN), Ornstein–Uhlenbeck (OU) noise, and static noise. The static noise bears the name due to its time independent stochastic variable $\chi(t)$ characterized by the flat probability distribution $P(\chi) = 1/\Delta_\chi$ for $|\chi - \chi_o| \le \Delta_\chi/2$ and vanishes for all other choices [31, 36, 37]. Here, Δ_χ is a measure of the degree of disorder of the environment, and χ_o gives the average value of the distribution. The autocorrelation function of the stochastic parameter $\chi(t)$ is given by $\langle\delta\chi(t)\delta\chi(0)\rangle = \Delta_\chi/2$. This results in a power spectrum described by a δ function that peaks up at zero frequency. Attributed to this is the longer characteristic time of the noise, longer than the system–environment coupling. Consequently, the static noise bears the characteristic of non-Markovian noise. Some detailed studies of the effects of static noise on various quantum systems such as the propagation of particles in optical coupled waveguides, quantum walks, as well as on the dynamics of different quantum correlations can be found in [31, 36, 37]. The overall effect of the static environment on the dynamics of quantum correlations is obtained by averaging the final density matrix over all the possible noise configurations. This goal is served by integrating the final density matrix over the stochastic variable $\chi(t)$ between $\chi_o - \Delta_\chi/2$ and $\chi_o + \Delta_\chi/2$ [31].

In the case of RTN noise, the stochastic parameter $\chi(t)$ randomly takes the values ± 1 at a particular rate γ, usually different for the two transitions. However, in this paper, we consider it to be the same for both transitions. Based upon the switching rate between the two allowed values and on the coupling strength with the system, the behavior of the noise could be Markovian or non-Markovian. The former behavior pertains to the so-called weak coupling regime and the latter to the strong coupling regime. The autocorrelation function is a time-dependent exponentially decaying function given by $\langle\delta\chi(t)\delta\chi(0)\rangle = e^{-2\gamma t}$. The power spectrum, in this case, is Lorentzian in character [38]. It is shown in [39] that a system evolving under the influence of RTN noise picks a random phase factor given by

$$\varphi(t) = -g\int_0^t dt'\,\chi(t'). \tag{3}$$

The overall effect of the noise on a system coupled with it for a time t can be obtained by averaging the density matrix of the system over the random phase factor $\varphi(t)$, that is, $\rho(t) = \langle\rho[\varphi(t)]\rangle_\varphi$.

Similarly, the OU process is characterized by the autocorrelation function $\langle\delta\chi(t)\delta\chi(0)\rangle = \dfrac{\gamma_p}{2}e^{-\gamma_p t}$ and is Gaussian in nature [40]. Here, γ_p is the inverse of the correlation time and thus defines the spectral width of the process. Under the influence of this environmental noise, the evolution operator accumulates a phase similar to the one given in Eq. (3). Again, the overall effect of the noise on the system can be obtained by averaging the final density matrix with respect to the accumulated phase factor. An explicit relation for the final density matrix of the system can be found by utilizing the characteristic function of Gaussian random process with zero mean as $\langle e^{n\Phi(t)}\rangle = e^{-n^2\mu(t)/2}$, with n being an integer, and $\mu(t)$ is given by [41,42],

$$\mu(t)=\frac{1}{\gamma_p}(e^{-\gamma_p t}+\gamma_p t-1). \tag{4}$$

Next, we briefly review the basic mechanism of the quantifiers for entanglement and quantum discord. Many entanglement measures for quantifying entanglement of bipartite states exist in the literature. However, we will use negativity, which is a reliable measure of entanglement of bipartite states of any dimensions, provided that the state has a negative partial transpose. The partial transpose of a bipartite density matrix $\rho_{m\nu,n\mu}$ over the second qubit B is given by $\rho^{T_B}_{m\mu,n\nu}=\rho_{m\nu,n\mu}$, and for the first qubit, it can similarly be defined. For a bipartite state ρ^{AB}, the negativity $\mathcal{N}(\rho^{AB})$ is defined as twice the absolute sum of the negative eigenvalues of partial transpose of ρ^{AB}; it can be expressed as follows:

$$\mathcal{N}(\rho^{AB})=\frac{1}{2}\left(\sum_i|\lambda_i|-1\right), \tag{5}$$

where λ_i are the eigenvalues of the partial transpose density matrix.

On the other hand, quantum discord $\mathcal{D}(\rho^{AB})$ for the bipartite state ρ^{AB} is defined as the difference between total correlations $\mathcal{I}(\rho^{AB})$ and the classical correlation $\mathcal{C}(\rho^{AB})$,

$$\mathcal{D}(\rho^{AB})=\mathcal{I}(\rho^{AB})-\mathcal{C}(\rho^{AB}). \tag{6}$$

The quantum mutual information $\mathcal{I}(\rho^{AB})$ is a measure of the total amount of classical and quantum correlations in a quantum state. Mathematically, it is given by

$$\mathcal{I}(\rho^{AB})=S(\rho^A)+S(\rho^B)-S(\rho^{AB}), \tag{7}$$

where $S(\rho)=-\mathrm{Tr}(\rho\log_2\rho)$ is the von Neumann entropy of the system in the state ρ, and $\rho^{A(B)}=\mathrm{Tr}_{B(A)}(\rho^{AB})$ are the reduced density matrices of the two subsystems of the bipartite composite system.

Similarly, by definition, the classical correlation is a measure of the maximum extractable information. For a bipartite system, it is given by [17–19]

$$\mathcal{C}_B(\rho^{AB})=S(\rho^B)-\min_{\{\mho_k^A\}}\sum_k p_k S(\rho_k^B), \tag{8}$$

where $\rho_k^B=\mathrm{Tr}_A\left[(\mho_k^A\otimes I)\rho^{AB}(\mho_k^A\otimes I)\right]/p_k$ is the post-measurement state of subsystem B after obtaining the outcome k on subsystem A with the probability $p_k=\mathrm{Tr}\left[\left(\mho_k^A\otimes I\right)\rho^{AB}\left(\mho_k^A\otimes I\right)\right]$. Here, the set $\left\{\mho_k^A\right\}$ stands for the projectors onto the space of qubit A and can be written as follows:

$$\mho_k^A=\frac{1}{2}(I\pm\sum_j n_j\sigma_j), \tag{9}$$

where the $\pm$ sign corresponds to $k=1,2$, respectively. The vector $\overrightarrow{n}$ defines a unit vector on the Bloch sphere having components $n=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)^t$ with $\theta\in[0,\pi]$ and $\phi\in[0,2\pi]$.

With all the necessary mathematical machinery in our hands, we are now in the position to present the dependence of the dynamics of quantum correlations on different parameters of the system.

3. Results and Discussion

In this section, we present the mathematical details and the graphical analysis of our results for each type of mixed classical noise on the dynamics of quantum correlations in bipartite qubit system.

3.1. Static and RTN

This part deals with the situation in which one qubit (qubit A) is coupled with static noise, and the other qubit (qubit B) evolves under the action of RTN noise. The evolution operator can be obtained from the Hamiltonian of Eq. (1) with the help of Eq. (2) while using the defined relations for the two random parameters given for static and RTN noises. It reads as $U(\chi_A, \chi_B, t) = e^{-i\int H(t)dt}$ with $\hbar$ set to unity, and the explicit form of it becomes

$$U(t) = e^{-2\epsilon t}\begin{pmatrix} CC & -iCS & -iSC & -SS \\ -iCS & CC & -SS & -iCS \\ -iCS & -SS & CC & -iCS \\ -SS & -iCS & -iCS & CC \end{pmatrix}, \tag{10}$$

with

$$CC = \cos(\chi_A g_A)\cos(\chi_B g_B), \qquad CS = \cos(\chi_A g_A)\sin(\chi_B g_B), \tag{11}$$

$$SC = \sin(\chi_A g_A)\cos(\chi_B g_B), \qquad SS = \sin(\chi_A g_A)\sin(\chi_B g_B). \tag{12}$$

As there is no direct interaction between the two environments, one can show that the unitary operator of Eq. (10) factorizes into two unitary operators, each describing the evolution of its own qubit. If the initial state of the two qubits is also factorizable, no interaction will ever occur between them, and they will evolve independently under their own environment.

On the other hand, if the initial state of the qubits is not factorizable, there is an indirect interaction between the two environments through the initial correlation between the state of the qubits. How this indirect interaction between the environments or the presence of a single environment will influence the dynamics of the initial correlations between the qubits is the subject of our study.

Let the system be initially prepared in the Bell state $\rho(0) = |\psi\rangle\langle\psi|$ with $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$; then the overall effect of the mixed noise on the evolved state $\rho(\chi_A, \chi_B, t) = U(\chi_A, \chi_B, t)\rho(0)U(\chi_A, \chi_B, t)^\dagger$ is given as

$$\rho'(\chi_A, \chi_B, t) = \left\langle \int_{\chi_A-\Delta_\chi/2}^{\chi_A+\Delta_\chi/2} d\chi_A P(\chi_A)\rho(\chi_A, \chi_B, t)\right\rangle_{\varphi_B}. \tag{13}$$

The explicit form of the final density matrix becomes

$$\rho'(t) = \begin{pmatrix} X_1 - 1/4 & -X_2 & -X_2 & X_1 - 1/4 \\ X_2 & -X_1 - 1/4 & -X_1 - 1/4 & X_2 \\ X_2 & -X_1 - 1/4 & -X_1 - 1/4 & X_2 \\ X_1 - 1/4 & -X_2 & -X_2 & X_1 - 1/4 \end{pmatrix}, \tag{14}$$

with

$$X_1 = \frac{1}{4\Delta_\chi g_A t}[(X_B)\cos(2\chi_o g_A t)\sin(\Delta_\chi g_A t)], \qquad X_2 = \frac{i}{4\Delta_\chi g_A t}[(X_B)\sin(2\chi_o g_A t)\sin(\Delta_\chi g_A t)]. \tag{15}$$

In the above equation, X_B represents the contribution arising from the average over the random phase factor $\langle e^{in\varphi_B(t)}\rangle$ and can explicitly be expressed as [39]

$$X_B = \begin{cases} e^{-\gamma t}\left[\cosh(\delta_{ng_B} t) + \dfrac{\gamma}{\delta_{ng_B}}\sinh(\delta_{ng_B} t)\right], & \text{for } \gamma \geq ng_B, \\ e^{-\gamma t}\left[\cos(\delta_{ng_B} t) + \dfrac{\gamma}{\delta_{ng_B}}\sin(\delta_{ng_B} t)\right], & \text{for } \gamma \leq ng_B, \end{cases} \tag{16}$$

where $\delta_{ng_B} = \sqrt{|\gamma^2 - (ng_B)^2|}$ with $n \in \{2, 4\}$. In the numerical simulations, we will use $n = 2$.

With the final density matrix of the system in our hands, it is straightforward to analyze the behavior of the two types of quantum correlations. As mentioned earlier, for the analysis of the dynamics of entanglement, we use negativity, which we calculate analytically using Eq. (5), first by taking the partial transpose of Eq. (14) with respect to qubit B. On the other hand, the behavior of quantum discord is analyzed numerically. The negativity in this case becomes

$$\mathcal{N}(\rho^{AB}) = 2\sqrt{\left|X_1^2 - X_2^2\right|}.$$

In Fig. 1, we plot both the negativity and quantum discord against time for different choices of the coupling strengths of the two noises. The dynamics of entanglement and quantum discord are given, respectively, in the top and bottom rows of Fig. 1. Each curve in the first column corresponds to a different value of g_A, and in the second column it corresponds to a different value of g_B. In the first column, the coupling strength of the marginal qubit with RTN is limited to the weak coupling regime ($g_B = 0.05$), and the coupling strength g_A is varied from zero to the weak and to the strong coupling region in steps of ($g_A = 0, 0.2, 0.4, 0.6, 0.8$). The values of other parameters are given in the caption of the figure.

The same strategy is followed for the second column of Fig. 1 but with values of both g_A and g_B interchanged. A comparison of the two columns reveals that both entanglement and quantum discord are very fragile against the strength of the coupling constant with static environment than with RTN. The rate of degradation of the correlations quickens and may lead to sudden death, as the coupling strength with the static noise (first column) increases.

In the strong coupling regime of g_A, the complete loss of the correlations is followed by periodic revivals of decreasing amplitudes with time. The revivals are more prominent for entanglement than for quantum discord, which shows robustness of entanglement over quantum discord in the periodic intervals. On the other hand, the relative weaker monotonic degradation of the correlations against the coupling strength g_B of RTN (second column) ensures avoiding complete loss or sudden death of them.

3.2. RTN and Ornstein–Uhlenbeck Process

In this section, we deal with the dynamics of quantum correlations when one qubit is coupled with OU noise and the other with RTN. Selecting the random variables for these two noises, we can find

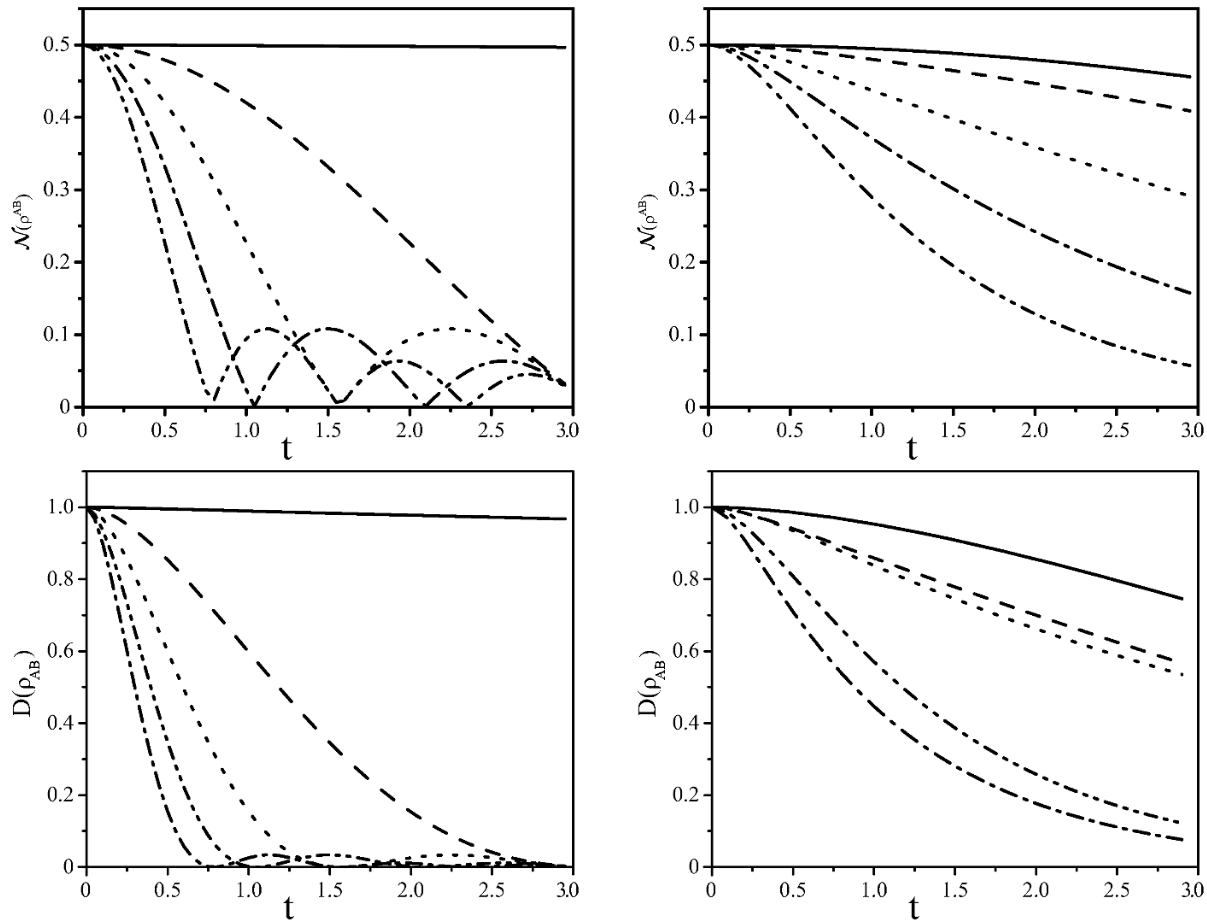

Fig. 1. Entanglement (first row) and quantum discord (second row) against time t for $\gamma = 2$, $\chi_o = 1$ and $\Delta_\chi = 5$. In the first column, $g_B = 0.05$ and $g_A = \{0, 0.2, 0.4, 0.6, 0.8\}$ from top to bottom. In the second column, the values of g_A and g_B are interchanged.

the evolution operator following the method of the previous case. However, the overall effect of the environment on the final density matrix of the system, in this case, is obtained by averaging it over both phase factors $\varphi_A(t)$ and $\varphi_B(t)$, where the former corresponds to RTN noise, and the latter to OU noise. That is, $\rho'(t)$ is obtained as $\rho'(t) = \langle\langle\rho(\chi_A, \chi_B, t)\rangle_{\varphi_A}\rangle_{\varphi_B}$, which leads to the following form:

$$\rho'(t) = \begin{pmatrix} (1/4)+Y & 0 & 0 & (1/4)+Y \\ 0 & (1/4)-Y & (1/4)-Y & 0 \\ 0 & (1/4)-Y & (1/4)-Y & 0 \\ (1/4)+Y & 0 & 0 & (1/4)+Y \end{pmatrix}, \tag{17}$$

where

$$Y = \begin{cases} \dfrac{1}{4}Y_1\left(\cosh(\delta_{ng_A}t) + \frac{\gamma}{\delta_{ng_B}}\sinh(\delta_{ng_A}t)\right)e^{-\gamma t}, & \text{for } \gamma \geq ng_A, \\ \dfrac{1}{4}Y_1\left(\cos(\delta_{ng_A}t) + \frac{\gamma}{\delta_{ng_A}}\sin(\delta_{ng_A}t)\right)e^{-\gamma t}, & \text{for } \gamma \leq ng_A, \end{cases} \tag{18}$$

with

$$Y_1 = \frac{2g_B}{\gamma_p}\left(1 - e^{-\gamma_p t} - \gamma_p t + \frac{\gamma_p}{2g_B}\right), \qquad \delta_{ng_A} = \sqrt{|\gamma^2 - (ng_A)^2|}.$$

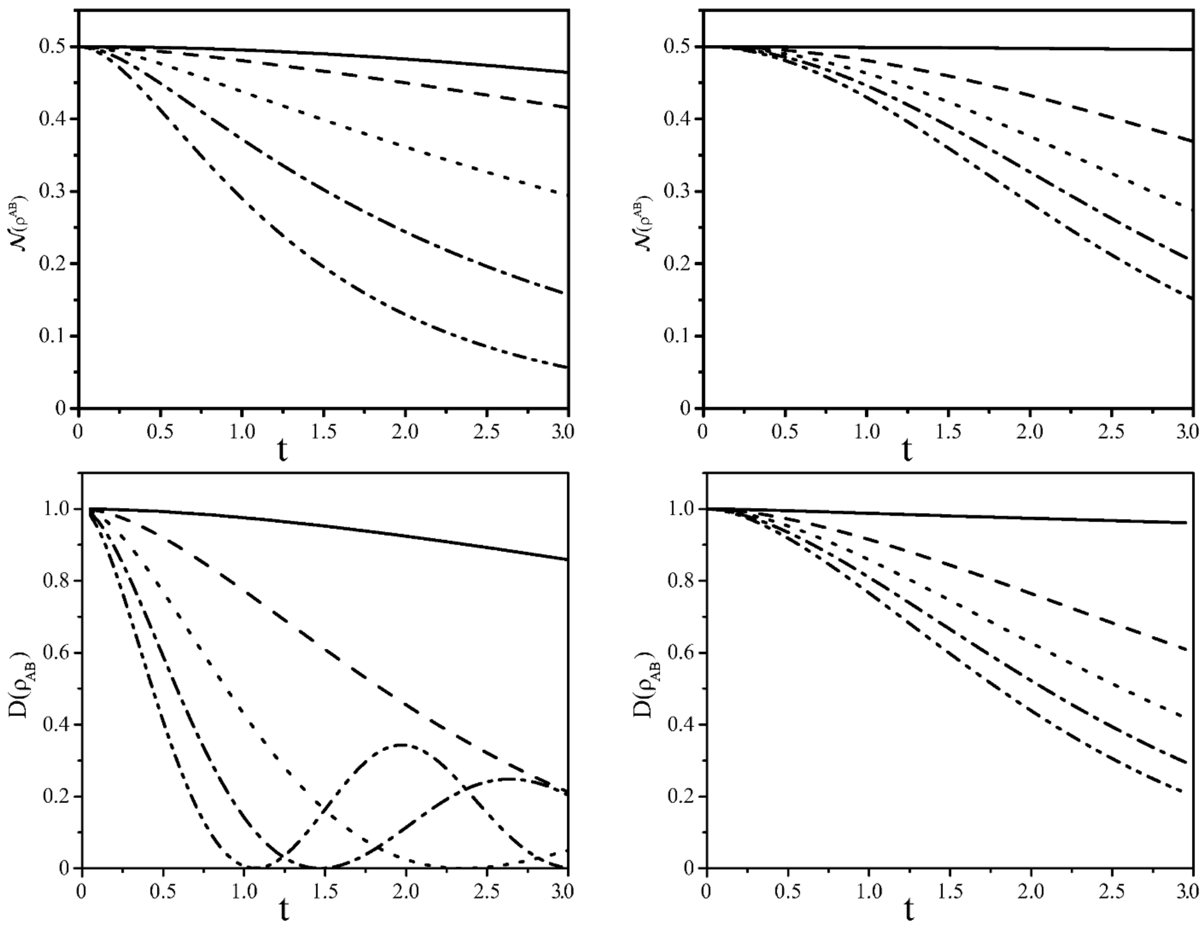

Fig. 2. Entanglement (first row) and quantum discord (second row) against time t for $\gamma = 2$, $\gamma_p = 0.2$, $\chi_o = 1$, and $\Delta_\chi = 5$. In the first column, $g_B = 0.05$ and $g_A = 0, 0.2, 0.4, 0.6, 0.8$ from top to bottom. In the second column, the values of g_A and g_B are interchanged.

Note that the final density matrix of Eq. (17) belongs to the family of X-type states whose discord can be found analytically [43,44]. However, in parallel with the previous case, we work it out numerically. Again, for the purpose of mathematical ease, we limit our analysis by employing the first condition in Eq. (18) with $n = 2$. The dynamics of the correlations against time for different coupling strengths between the system and the environments, in this case, are shown in Fig. 2, where the rows and columns bear the same meaning as explained in Fig. 1. In contrast to Fig. 1, the entanglement monotonously degrades, with different rates, both in the weak and strong coupling regimes, irrespective of the nature of environments. A comparison of Figs. 1 b and 2 a reveals that the weak coupling of the system with static and OU noises has no noticeable effect on the dynamics of entanglement with increasing coupling strength of the system with RTN. However, the difference of the effects of increasing coupling strengths of the static and OU environments in the presence of weak RTN coupling can easily be seen from a comparison of Figs. 1 a and 2 b. It can be seen that entanglement is comparatively robust against OU noise, as there is neither entanglement sudden death nor its complete loss in the given coupling time.

On the other hand, the effects of weak couplings of static and OU noises on the behavior of quantum discord with increasing strength of RTN is quite obvious. A comparison of Figs. 1 d and 2 c reveals that the dynamics of quantum discord in the strong coupling regime with RTN depends heavily on whether the marginal system is weakly coupled to static or OU noise. In the case of weak coupling of the marginal system with OU noise (Fig. 2 c), unlike entanglement, the discord is more fragile than in the case of weak coupling of the marginal system with static noise (Fig. 1 d). In the presence of weak coupling of one marginal system with OU noise, the increasing coupling strength of the second marginal system with RTN quickly destroys quantum discord. This, however, is followed by a sudden giant revival in the strong coupling regime.

4. Conclusions

In this paper, we investigated the dynamics of entanglement and quantum discord of a maximum entangled bipartite quantum system in the presence of different mixed classical environments. The static and OU noises are mixed up with RTN, and a comparative study of the behaviors of entanglement and quantum discord in both weak and strong coupling regimes is demonstrated. We studied the behavior of entanglement analytically, whereas we used a numerical approach for the study of quantum discord. Both the correlations are very fragile and go through sudden death, followed by periodic revivals, when one qubit is strongly coupled with static noise and the other is weakly coupled to RTN noise.

On the other hand, when the coupling with RTN noise is strong, both correlations degrade monotonically and completely avoid sudden death. This means that both correlations can survive for a long enough time and can be utilized for realization of different quantum information tasks. Moreover, we showed that the behavior of quantum discord, when one qubit is strongly coupled with RTN noise, is very sensitive to the nature of the weakly coupled environment. It is robust when the weakly coupled environment is static and is fragile when the weakly coupled environment is OU. On the contrary, unlike quantum discord, the behavior of entanglement is independent of the nature of the weakly coupled environment. Our study provides a significant insight into which quantum correlation is better to use for different quantum information tasks in various mixed classical environments.

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