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CONTINUOUS SETS OF DEQUANTIZERS AND QUANTIZERS FOR ONE-QUBIT STATES*

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Abstract

We show the star-product quantization procedure for spin-1/2 particles (qubits) employing the construction of a pair of operators – dequantizers and quantizers. We present an explicit description of all minimal systems of such dequantizers and quantizers and discuss their relation to the probability representation of spin states where the fair probability distribution is identified with the spin states. We give some examples and discuss the possibility of constructing a symplectic structure in the finite-dimensional phase space.

Keywords: discrete Wigner function, operator symbols, spin states, star-product, quantizers, dequantizers, symplectic structure.

1. Introduction

In quantum mechanics, the notion of state of a system differs from the notion of state in classical mechanics. The classical particle state, if there is no fluctuations due to the interaction with an environment, is associated with a pair of numbers — the particle position q and velocity $\dot{q}$ or the particle position q and momentum p. If there are fluctuations of the particle position and momentum, the particle state is identified with the probability density in the phase space, which is a nonnegative normalized function $f(q,p)$ with intuitively clear physical meaning.

In quantum mechanics, the particle's pure states are identified with the complex wave function $\psi(x)$ depending on the particle position x, and mixed states are identified with the complex function $\rho(x,x')$ of two variables x and x', which is the matrix element of the state density operator $\hat{\rho}$ in the position representation, i.e., $\rho(x,x') = \langle x \mid \hat{\rho} \mid x' \rangle$.

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In classical mechanics, the observables like the particle position and momentum are described by c-numbers. In quantum mechanics, the observables (position and momentum) are identified with the Hermitian operators $\hat{q}$ and $\hat{p}$ acting in the Hilbert space of the particle states $\mid \psi\rangle$. Intuitively such description of states and observables in quantum mechanics is quite different from the clear classical picture of states and observables. Due to this fact, the formulations of quantum mechanics based on star-product schemes of quantization were suggested; here, invertible maps of operators of physical observables and the state density operators onto the functions (analogous to the corresponding functions associated with classical particles) were suggested; see, for example, the recent review [1].

The most general formulation of the star-product scheme of quantization, in which the basic notion of pairs of operators called dequantizers $\hat{U}(\alpha)$ and quantizers $\hat{Q}(\alpha)$ (depending on a variable α) was introduced and applied to construct the symbols of an arbitrary operator acting in the Hilbert space of states, was elaborated in [2]. The pairs of dequantizer and quantizer operators determine the rule of nonlocal associative multiplication of the functions, which are the corresponding symbols of the operators associated with quantum observables and the state density operators. Among the known formulations of quantum mechanics with continuous variables like the position and momentum, the formulations based on the phase-space representations of the state play the important role.

In the phase-space formulation of quantum mechanics, the quasiprobability distributions are widely used for describing the quantum states. The problem of representation of quantum mechanics in the phase space was also discussed long ago by Stratonovich [3]. The properties of quantum quasidistribution functions for systems with continuous variables are studied in the reviews [4–7] and books [8–11]. The most well-known and frequently used among them are the Wigner function $W(q,p)$ [12], the Husimi–Kano function $Q(q,p)$ [13, 14], and the Glauber–Sudarshan $P(q,p)$ quasidistribution [15, 16]. In this approach, some special functions defined on the phase space correspond to the operators called the symbols of operators.

In addition to various quasiprobability distributions in quantum mechanics, there exists a symplectic tomogram of the quantum state [17–20]; it is a real probability-distribution function of the position measured in an ensemble of reference frames in the phase space. The symplectic tomogram completely determines the quantum state. and this probability distribution is the tomographic symbol of the density operator. The connection between symplectic tomograms and the Husimi functions for systems with continuous variables was studied in [21].

In this paper, we consider quantum systems with discrete (spin) variables. The approach to these systems based on quasiprobability distributions in a finite phase space was developed in [22–27]. Such quasiprobability and probability distributions were studied in [28–33]. Just as in the case of quantum systems with continuous variables, the systems with discrete (spin) variables can be described in view of the probability distributions. These probability distributions are called spin tomograms introduced in [34–37] and unitary-matrix tomograms introduced in [38].

With the help of dequantizers, one can find the symbols of operators, and with the help of quantizers one can restore the operators by their symbols. It is worth noting that the pair of quantizer–dequantizer operators introduced in [2] for the description of an arbitrary star-product scheme is a generalization of the method of using the quantizer discussed in [3] for the cases where the quantizer and dequantizer operators are substantially different. For some special set of dequantizers, the symbols of the density operators are called the Wigner function [25]. The algebra of symbols is based on the star-product method [2, 39]. The technique of star-products of symbols in the case of spin states is described in detail, e.g., in [40].

The Wigner functions and tomograms of the density matrix of quantum spin states are studied in many papers. In this paper, we propose a method of constructing all possible minimal systems of dequantizers and quantizers, both Hermitian and non-Hermitian, self-dual and non-self-dual ones. We prove several statements concerning their properties. For this purpose, we use a special technique [41, 42] described in Sec. 2, which allows one to consider instead of matrices some vectors corresponding to the matrices. We define linear transforms in the space of these vectors and consider orbits of these transforms. In Sec. 3, we present a general description of the dequantizer and quantizer constructions. We give the general approach to constructing minimal sets of dequantizers and quantizers in the case of one-qubit states in Sec. 4. We present our conclusions in Sec. 5.

Most of the results obtained can be generalized to the case of N qubits.

2. Representation of Matrices as Vectors

While constructing systems of dequantizers and quantizers, which are realized in the form of matrices, one requires that the sets of matrices should satisfy some properties of orthogonality. To this end, one needs to calculate products of these matrices and check if traces of the products are equal to zero or not. In some cases, for this purpose, it is convenient to represent matrices in the form of vectors. In this representation, the traces of the products of matrices are equal to scalar products of the vectors (see, e.g., [41]).

Assume that we have a $n\times n$ matrix $\hat{A} = ||a_{ij}||,\ i,j = 1,\ldots,n$. With this matrix, we associate a vector of the n^2-dimensional space. This vector, being written as a column, has appropriate matrix elements of $\hat{A}$ as components located in the same order as they appear in the columns of the matrix $\hat{A}$. In other words, in order to build a column vector corresponding to the matrix $\hat{A}$, we should take the columns of the matrix and arrange them under each other. We explain this procedure on the example of a real 2×2 matrix.

If we have a 2×2 matrix

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \tag{1}$$

the 4-vector can be constructed as follows:

$$\vec{A} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{12} \\ a_{22} \end{pmatrix}. \tag{2}$$

We see that in (2) the columns of matrix (1) are one under the other, and it is just the rule of invertible mapping of vectors onto matrices. We denote this map of matrices into vectors by the symbol MV, and the reverse mapping by the symbol VM; they read

$$(MV) : \hat{A} \to \vec{A}, \qquad (VM) : \vec{A} \to \hat{A}. \tag{3}$$

We can determine the scalar product $(\vec{A}, \vec{B})_M$ of such vectors as follows.

Assume that we have two real vectors $\vec{A}$ and $\vec{B}$ with n^2 components. First, we construct the matrix $\hat{B}$ by a vector $\vec{B}$, then transpose the matrix $\hat{B}$ and pass from the transposed matrix $\hat{B}^T$ to the vector

$\vec{B^T}$. Then we can calculate the scalar product of vectors $\vec{A}$ and $\vec{B^T}$ in the usual way by multiplying their components; we obtain

$$(\vec{A},\vec{B})_M = (\vec{A},\vec{B^T}) = \sum_{i=1}^{n^2} A_i \vec{B^T}_i, \tag{4}$$

where A_i and $\vec{B^T}_i$ are the components of vectors $\vec{A}$ and $\vec{B^T}$, respectively.

It is easy to verify the relation

$$(\vec{A},\vec{B})_M = (\vec{A},\vec{B^T}) = \mathrm{Tr}\,(\hat{A}\hat{B}). \tag{5}$$

We showed that, instead of calculating the matrix product trace, one can calculate the scalar product of the corresponding vectors. In some cases, it is easier to work with vectors than with matrices, in particular, for checking the orthogonality property. An analogous rule can be formulated for complex vectors.

3. Constructions of Dequantizers and Quantizers

In this section, we consider finite-dimensional quantum systems such as, for example, the spin systems. The physical states of such systems correspond to the vectors of some d-dimensional Hilbert space $\mathcal{H}_d$. Also $\mathcal{B}(\mathcal{H}_d)$ is a set of linear operators in the space $\mathcal{H}_d$. Each operator $\hat{A} \in \mathcal{B}(\mathcal{H}_d)$ is bounded and can be represented as a $d{\times}d$-dimensional matrix with complex elements. This Hermitian matrix can be associated with values of some observables.

In the space $\mathcal{H}_d$, we consider a set of Hermitian operators $\hat{U}(\alpha)$ assuming the parameter α to be either discrete or continuous. If one considers an operator $\hat{A} \in \mathcal{B}(\mathcal{H}_d)$, the c-function $f_A(\alpha)$ defined by the formula

$$f_A(\alpha) = \mathrm{Tr}\left(\hat{U}(\alpha)\hat{A}\right) \tag{6}$$

is called the symbol of operator $\hat{A} \in \mathcal{B}(\mathcal{H}_d)$ built by a set of operators $\hat{U}(\alpha)$. The operators $\hat{U}(\alpha)$ are called the dequantizers. Let α be a discrete variable, which we denote as k.

For a given system of dequantizers $\hat{U}(k)$, $k = 1,\ldots,N$, formula (6) defines a map of the space of operators $\mathcal{B}(\mathcal{H}_d)$ into the space of functions $\mathcal{F} = \{f(k) : \{k = 1,\ldots,N\} \to \mathtt{C}\}$. It is a linear map, i.e., the relations

$$\hat{C} = \hat{A} + \hat{B} \Leftrightarrow f_C(k) = f_A(k) + f_B(k), \qquad \hat{C} = c\hat{A} \Leftrightarrow f_C(k) = cf_A(k),$$

are valued for each number $c \in \mathtt{C}$.

If map (6) of the operator $\hat{A}$ onto the function $f_A(k)$ is a one-to-one map, the symbol $f_A(k)$ contains complete information on the operator $\hat{A}$, and the latter can be reconstructed as follows:

$$\hat{A} = \sum_{k=1}^{N} f_A(k)\hat{D}(k). \tag{7}$$

The operators $\hat{D}(k) \in \mathcal{B}(\mathcal{H}_d)$ are called the quantizers, and their relation to the dequantizers is

$$\mathrm{Tr}\left(\hat{U}(k)\hat{A}\right) = \sum_{k'=1}^{N} \mathrm{Tr}\left(\hat{U}(k')\hat{A}\right)\,\mathrm{Tr}\left(\hat{U}(k)\hat{D}(k')\right). \tag{8}$$

Relation (8) holds for all sets of dequantizers and quantizers. For minimal sets of dequantizers and quantizers, it can be simplified,

$$\mathrm{Tr}\left(\hat{U}(k)\hat{D}(k')\right) = \delta(k,k'). \tag{9}$$

Such minimal sets of dequantizers contain d^2 linearly independent operators.

In the case of discrete systems, the sets of functions $\mathrm{Tr}\left(\hat{U}(k)\hat{\rho}\right)$ are analogs of the quasiprobability distributions, which are used to study the systems with continuous variables such as the Wigner and Husimi functions. For this reason, the functions $\mathrm{Tr}\left(\hat{U}(k)\hat{\rho}\right)$ with appropriate dequantizers are also often called the discrete Wigner functions.

These functions, which are symbols of the density operators $\hat{\rho}$, i.e., $f_\rho(k) = \mathrm{Tr}\left(\hat{U}(k)\hat{\rho}\right)$, can be negative; in these cases, they cannot be interpreted as the probabilities.

4. Minimal Sets of Dequantizers and Quantizers

Let us consider a one-qubit state. Its density matrix reads

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+z & x+iy \\ x-iy & 1-z \end{pmatrix}, \qquad x^2+y^2+z^2 \leq 1. \tag{10}$$

The matrix is Hermitian, i.e., $\rho^\dagger = \rho$, its trace is equal to the unity, and eigenvalues are nonnegative numbers. The three real parameters in the matrix determine the probabilities to have spin projectors $\pm 1/2$ on the axes x, y, and z, respectively.

In this case, the dimension $d = 2$, and the minimum set of dequantizers contains four operators.

In the general case, dequantizers $\hat{U}^{(k)}$ have the form of arbitrary linearly independent 2×2 matrices,

$$\hat{U}^{(k)} = \begin{pmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{pmatrix}, \qquad k = 1,2,3,4. \tag{11}$$

The matrices form a basis in the space of all 2×2 matrices.

The four-vectors $\vec{U}^{(k)},\ k = 1,2,3,4$ corresponding to the above matrices can be written in the form of column-vectors; they read

$$\vec{U}^{(k)} = \begin{pmatrix} U_{11} \\ U_{21} \\ U_{12} \\ U_{22} \end{pmatrix}, \qquad k = 1,2,3,4. \tag{12}$$

For the vector $\vec{U}^{(k)}$, we use the notation omitting index k in the column having in mind that such structure of the vectors corresponds to an arbitrary value of k. These four vectors form a basis in the four-dimensional complex linear space.

Now we construct the quantizers $\hat{D}^{(k)},\ k = 1,2,3,4$, which are 2×2 matrices. To this end, we consider four 2×2 matrices

$$\hat{D}^{(k)} = \begin{pmatrix} D_{11}^{(k)} & D_{12}^{(k)} \\ D_{21}^{(k)} & D_{22}^{(k)} \end{pmatrix}, \qquad k = 1,2,3,4. \tag{13}$$

The four vectors $\vec{D}^{(k)},\ k=1,2,3,4$ read

$$\vec{D}^{(k)}=\begin{pmatrix} D_{11}\\ U_{21}\\ D_{12}\\ D_{22}\end{pmatrix},\qquad k=1,2,3,4. \tag{14}$$

As in the case of dequantizer, here, we again omit the index k in the column-vector components.

In order to be the sets of dequantizers and quantizers, the matrices (11) and (13) should satisfy relation (9). With the help of vectors (12) and (14), relation (9) can be written in the form of the orthogonality condition for these vectors

$$\big(\vec{U}^{(k)},\vec{D}^{(k')}\big)_M=\delta(k,k'). \tag{15}$$

Let us choose a set of linearly independent vectors (14), i.e., we assume that the values $U_{ij}^{(k)}$ $(i,j=1,2;\ k=1,2,3,4)$ are given and, for simplicity, are real. Now we can rewrite the orthogonality condition (15) as a system of equations for unknown parameters $D_{ij}^{(k')}$ $(i,j=1,2;\ k'=1,2,3,4)$. If $k'=1$, one arrives at four linear equations

$$\begin{aligned} &U_{11}^{(1)}D_{11}^{(1)}+U_{21}^{(1)}D_{12}^{(1)}+U_{12}^{(1)}D_{21}^{(1)}+U_{22}^{(1)}D_{22}^{(1)}=1,\quad U_{11}^{(2)}D_{11}^{(1)}+U_{21}^{(2)}D_{12}^{(1)}+U_{12}^{(2)}D_{21}^{(1)}+U_{22}^{(2)}D_{22}^{(1)}=0,\\ &U_{11}^{(3)}D_{11}^{(1)}+U_{21}^{(3)}D_{12}^{(1)}+U_{12}^{(3)}D_{21}^{(1)}+U_{22}^{(3)}D_{22}^{(1)}=0,\quad U_{11}^{(4)}D_{11}^{(1)}+U_{21}^{(4)}D_{12}^{(1)}+U_{12}^{(4)}D_{21}^{(1)}+U_{22}^{(4)}D_{22}^{(1)}=0. \end{aligned} \tag{16}$$

Thus, the problem of obtaining the quantizer is reduced to the problem of solving the system of the above linear equations.

Vectors (12) are linearly independent, so the determinant $\Delta(\hat{A})$ of matrix

$$\hat{A}=||a_{mn}||=\begin{pmatrix} U_{11}^{(1)} & U_{21}^{(1)} & U_{12}^{(1)} & U_{22}^{(1)}\\ U_{11}^{(2)} & U_{21}^{(2)} & U_{12}^{(2)} & U_{22}^{(2)}\\ U_{11}^{(3)} & U_{21}^{(3)} & U_{12}^{(3)} & U_{22}^{(3)}\\ U_{11}^{(4)} & U_{21}^{(4)} & U_{12}^{(4)} & U_{22}^{(4)}\end{pmatrix},\qquad m,n=1,2,3,4 \tag{17}$$

is not equal to zero: $\Delta(\hat{A})\neq 0$.

Using the standard procedure, we solve the system of equations (16) and obtain the components of the vector $\vec{D}^{(1)}$,

$$D_{11}^{(1)}=A_{11}\Delta(\hat{A})^{-1},\quad D_{12}^{(1)}=A_{12}\Delta(\hat{A})^{-1},\quad D_{21}^{(1)}=A_{13}\Delta(\hat{A})^{-1},\quad D_{22}^{(1)}=A_{14}\Delta(\hat{A})^{-1}. \tag{18}$$

These components provide the first column in the quantizer matrix.

Similarly, one obtains the components of vectors $\vec{D}^{(2)}$, $\vec{D}^{(3)}$, and $\vec{D}^{(4)}$,

$$\begin{aligned} &D_{11}^{(2)}=A_{21}\Delta(\hat{A})^{-1},\quad D_{12}^{(2)}=A_{22}\Delta(\hat{A})^{-1},\quad D_{21}^{(2)}=A_{23}\Delta(\hat{A})^{-1},\quad D_{22}^{(2)}=A_{24}\Delta(\hat{A})^{-1},\\ &D_{11}^{(3)}=A_{31}\Delta(\hat{A})^{-1},\quad D_{12}^{(3)}=A_{32}\Delta(\hat{A})^{-1},\quad D_{21}^{(3)}=A_{33}\Delta(\hat{A})^{-1},\quad D_{22}^{(3)}=A_{34}\Delta(\hat{A})^{-1},\\ &D_{11}^{(4)}=A_{41}\Delta(\hat{A})^{-1},\quad D_{12}^{(4)}=A_{42}\Delta(\hat{A})^{-1},\quad D_{21}^{(4)}=A_{43}\Delta(\hat{A})^{-1},\quad D_{22}^{(4)}=A_{44}\Delta(\hat{A})^{-1}. \end{aligned} \tag{19}$$

In (18) and (19), A_{mn} is the cofactor of an element a_{mn} of matrix (17).

Thus, we obtained all matrix elements of the quantizer matrix. Analogous formulas can be written for complex matrices.

Now we are in the position to prove some statements concerning the dequantizers and quantizers.

Statement 1.

If vectors (12) are linearly independent, vectors (14) are linearly independent too.

In fact, the coefficients $D_{ij}^{(k')}$ $(i,j=1,2;\ k'=1,2,3,4)$ of vectors (14) form the matrix

$$\hat{B}=\begin{pmatrix} D_{11}^{(1)} & D_{12}^{(1)} & D_{21}^{(1)} & D_{22}^{(1)} \\ D_{11}^{(2)} & D_{12}^{(2)} & D_{21}^{(2)} & D_{22}^{(2)} \\ D_{11}^{(3)} & D_{12}^{(3)} & D_{21}^{(3)} & D_{22}^{(3)} \\ D_{11}^{(4)} & D_{12}^{(4)} & D_{21}^{(4)} & D_{22}^{(4)} \end{pmatrix}=\Delta(\hat{A})^{-1}\begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}, \tag{20}$$

i.e., $\hat{A}\hat{B}^T=1$. Therefore, $\Delta(\hat{B})=\Delta(\hat{A})^{-1}\neq 0$, and vectors $\vec{D}^{(k)}$, $k=1,2,3,4$ (14), are linearly independent.

Statement 2.

If matrices (11) are Hermitian, matrices (13) are Hermitian as well.

We assume now that dequantizers $\hat{U}^{(k)}$ are Hermitian operators. Then the matrix element $U_{12}^{(k)}=U_{21}^{(k)*}$. If we introduce the notation $U_{12}^{(k)}=P^{(k)}+iQ^{(k)}$, Eqs. (16) read

$$\begin{aligned} U_{11}^{(1)}D_{11}^{(1)}+(P^{(1)}-iQ^{(1)})D_{12}^{(1)}+(P^{(1)}+iQ^{(1)})D_{21}^{(1)}+U_{22}^{(1)}D_{22}^{(1)}&=1,\\ U_{11}^{(2)}D_{11}^{(1)}+(P^{(2)}-iQ^{(2)})D_{12}^{(1)}+(P^{(2)}+iQ^{(2)})D_{21}^{(1)}+U_{22}^{(2)}D_{22}^{(1)}&=0,\\ U_{11}^{(3)}D_{11}^{(1)}+(P^{(3)}-iQ^{(3)})D_{12}^{(1)}+(P^{(3)}+iQ^{(3)})D_{21}^{(1)}+U_{22}^{(3)}D_{22}^{(1)}&=0,\\ U_{11}^{(4)}D_{11}^{(1)}+(P^{(4)}-iQ^{(4)})D_{12}^{(1)}+(P^{(4)}+iQ^{(4)})D_{21}^{(1)}+U_{22}^{(4)}D_{22}^{(1)}&=0. \end{aligned} \tag{21}$$

The values $U_{11}^{(k)}$, $D_{11}^{(k)}$, $U_{22}^{(k)}$, $D_{22}^{(k)}$, $P^{(k)}$, and $Q^{(k)}$ are real; therefore, it follows from Eqs. (21) that $U_{12}^{(k)}=U_{21}^{(k)*}$ and, therefore, the quantizers $\hat{D}^{(k)}$ are Hermitian operators.

Let us consider some arbitrary set of dequantizers $\hat{U}^{(k)}$ and quantizers $\hat{D}^{(k')}$; they satisfy relation (15). We make a nondegenerate linear transform of the vectors $\big(\hat{U}^{(1)},\,\hat{U}^{(2)},\,\hat{U}^{(3)},\,\hat{U}^{(4)}\big)^T$,

$$\hat{V}^{(k)}=\sum_{i=1}^{4}L_{ki}\hat{U}^{(i)},\qquad k=1,2,3,4. \tag{22}$$

Our goal is to find a new set of quantizers $\hat{E}^{(k')}$ such that its members satisfy relation (9),

$$\mathrm{Tr}\,(\hat{V}^{(k)}\hat{E}^{(k')})=\delta(k,k'). \tag{23}$$

So we are considering the problem using a new reference frame in our linear space.

We are looking for these new operators as linear combinations of the previous quantizers $\hat{D}^{(k)}$,

$$\hat{E}^{(l)} = \sum_{j=1}^{4} M_{lj}\hat{D}^{(j)}, \qquad k = 1, 2, 3, 4. \tag{24}$$

In view of relation (15), one can obtain a system of equations for the unknown coefficients M_{lj}. This system of equations is similar to the system of equations (16). If $k' = 1$, one has four linear equations, which can be solved using the standard procedure:

$$\begin{gathered} L_{11}M_{11} + L_{12}M_{12} + L_{13}M_{13} + L_{14}M_{14} = 1, \quad L_{21}M_{11} + L_{22}M_{12} + L_{23}M_{13} + L_{24}M_{14} = 0, \\ L_{31}M_{11} + L_{32}M_{12} + L_{33}M_{13} + L_{34}M_{14} = 0, \quad L_{41}M_{11} + L_{42}M_{12} + L_{43}M_{13} + L_{44}M_{14} = 0 \end{gathered} \tag{25}$$

Since the transform (22) is nondegenerate, the determinant $\hat{\Delta(L)}$ of the matrix

$$\hat{L} = ||L_{mn}|| = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \\ L_{41} & L_{42} & L_{43} & L_{44} \end{pmatrix}, \qquad m, n = 1, 2, 3, 4 \tag{26}$$

is not equal to zero, i.e., $\hat{\Delta(L)} \neq 0$.

Now after solving the system of equations (25), we obtain the coefficients M_{1k} $(k = 1, 2, 3, 4)$,

$$M_{11} = l_{11}\Delta(\hat{L})^{-1}, \quad M_{12} = l_{12}\Delta(\hat{L})^{-1}, \quad M_{13} = l_{13}\Delta(\hat{L})^{-1}, \quad M_{14} = l_{14}\Delta(\hat{L})^{-1}. \tag{27}$$

Similarly, we obtain the coefficients M_{2k}, M_{3k}, and M_{4k} $(k = 1, 2, 3, 4)$,

$$\begin{array}{llll} M_{21} = l_{21}\Delta(\hat{L})^{-1}, & M_{22} = l_{22}\Delta(\hat{L})^{-1}, & M_{23} = l_{23}\Delta(\hat{L})^{-1}, & M_{24} = l_{24}\Delta(\hat{L})^{-1}, \\ M_{31} = l_{31}\Delta(\hat{L})^{-1}, & M_{32} = l_{32}\Delta(\hat{L})^{-1}, & M_{33} = l_{33}\Delta(\hat{L})^{-1}, & M_{34} = l_{34}\Delta(\hat{L})^{-1}, \\ M_{41} = l_{41}\Delta(\hat{L})^{-1}, & M_{42} = l_{42}\Delta(\hat{L})^{-1}, & M_{43} = l_{43}\Delta(\hat{L})^{-1}, & M_{44} = l_{44}\Delta(\hat{L})^{-1}. \end{array} \tag{28}$$

Here, l_{mn} is the cofactor of element L_{mn} of matrix (26).

The coefficients M_{mn} form a matrix $\hat{M}$,

$$\hat{M} = ||M_{mn}|| = \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}, \qquad m, n = 1, 2, 3, 4. \tag{29}$$

Now we are in the position to formulate the next statement.

Statement 3.

If the set of dequantizers $\hat{U}^{(k)}$ is transformed by the linear transform (22), which is defined by the matrix $\hat{L}$ (26), the set of quantizers $\hat{D}^{(k)}$ is transformed by the linear transform (24), which is defined by the matrix $\hat{M}$ (29).

We can consider the values $\hat{V}^{(k)}$ and $\hat{E}^{(l)}$ as elements of the four-dimensional vector space and coefficients L_{ki} and M_{lj} as coordinates of these vectors. We also consider the coefficients L_{ki} and M_{lj} as coordinates of points of two four-dimensional space that we call phase spaces. These spaces are conjugate to each other. Our aim is to construct a symplectic structures in such phase spaces. Formulas (22) and (24) can be interpreted as linear transforms of these phase spaces. One can construct an orbit of this group of linear transform and consider symbols of operators and Wigner functions of quantum states as continuous functions, which are defined on this orbit. Now it is possible to construct a symplectic structure on this orbit using the Kirillov–Kostant bracket [43, 44].

The other possibility of constructing the simplectic structure is related to the existence of duality between orbits of transforms (22) and (24).

We will study in detail such a construction in another paper.

Statement 4.

If dequantizers $\hat{U}^{(k)}$ are orthogonal to each other, i.e., $\mathrm{Tr}\,(\hat{U}^{(k)}\hat{U}^{(k')}) = \delta(k,k')$, the corresponding quantizers $\hat{D}^{(k)}$ are orthogonal to each other too, i.e., $\mathrm{Tr}\,(\hat{D}^{(k)}\hat{D}^{(k')}) = \delta(k,k')$, and coincide with dequantizers $\hat{U}^{(k)}$. So they form a self-dual system.

In fact, according to Statement 3, each set of dequantizers $\hat{U}^{(k)}$ can be transformed with the help of linear transforms to the set of the form

$$\begin{aligned} \hat{U}^{(1)} &= \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \hat{U}^{(2)} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1+i \\ 1-i & 0 \end{pmatrix}, \\ \hat{U}^{(3)} &= \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1-i \\ 1+i & 0 \end{pmatrix}, \qquad \hat{U}^{(4)} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}. \end{aligned} \tag{30}$$

In this case, matrix $\hat{A}$ (17) reads

$$\hat{A} = \frac{1}{2\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1-i & 1+i & 0 \\ 0 & 1+i & 1-i & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{31}$$

One can easily verify that for dequantizers (30) the corresponding quantizers read

$$\hat{D}^{(k)} = \hat{U}^{(k)}, \qquad k = 1,2,3,4. \tag{32}$$

As an example, we consider the tomographic-probability scheme determined by specific dequantizers and quantizers.

For spin $s = 1/2$, the dequantizers denoted as $\hat{Q}(x) \equiv \hat{Q}(m,\vec{n})$ are

$$\hat{Q}(m,\vec{n}) = U^{\dagger}|\,m\,\rangle\langle\,m\,|U, \tag{33}$$

where the spin projection $m = \pm 1/2$, and the unitary matrix U reads

$$U = \begin{pmatrix} \cos\vartheta/2\; e^{i(\varphi+\psi)/2} & \sin\vartheta/2\; e^{i(\varphi-\psi)/2} \\ -\sin\vartheta/2\; e^{i(-\varphi+\psi)/2} & \cos\vartheta/2\; e^{-i(\varphi+\psi)/2} \end{pmatrix}. \tag{34}$$

Here, the angles parameterizing the unitary matrix U are the Euler angles.

The dequantizer operators $\hat{Q}(x) \equiv \hat{Q}(m, \vec{n})$ (33) with a unit vector $\vec{n} = \big(\sin\vartheta\sin\psi,\, \sin\vartheta\cos\psi,\, \cos\vartheta\big)$ and unitary matrix (34) have the explicit matrix form

$$\begin{aligned}
\hat{Q}\,(1/2, \vartheta, \psi) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\psi} \\ \sin\vartheta e^{i\psi} & -\cos\vartheta \end{pmatrix}, \\
\hat{Q}\,(-1/2, \vartheta, \psi) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\psi} \\ \sin\vartheta e^{i\psi} & -\cos\vartheta \end{pmatrix}.
\end{aligned} \tag{35}$$

The dequantizer operator depends only on two Euler angles, which follows from relation (33).

The quantizer operators $\hat{D}$ are given by explicit relations

$$\begin{aligned}
\hat{D}\,(1/2, \vartheta, \psi) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\psi} \\ \sin\vartheta e^{i\psi} & -\cos\vartheta \end{pmatrix}, \\
\hat{D}\,(-1/2, \vartheta, \psi) &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} \cos\vartheta & \sin\vartheta e^{-i\psi} \\ \sin\vartheta e^{i\psi} & -\cos\vartheta \end{pmatrix}.
\end{aligned} \tag{36}$$

The continuous set of dequantizers (35) and quantizers (36) satisfy the orthogonality condition (8) but do not satisfy the condition (9). Thus. we obtain a version of the spin-1/2 tomography determined by a pair of the dequantizer and quantizer, which are not self-dual operators.

5. Conclusions

We developed a general approach to constructing the sets of dequantizers and quantizers introduced in the general scheme for description of associative products in [2] in the finite-dimensional space. We apply an invertible map of the matrices onto vectors elaborated in [41], i.e., instead of matrices, which act in this space, we use the vectors, which correspond to the matrices, and investigate the properties of the space of these vectors. Calculating the scalar products of such vectors, one can find the equations for elements of matrices that represent the dequantizers and quantizers. We realized this program for one-qubit states. We presented the algorithm for constructing all minimal sets of dequantizers and quantizers in the one-qubit case and described some properties of these sets; all these results can be generalized for the N-qubit case.

The coefficients of linear transforms (22) and (24) can be considered as coordinates in a finite-dimensional phase space. The construction under discussion has a structure of an orbit of a group of linear transform. In particular, in the case of tomographic symbols of one-qubit states, the corresponding orbit has the form of two-dimensional sphere [45, 46]. In a future paper, we will construct a symplectic structure on this orbit using the Kirillov–Kostant bracket [43, 44].

In view of the spin-tomography methods, the coordinates of four-dimensional phase spaces can be connected with the probabilities of measurable observables. We plan to investigate the connection between the coordinates of symplectic space and measurable tomographic symbols [47–50].

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