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OPTICAL GAUSSIAN STATES CARRYING ORBITAL ANGULAR MOMENTUM

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Abstract

Within the framework of the tomographic probability representation, we introduce specific optical Gaussian states, which were recently proved to carry the orbital angular momentum. We obtain the symplectic and optical tomograms defining uniquely both quantum and classical states for the rotating Gaussian states of light. This approach needs to be developed and applied to the mentioned states due to the convenience of using in the state reconstructions and measurements. Having in mind this aim, we obtain the mean values and variances of the amplitude quadratures directly measurable in the homodyne optical-tomography experiments. Also we consider the time evolution of the rotating Gaussian states in terms of the tomograms and obtain the corresponding tomographic propagator.

Keywords: orbital angular momentum (OAM) states, Gaussian packets, tomographic probability representation, time evolution, propagator.

1. Introduction

Optical states carrying the orbital angular momentum (OAM) are widely used nowadays in quantum communication and quantum information processing. There is a large bibliography concerning these types of OAM-states implementation. Among the recent works, it is worth mentioning [1–4]. The most frequently used and therefore well-studied OAM states are the Laguerre–Gaussian packets, but there exist also other types of states having nonzero orbital angular momentum, such as Ince–Gauss beams [5–7], Bessel beams [9, 10], etc. Recently, the properties of the rotating Gaussian states were employed in [10], where it was also proved that these states carry the orbital angular momentum.

The applicability of the OAM states to quantum information technologies motivates the task to introduce an appropriate method of OAM-packet measurements and their tomography. In most experimental works, tomography is based on the density-matrix reconstruction with the help of the maximum-likelihood method. In [11] it was proposed to use the OAM-state reconstruction by using the Wigner function defined on a discrete cylinder, that is, the phase space for the pair angle – angular momentum. The proposed approaches have several disadvantages, because the statistical protocols of the density matrix (or wave function) reconstruction have a finite fidelity and are computationally complicated.

An alternative approach called the tomographic probability representation [12] provides a more appropriate method to measure quantum states avoiding statistical reconstruction. In this representation, quantum states are defined by a function called symplectic tomogram; it has all the properties of a probability distribution. For optical packets, a particular case of symplectic tomogram is the optical tomogram which has been proved to be measurable directly in homodyne detection experiments with a

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high accuracy [13]. Obviously, this approach can be also applied to the OAM-state tomography, and the homodyne detection of OAM states is nowadays a well-developed method [14]. In order to introduce the tomographic probability approach to the problem of OAM-state reconstruction, one should consider the properties of these states in the mentioned representation.

One more reason to suggest the tomographic probability description of OAM states is to provide a universal notation for representing both classical and quantum light. The form of OAM states was formulated in terms of classical physics [15] and extended to the quantum case due to the analogy between the paraxial optics and quantum mechanics [16]. We demonstrate this well-known analogy in Sec. 2; the problem of light classicality or quantumness was discussed, for example, in [17].

Providing a universal description of both classical and quantum states in terms of a phase-space-like function, the tomographic probability approach allows one to define whether one deals with a classical or a quantum case directly from the form of the corresponding symplectic tomogram [18, 19]. The employment of quantum language for the description of classical OAM light is of a practical interest; as was emphasized in [20, 21], such quantumlike states can be successfully used for simulating quantum computations.

Due to the reasons mentioned, the tomographic probability representation is the most important way to describe the OAM states. The aim of this research is to provide the tomographic probability description of the rotating Gaussian states following [10].

This paper is organized as follows.

In Sec. 3, we give a short insight into the tomographic probability approach and its physical meaning and properties. In Sec. 4, we obtain the symplectic and optical tomograms of a quantum rotating Gaussian state in the general case and, as a particular case, of a minimum-energy rotating Gaussian state possessing a fixed angular momentum. In Sec. 5, we discuss the quantum evolution of the OAM Gaussian states in terms of the tomographic probabilities.

2. Formulation of OAM States: Paraxial Approximation and Quantum Mechanics

The mentioned connection between classical and quantum optics was emphasized by Fock and Leontovich [16]. For clarity, in this section, we list the main points of this reasoning and introduce an approximation that will hold throughout this article.

The Maxwell equations for the electromagnetic wave in the region without charges and currents yield the following wave equation for the electric component:

$$\nabla^2 E - \frac{\varepsilon}{c^2}\frac{\partial^2 E}{\partial t^2} = 0, \tag{1}$$

where we use the common notation for ε as the permittivity and c is the speed of light. For a monochromatic wave, Eq. (1) reduces to the Helmholtz equation

$$\nabla^2 E + k^2 E = 0, \tag{2}$$

where k is the wave number related to the wave frequency ω and the refractive index n ($k = \omega n/c$).

We consider the electric field (without taking into account the y direction)

$$E\left(x,z\right) = \frac{\psi\left(x,z\right)}{\sqrt{n\left(0,z\right)}}\exp\left(ik\int_0^z n\left(0,\zeta\right)d\zeta\right), \tag{3}$$

where we assume $\psi(x,z)$ is a complex amplitude of the classical electric field.

Substitution of (3) into the Helmholtz equation (2) provides a Schrodinger-like equation with respect to $\psi(x,z)$

$$i\lambda\frac{\partial\psi(x,z)}{\partial z}+\frac{\lambda^2}{4\pi n(0,z)}\frac{\partial^2\psi(x,z)}{\partial x^2}-\frac{\pi\left[n^2(0,z)-n^2(x,z)\right]}{n(0,z)}\psi(x,z)=0, \tag{4}$$

where we neglected the second-order derivatives of $\psi(x,z)$ with respect to z and assumed that

$$\frac{\lambda}{n^2(0,z)}\left|\frac{dn(0,z)}{dz}\right|\ll 1, \tag{5}$$

with $\lambda=2\pi/k$ being the wavelength. The mentioned condition follows from the paraxial approximation.

If one introduces the formal Hamiltonian

$$\hat{H}=\frac{\hat{p_x}^2}{2M}+U(x,t),$$

relation (4) coincides with the Schrödinger equation defining the wave function of a quantum object, where one assumes that z plays the role of time ($z\to t$) and λ, the role of the Planck constant ($\lambda\to\hbar$), the mass M is denoted as $2\pi n(0,z)$, $U(x,z)=\pi\left[n^2(0,z)-n^2(x,z)\right]n^{-1}(0,z)$ is an effective potential, and $\psi(x,z)$ is the wave function. This means that quantum OAM states and classical beams in optical fibers are described by equations of the same form, but we need to take into account the y dependence of the wave function and the potential.

In the following sections, we employ the potential of the two-dimensional harmonic oscillator.

3. Tomographic Probability Representation of Quantum Mechanics

The tomographic probability representation [12] of quantum mechanics was aimed at determining quantum states by probability-distribution functions in analogy with phase-space functions in classical physics. The quantum phase-space functions introduced before, including the widely used Wigner function [22], cannot be considered as probability distributions due to the dependence on noncommuting operators of coordinate and momentum.

In contrast, within the tomographic probability representation, a quantum state is described by a function $w(X,\mu,\nu)$ depending on a coordinate X, which is measured in a rotated and scaled phase space and on the coefficients μ and ν of this rotation and scaling ($X=\mu q+\nu p$). Whereas the mentioned phase spaces determining the variable X are connected to each other by symplectic transforms, the function $w(X,\mu,\nu)$ was named symplectic tomogram.

The main advantage of the symplectic tomogram is that it is positive, real, and has all other properties of the probability distribution. Moreover, a symplectic tomogram can determine pure or mixed states as well, instead of the wave function or the density matrix. The symplectic tomogram and the Wigner function are connected through the invertible Radon transform [12]

$$w(X,\mu,\nu)=\frac{1}{2\pi}\int W(q,p)\,\delta(X-\mu q-\nu p)\;dq\,dp.$$

Due to the homogeneity of the δ-function, the symplectic tomogram possesses this property as well,

$$w(\lambda X,\lambda\mu,\lambda v)=\frac{1}{|\lambda|}w(X,\mu,v). \tag{6}$$

The Wigner function is related to the corresponding wave function (or the density matrix) by the Fourier transform [19, 22]

$$W(q,p) = \frac{1}{\pi}\int \psi^*(q+u)\,\psi(q-u)\,e^{2ipu}\,du. \tag{7}$$

The connection between the symplectic tomogram and the wave function for the one-dimensional case reads (see [23])

$$w(X,\mu,\nu) = \frac{1}{2\pi|\nu|}\left|\int \psi(x)\exp\left(\frac{i\mu}{2\nu}x^2 - \frac{iX}{\nu}x\right)dx\right|^2, \tag{8}$$

and in the two-dimensional case it is

$$w(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2) = \frac{1}{4\pi^2|\nu_1\nu_2|}\left|\int \psi(x,y)\exp\left(\frac{i\mu_1}{2\nu_1}x^2 - \frac{iX_1}{\nu_1}x + \frac{i\mu_2}{2\nu_2}y^2 - \frac{iX_2}{\nu_2}y\right)dx,dy\right|^2. \tag{9}$$

From the definition of the symplectic tomogram it follows that it can be used for obtaining the moments of the operator $\hat{X}$ [24]

$$\left\langle \hat{X}^n \right\rangle = \int X^n w(X,\mu,\nu)\,dX. \tag{10}$$

Originally, the tomographic probability approach had the aim to reconstruct the Wigner function by measurements of coordinates along different directions in the phase space using the inverse Radon transform. Experimentally, these measurements are realized in the scope of homodyne detection, where a particular case of the symplectic tomogram is used.

The so-called optical tomogram is defined on the set of rotated but unscaled phase spaces; it depends on two variables – the quadrature X and the parameter θ, which is connected with μ and ν as follows: $\mu = \cos\theta$ and $\nu = \sin\theta$. Although at the beginning, the technique mentioned was used for the Wigner-function reconstruction, due to further active development of the tomographic probability approach, the other concept based on the substitution of the wave function (or density matrix) by symplectic or optical tomogram in the description of quantum states appeared. The current research was carried out within the scope of this concept and is aimed at presentation of the OAM Gaussian states and their properties within the framework of the tomographic probability approach.

4. Rotating Gaussian Packets in the Tomographic Probability Representation

The Gaussian states with orbital angular momentum were recently introduced for the pure states in [10] and for mixed states in [25]. It was shown that the two-dimensional Gaussian packets can have a rotating structure that consists of the so-called "external" and "internal" rotations. In overview, a rotating Gaussian packet has the form

$$\psi(x,y) = N\exp\left[-\kappa\left(ax^2+bxy+cy^2\right)+Fx+Gy\right], \tag{11}$$

where κ is a constant scale factor, N is the normalizing factor, and a, b, c, F, and G are complex coefficients: $a = \dfrac{\alpha}{2} + i\chi_a$, $b = \beta + i\rho$, $c = \dfrac{\gamma}{2} + i\chi_c$, $F = F_1 + iF_2$, and $G = G_1 + iG_2$.

This relation corresponds to a Gaussian packet that assumes a rotating form under particular values of the coefficients in the exponent for the wave-function expression. The probability density $|\psi(x,y)|^2$ of the rotating Gaussian state under consideration is the probability density in the packet center $|\psi(x_0,y_0)|^2$ multiplied by a time-dependent exponent $|\psi(x,y,t)|^2=e^{-f(\tilde{x},\tilde{y},t)}\,|\psi(x_0(t),y_0(t))|^2$, where $\tilde{x}=x-x_0(t)$ and $\tilde{y}=y-y_0(t)$. This product corresponds to the "external" and "internal" rotations mentioned above, and its structure provides one with the condition for the wave-function coefficients.

Following this approach, we obtain the Gaussian state in the tomographic picture and then apply restrictions obtained for the wave-function coefficients in order to proceed to a particular case of rotating packets. The integral relation (9) and the expression for the wave function of the two-dimensional Gaussian state (11) allow us to derive the symplectic tomogram of this packet

$$w_s\left(\vec{X},\vec{\mu},\vec{\nu}\right)=\frac{C\sqrt{\Delta}}{\pi\,|\nu_1\nu_2|}\left|\frac{\exp\left[-\vec{X}^T\Sigma\vec{X}+\vec{\Theta}^T\vec{X}\right]}{\sqrt{\xi}}\right|^2. \tag{12}$$

Here, the two-vectors $\vec{X}$, $\vec{\mu}$, and $\vec{\nu}$ are expected to consist of the tomographic variables X_1, X_2, μ_1, μ_2, ν_1, and ν_2, the factor Δ has the meaning of $\Delta=\alpha\gamma-\beta^2$, and the normalization factor reads

$$C=\exp\left[F_1^2\gamma\left(\kappa-\Delta^{-1}\right)-F_2^2\kappa\gamma+G_1^2\alpha\left(\kappa-\Delta^{-1}\right)-G_2^2\kappa\alpha\right]$$
$$\times\exp\left[-4F_1F_2\left(\kappa\chi_c-\frac{\mu_2}{2\nu_2}\right)-4G_1G_2\left(\kappa\chi_a-\frac{\mu_1}{2\nu_1}\right)+2\kappa F_1\left(\beta G_1-\rho G_2\right)-2\kappa F_2\left(\rho G_1+\beta G_2\right)\right].$$

For brevity, we denoted by ξ a combination of the tomographic variables

$$\xi=\left|4\left(\kappa a+\frac{i\mu_1}{2\nu_1}\right)\left(\kappa c+\frac{i\mu_2}{2\nu_2}\right)-\kappa^2b^2\right|^2,$$

the notation $\vec{\Theta}$ is used for a two-vector with the components

$$\Theta_1=\frac{i}{\xi}\left(\frac{2F}{\nu_1}\left(\frac{i\mu_2}{2\nu_2}-\kappa c\right)-\frac{\kappa bG}{\nu_1}\right)\qquad\text{and}\qquad\Theta_2=\frac{i}{\xi}\left(\frac{2G}{\nu_2}\left(\frac{i\mu_1}{2\nu_1}-\kappa a\right)-\frac{\kappa bF}{\nu_2}\right),$$

and Σ is the matrix of the tomographic variables

$$\Sigma=\frac{1}{\xi}\begin{pmatrix}\dfrac{\kappa c}{\nu_1^2}-\dfrac{i\mu_2}{2\nu_1^2\nu_2} & \dfrac{\kappa b}{2\nu_1\nu_2}\\ \dfrac{\kappa b}{2\nu_1\nu_2} & \dfrac{\kappa a}{\nu_2^2}-\dfrac{i\mu_1}{2\nu_1\nu_2^2}\end{pmatrix}.$$

Obviously, the symplectic tomogram (12) also has a Gaussian form. In terms of the optical tomogram, which is applicable for homodyne detection, the general form of a Gaussian state is

$$w_{\rm op}\left(\vec{X},\vec{\theta}\right)=\frac{\tilde{C}\sqrt{\Delta}}{\pi\,|\cos\theta_1\cos\theta_2|}\left|\frac{\exp\left[-\vec{X}^T\tilde{\Sigma}\vec{X}+\vec{\tilde{\Theta}}^T\vec{X}\right]}{\sqrt{\tilde{\xi}}}\right|^2, \tag{13}$$

where the parameters of the normal probability distribution are expressed in terms of angles θ_1 and θ_2. These parameters are the components of the two-vector $\vec{\theta}$, and the new normalization factor reads

$$\tilde{C}=\exp\left[F_1^2\gamma\left(\kappa-\Delta^{-1}\right)-F_2^2\kappa\gamma+G_1^2\alpha\left(\kappa-\Delta^{-1}\right)-G_2^2\kappa\alpha\right]$$
$$\times\exp\left[-4F_1F_2\left(\kappa\chi_c-\frac{tg\theta_2}{2}\right)-4G_1G_2\left(\kappa\chi_a-\frac{tg\theta_1}{2}\right)+2\kappa F_1\left(\beta G_1-\rho G_2\right)-2\kappa F_2\left(\rho G_1+\beta G_2\right)\right].$$

The coefficient $\tilde{\xi}$, the vector $\tilde{\vec{\Theta}}$, and the matrix $\tilde{\Sigma}$ have the same form as the initial ones without "tilde" by the substitutions $\mu_1 \to \cos\theta_1$, $\nu_1 \to \sin\theta_1$, $\mu_2 \to \cos\theta_2$, and $\nu_2 \to \sin\theta_2$.

As was demonstrated in [10], the wave function of a minimum-energy rotating Gaussian state has the form (11) under the following conditions:

$$a = \frac{1}{2}\left[1 + \eta \exp\left(-i\lambda_i u\right)\right], \qquad b = i\lambda_i \eta \exp\left(-i\lambda_i u\right), \qquad c = \frac{1}{2}\left[1 - \eta \exp\left(-i\lambda_i u\right)\right],$$
$$F = \kappa R\left[\exp\left(-i\lambda_c v\right) + \eta \exp\left(i\lambda_i \left(v - u\right)\right)\right], \qquad G = i\kappa R\left[\lambda_c \exp\left(-i\lambda_c v\right) + \lambda_i \eta \exp\left(i\lambda_i\left(v-u\right)\right)\right].$$

Here, $\eta = \sqrt{\left(\frac{\alpha-\gamma}{2}\right)^2 + \beta^2}$, $R = \sqrt{\left(\frac{\gamma F_1 - \beta G_1}{\kappa\Delta}\right)^2 + \left(\frac{\alpha G_1 - \beta F_1}{\kappa\Delta}\right)^2}$, $v = \arccos\left(\frac{\gamma F_1 - \beta G_1}{R\kappa\Delta}\right)$, λ_c and λ_i equal ± 1 relating to the sign of mean "classical" (responsible for the motion of the packet center) and intrinsic angular momentum, respectively, and u is an arbitrary phase. Obviously, the symplectic and optical tomograms of rotating Gaussian states have the form (12) and (13) under the mentioned coefficients. Further on, we use this notation.

In order to provide the compatibility of the tomographic picture with experiments where the rotating Gaussian packets were measured, say, in the homodyne-detection experiments, we calculate the mean values and variances of the tomographic variables X_1 and X_2,

$$\begin{aligned} \langle X_1 \rangle &= \frac{\mu_1\left(\gamma F_1 - \beta G_1\right) + \nu_1\left[F_2\kappa\Delta - 2\kappa\chi_a\left(\gamma F_1 - \beta G_1\right) - \kappa\rho\left(\alpha G_1 - \beta F_1\right)\right]}{\kappa\Delta}, \\ \langle X_2 \rangle &= \frac{\mu_2\left(\alpha G_1 - \beta F_1\right) + \nu_2\left[G_2\kappa\Delta - 2\kappa\chi_c\left(\alpha G_1 - \beta F_1\right) - \kappa\rho\left(\gamma F_1 - \beta G_1\right)\right]}{\kappa\Delta}. \end{aligned} \tag{14}$$

Here one can use the property (10) of the symplectic tomogram assuming $n = 1$. From the same relation one can use the second moments of the quadratures X_i:

$$\begin{aligned} \left\langle X_1^2 \right\rangle &= \mu_1^2 \frac{\gamma}{2\kappa\Delta} + \nu_1^2 \frac{\kappa\gamma\left(\alpha^2 + 4\chi_a^2\right) + \kappa\alpha\left(\rho^2 - \beta^2\right) - 4\kappa\beta\rho\chi_a}{2\Delta} + \mu_1\nu_1\frac{\beta\rho - 2\gamma\chi_a}{\Delta}, \\ \left\langle X_2^2 \right\rangle &= \mu_2^2 \frac{\alpha}{2\kappa\Delta} + \nu_2^2 \frac{\kappa\alpha\left(\gamma^2 + 4\chi_c^2\right) + \kappa\gamma\left(\rho^2 - \beta^2\right) - 4\kappa\beta\rho\chi_c}{2\Delta} + \mu_2\nu_2\frac{\beta\rho - 2\alpha\chi_c}{\Delta}, \\ \left\langle X_1 X_2 \right\rangle &= -\mu_1\mu_2\frac{\beta}{2\kappa\Delta} + \mu_1\nu_2\frac{2\beta\chi_c - \rho\gamma}{2\Delta} + \mu_2\nu_1\frac{2\beta\chi_a - \rho\alpha}{2\Delta} \\ &\quad + \nu_1\nu_2\frac{\kappa\beta\left(\Delta - \rho^2 - 4\chi_a\chi_c\right) + 2\kappa\rho\left(\alpha\chi_c + \gamma\chi_a\right)}{2\Delta}. \end{aligned} \tag{15}$$

From expressions (14) and (15), we can obtain dispersions of the tomographic variables

$$\sigma_{X_i X_j} = \langle X_i X_j \rangle - \langle X_i \rangle \langle X_j \rangle.$$

5. Tomographic Quantum Propagator for Rotating Gaussian Packets

The evolution of quantum states in the tomographic probability approach is described by a Moyal-type equation in full analogy with the Schrödinger equation [12]. It corresponds to a quantum propagator representing the time evolution of the symplectic tomogram:

$$w\left(X, \mu, \nu, t\right) = \int \Pi\left(X, \mu, \nu, t, X', \mu', \nu', t_0\right) w\left(X', \mu', \nu', t_0\right)\, dX'\, d\mu'\, d\nu'. \tag{16}$$

Since the tomogram is related to the wave function, the tomographic propagator should be connected with the corresponding Green function as well. After deriving this general relation and using the Green function of the rotating Gaussian state, one obtains the tomographic propagator for the system under consideration. For the one-dimensional case, the transform connecting the tomographic propagator and the Green function $G(x,y,t)$ was found in an explicit form in [26]; it reads

$$\Pi\left(X,\mu,\nu,X',\mu',\nu',t\right)=\frac{1}{4\pi^2}\int k^2 G\left(r+\frac{k\nu}{2},y,t\right)G^*\left(r-\frac{k\nu}{2},z,t\right)\delta\left(y-z-k\nu'\right)$$
$$\times\exp\left[ik\left(X'-X+\mu r-\mu'\frac{y+z}{2}\right)\right]dk\,dy\,dz\,dr. \tag{17}$$

Here, we generalize this relation for the two-dimensional case to obtain the tomographic propagator for rotating Gaussian states. Due to the homogeneity of the tomogram (6), the tomographic propagator possesses this property as well, i.e.,

$$\Pi\left(sX,s\mu,s\nu,sX',s\mu',s\nu',t\right)=|s|^{-3}\Pi\left(X,\mu,\nu,X',\mu',\nu',t\right). \tag{18}$$

This fact leads to the important property of its Fourier components that was derived in [26] for the one-dimensional case,

$$\Pi_F\left(k,\mu,\nu,X',\mu',\nu',t\right)=k^2\Pi_F\left(1,k\mu,k\nu,kX',k\mu',k\nu',t\right), \tag{19}$$

where

$$\Pi_F\left(k,\mu,\nu,X',\mu',\nu',t\right)=\int\Pi\left(X,\mu,\nu,X',\mu',\nu',t\right)e^{ikX}dX,$$

$$\Pi_F\left(1,k\mu,k\nu,kX',k\mu',k\nu',t\right)=\int\Pi\left(kX,k\mu,k\nu,kX',k\mu',k\nu',t\right)e^{iX}dX.$$

Relation (19) can be easily generalized for the two-dimensional case as follows:

$$\Pi_F\left(k_1,k_2,\mu_1,\mu_2,\nu_1,\nu_2,X_1',X_2',\mu_1',\mu_2,'\nu_1',\nu_2',t\right)$$
$$=k_1^2k_2^2\Pi_F\left(1,1,k_1\mu_1,k_2\mu_2,k_1\nu_1,k_2\nu_2,k_1X_1',k_2X_2',k_1\mu_1',k_2\mu_2,'k_1\nu_1',k_2\nu_2',t\right). \tag{20}$$

In view of relation (20) and definitions of the tomographic propagator

$$w\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,t\right)=\int\Pi\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X_1',\mu_1',\nu_1',X_2',\mu_2',\nu_2',t,t_0\right)$$
$$\times w\left(X_1',\mu_1',\nu_1',X_2',\mu_2',\nu_2',t_0\right)dX_1'\,d\mu_1'\,d\nu_1'\,dX_2'\,d\mu_2'\,d\nu_2' \tag{21}$$

and the Green function in the two-dimensional case

$$\psi(x,y,t)=\int G\left(x,y,x',y',t,t_0\right)\psi\left(x',y',t_0\right)dx'\,dy', \tag{22}$$

we derive the following integral relation:

$$\Pi\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X_1',\mu_1',\nu_1',X_2',\mu_2',\nu_2',t\right)$$
$$=\frac{1}{(2\pi)^4}\int k_1^2k_2^2G\left(a_1+\frac{k_1\nu_1}{2},a_2+\frac{k_2\nu_2}{2},y_1,y_2,t\right)G^*\left(a_1-\frac{k_1\nu_1}{2},a_2-\frac{k_2\nu_2}{2},z_1,z_2,t\right)$$
$$\times\exp\left[ik_1\left(X_1'-X_1+\mu_1a_1-\mu_1'\frac{y_1+z_1}{2}\right)+ik_2\left(X_2'-X_2+\mu_2a_2-\mu_2'\frac{y_2+z_2}{2}\right)\right]$$
$$\times\delta\left(y_1-z_1-k_1\nu_1'\right)\delta\left(y_2-z_2-k_2\nu_2'\right)dk_1\,dk_2\,dy_1\,dy_2\,dz_1\,dz_2\,da_1\,da_2. \tag{23}$$

We can use this relation for considering the time evolution of the optical rotating Gaussian states. Since the Hamiltonian $\hat{H} = \frac{1}{2M}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}M\omega^2\left(x^2+y^2\right)$ of the corresponding system coincides with the Hamiltonian of a particle with mass M moving in the isotropic harmonic potential, the Green functions of these systems also coincide.

The time evolution of the OAM Gaussian packet is determined by the Green function of an isotropic harmonic oscillator [10], namely,

$$G\left(x,y,x',y',t\right) = \frac{\kappa}{2\pi i \sin(\omega t)} \exp\left\{\frac{i\kappa}{2\sin(\omega t)}\left[\cos(\omega t)\left(x^2+y^2+x'^2+y'^2\right) - 2\left(xx'+yy'\right)\right]\right\}, \tag{24}$$

where κ is the factor characterizing the wave function in (11); for the case of an isotropic harmonic oscillator, $\kappa = M\omega/\hbar$. In view of relation (24), we obtain the expression for the tomographic propagator corresponding to rotating Gaussian states; it is

$$\begin{aligned}\Pi\left(X_1,\mu_1,\nu_1,X_2,\mu_2,\nu_2,X_1',\mu_1',\nu_1',X_2',\mu_2',\nu_2',t\right) &= 4\kappa^2\sin^2(\omega t)\,\delta(X_1'-X_1)\delta(X_2'-X_2)\\ &\times\delta\left(\kappa\nu_1\cos(\omega t) - \kappa\nu_1' + \mu_1\sin(\omega t)\right)\delta\left(\kappa\nu_2\cos(\omega t) - \kappa\nu_2' + \mu_2\sin(\omega t)\right)\\ &\times\delta\left(\kappa\nu_1'\cos(\omega t) - \kappa\nu_1 - \mu_1'\sin(\omega t)\right)\delta\left(\kappa\nu_2'\cos(\omega t) - \kappa\nu_2 + \mu_2'\sin(\omega t)\right).\end{aligned} \tag{25}$$

6. Summary

To conclude, we list our main results.

We described the OAM states in terms of the tomographic probabilities using the example of the OAM Gaussian states, whose properties were recently explored in [10]. We obtained the symplectic and optical tomograms of a rotating Gaussian state and showed that the symplectic tomogram is responsible for providing a universal language to describe both classical and quantum states, which is relevant for considering the OAM light.

Optical tomograms are important for experimental implementation since they provide the possibility of determining quantum states by functions, which do no need to be statistically reconstructed. The optical tomogram can be measured directly in the homodyne detection experiments due to the fact that it depends only on two variables. In the one-dimensional case, we have the quadrature X having the meaning of the difference between the number of the recorded photoelectrons in the subtraction scheme in a homodyne setup normalized by the local oscillator (LO) amplitude and the parameter θ, the LO frequency. This scheme can be generalized (see, for example, [14]) to a two-dimensional case in terms of the quadratures X_1 and X_2 considered in this work. In order to provide a consistent representation needed for the experiments, we obtained the mean values and variances of the quadratures X_i.

We performed the evolution of the OAM Gaussian states in the tomographic probability approach. We obtained the relation between the tomographic propagator and the Green function for the two-dimensional case and derived the tomographic propagator in an explicit form for the problem under consideration.

References

1. R. Fickler, M. Krenn, R. Lapkiewicz, et al., *Sci. Rep.*, **3**, 1914 (2013).

2. R. Fickler, R. Lapkiewicz, M. Huber, et al., *Nature Commun.*, **5**, 28 (2014).
3. R. Fickler, R. Lapkiewicz, W. N. Plick, et al., *Science*, **338**, 640 (2012).
4. A. Nicolas, L. Veissier, L. Giner, et al., *Nature Photon.*, **8**, 234 (2014).
5. M. A. Bandres and J. C. Gutierrez-Vega, *Opt. Lett.*, **29**, 144 (2004).
6. M. A. Bandres and J. C. Gutierrez-Vega, *J. Opt. Soc. Am. A*, **21**, 873 (2004).
7. M. Krenn, R. Fickler, M. Huber, et al., *Phys. Rev. A*, **87**, 012326 (2013).
8. K. Volke-Sepulveda, V. Garces-Chavez, S. Chavez-Cerda, et al., *J. Opt. B: Quantum Semiclass. Opt.*, **4**, S82 (2002).
9. D. McGloin and K. Dholakia, *Contemp. Phys.*, **46**, 15 (2005).
10. V. V. Dodonov, *J. Phys. A: Math. Theor.*, **48**, 435303 (2015).
11. I. Rigas, L. L. Sanchez-Soto, A. B. Klimov, et al., *Phys. Rev. A*, **78**, 060101 (2008).
12. S. Mancini, V. I. Man'ko, and P. Tombesi, *Phys. Lett. A*, **213**, 1 (1996).
13. M. Bellini, A. S. Coelho, S. N. Filippov, et al., *Phys. Rev. A*, **85**, 052129 (2012).
14. M. Lassen, G. Leuchs, and U. L. Andersen, *Phys. Rev. Lett.*, **102**, 163602 (2009).
15. L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, *Phys. Rev. A*, **45**, 8185 (1992).
16. M. A. Leontovich and V. A. Fock, *Zh. Éksp. Teor. Fiz.*, **16**, 557 (1946).
17. D. L. Andrews, *Structured Light and Its Applications: An Introduction to Phase-Structured Beams and Nanoscale Optical Forces*, Academic Press, New York (2011).
18. A. Ibort, V. I. Man'ko, G. Marmo, et al., *Phys. Scr.*, **79**, 065013 (2009).
19. V. I. Man'ko, M. Moshinsky, and A. Sharma, *Phys. Rev. A*, **59**, 1809 (1999).
20. M. A. Man'ko, V. I. Man'ko, and R. V. Mendes, *Phys. Lett. A*, **288**, 132 (2001).
21. S. Chavez-Cerda and J. R. Moya-Cesa, and H. M. Moya-Cesa, *J. Opt. Soc. Am. B*, **24**, 404 (2007).
22. E. Wigner, *Phys. Rev.*, **40**, 749 (1932).
23. V. I. Man'ko and R. V. Mendes, *Phys. Lett. A*, **263**, 53 (1999).
24. S. Mancini, V. I. Man'ko, and P. Tombest, *Found. Phys.*, **27**, 801 (1997).
25. V. V. Dodonov, *Phys. Rev. A*, **93**, 022106 (2016).
26. O. V. Man'ko and V. I. Man'ko, *J. Russ. Laser Res.*, **20**, 67 (1999).