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HIDDEN QUANTUM CORRELATIONS IN SINGLE QUDIT SYSTEMS†

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Abstract

We introduce the notion of hidden quantum correlations. We present the mean values of observables depending on one classical random variable described by the probability distribution in the form of correlation functions of two (three, etc.) random variables described by the corresponding joint probability distributions. We develop analogous constructions for the density matrices of quantum states and quantum observables. We consider examples of four-dimensional Hilbert space corresponding to the "quantum roulette" and "quantum compass."

Keywords: entanglement, hidden quantum correlations, information and entropic inequalities, qudits, noncomposite systems.

1. Introduction

Quantum correlation phenomena, like the entanglement [1] present in composite systems, for example, in the system of several qubits, are known to play an important role in developing new quantum technologies, including quantum computing [2]. Strong quantum correlations in two-qubit systems responsible for the violation of Bell inequalities [3, 4] were checked experimentally [5, 6]. In [7], it was suggested to extend the notion of entanglement in order to relate this phenomenon to correlation properties of single qudits.

The new entropic inequalities reflecting the presence of correlations and analogous to the subadditivity and strong subadditivity conditions known for bipartite and tripartite systems [8–10] were found for noncomposite systems like single qudits or multilevel atoms [11–17]. Examples of qudits, including $j = 3/2$, were considered in this context in [18–22], and the results obtained show that the correlations in composite systems and the correlations in noncomposite systems can formally be considered as identical, using a common mathematical framework.

The aim of this work is to develop the approach for describing both classical and quantum correlations in composite and noncomposite systems, using the same scheme based on the application of invertible maps of integer numbers s onto pairs (triples, etc.) of the integers (j, k) employed in [16]. In view of these maps, we demonstrate that a single variable and its statistical properties, such as mean values, can be considered as the properties of several random variables described by the corresponding joint probability

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distributions and given in terms of the correlation functions calculated for these several random variables. We show this property for both classical and quantum systems.

This paper is organized as follows.

In Sec. 2, we discuss the means and correlations in classical systems. In Sec. 3, we present examples of four- and eight-dimensional probability distributions. In Sec. 4, we study quantum states, and in Sec. 5 we consider in detail the case of $N = 4$ along with entropic inequalities. We give our conclusions in Sec. 6.

2. Means as Correlation Functions

Our aim now is to consider correlations in a single qudit as correlations in artificial multiqudit systems. We start from classical states.

Following [11, 16] we consider a set of nonnegative numbers $A_1, A_2, \ldots, A_N$ which, in turn, provides a set of other nonnegative numbers $0 \leq p_s = \dfrac{A_s}{\sum_{j=1}^{N} A_j} \leq 1$ satisfying the normalization condition $\sum_{s=1}^{N} p_s = 1$. The numbers p_s can be interpreted as the probability distributions of one random variable.

Let us measure the observable $F(s)$. For each value of the integer $s = 1, 2, \ldots, N$, one obtains the result of the measurement $F(s)$. Repeating the measurement L times, where L is a large enough integer, one obtains such statistical characteristic as the mean value of the measured observable

$$\langle F\rangle = \sum_{s=1}^{N} p_s F(s). \tag{1}$$

We consider s as *a random variable*, $F(s)$ as *an observable*, and p_s as the probability distribution of one random variable, which we call *the state*. The other statistical characteristics described by the highest moments like, for example, variances

$$\langle F^2\rangle - \langle F\rangle^2 = \sum_{s=1}^{N} p_s F^2(s) - \Big(\sum_{s=1}^{N} p_s F(s)\Big)^2, \tag{2}$$

can also be obtained.

Formally, one has two functions $F(s)$ and p_s defined on the set of integers $s = 1, 2, \ldots, N$.

We call the function p_s *the state* and the function $F(s)$ *the observable* due to the following reason.

If one has the continuous variable x $\big($a position of the particle with the Gaussian distribution $P(x)\big)$, we call x the random variable and the distribution $P(x)$ the state of the system. We extend this terminology to a discrete variable s and the distribution p_s.

The mean value $\langle F\rangle$ and variance $\langle F^2\rangle - \langle F\rangle^2$ are the functionals given by Eqs. (1) and (2). In principle, one can formally define the functions $F(s)$ and p_s, as well as the functionals (1) and (2), without the probabilistic interpretation of these objects.

On the other hand, there exist functionals determined not by both functions $F(s)$ and p_s but by only one function p_s. For example, Shannon entropy [23] associated with the probability distribution p_s is given by the expression

$$H = -\sum_{s} p_s \ln p_s; \tag{3}$$

the entropy being the functional of the state.

Meanwhile, the entropy H is the functional which can also be considered formally without a probabilistic interpretation of the numbers p_s. We point out the possibility to treat the functionals $\langle F\rangle$, $\langle F^2\rangle - \langle F\rangle^2$, and H as objects that can be considered without their probabilistic interpretation because the numerical properties of these and other analogous functionals, e.g., all highest moments

$$\langle F^k\rangle = \sum_{s=1}^{N} p_s F^k(s), \tag{4}$$

like equalities and inequalities for these objects, exist independently of their relation to probabilities.

One can repeat the above consideration for nonnegative numbers p_{jk}. This simple observation can be used for obtaining some new equalities and inequalities for functionals (entropies, correlations, means, variances, and covariances) associated with tables of nonnegative numbers $0 \leq p_{jk} \leq 1$, $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, m$, and $N = nm$, since the table can be considered as a joint probability distribution for two random variables.

Within the framework of the interpretation of the table p_{jk} as a joint probability distribution, the characteristics like entropy, mutual information, etc. naturally appear. These characteristics are known to satisfy the entropic inequalities for bipartite classical systems; see [24].

On the other hand, the numerical expressions of these inequalities are valid independently of the probabilistic interpretation of the numbers in the table p_{jk}. We employ this fact for obtaining new inequalities for *the state* (the probability distribution p_s) associated with one random variable and the function $F(s)$ $\big($*observable* $F(s)\big)$. The key tool to achieve this result is introducing the map of integers, namely, for $s = 1, 2, \ldots, N$ we construct the invertible map

$$1 \leftrightarrow 1,1;\quad 2 \leftrightarrow 2,1;\quad \ldots;\quad n \leftrightarrow n,1;\quad n+1 \leftrightarrow 1,2;\quad n+2 \leftrightarrow 2,2;\quad \ldots; N-1 \leftrightarrow n-1,m;\quad N \leftrightarrow n,m.$$

This map could be described as a procedure for introducing the function $s(j,k)$. Such a function defined in the domain of integers $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$ provides for each pair of the integers j,k the value of the function equal to the integer s. The function is constructed using the invertibility condition; this means that for each value of the integer s one has only one pair of integers j,k corresponding to this value. Such construction was used in [16] to derive new entropic inequalities for qudit states. Here, we extend this construction to study the properties of observables associated with functions $F(s)$.

In fact, these observables associated with one random variable can be treated as observables connected with two random variables. To demonstrate this fact, we define the function $\Phi(j,k) \equiv F\big(s(j,k)\big)$. One can choose this function in the product form

$$\Phi(j,k) = \phi(j)\chi(k). \tag{5}$$

The form of *observable* $F\big(s(j,k)\big)$ provides the possibility to interpret the observable as the existence of two *observables* $\phi(j)$ and $\chi(k)$ associated with two random variables j and k. Also the probability distribution p_s can be chosen in the product form $p_{jk} = \Pi_j \mathcal{P}_k$; this representation can be chosen with high ambiguity. In view of this representation, one can rewrite formula (1) for the mean value $\langle F\rangle$ as follows:

$$\langle F\rangle = \sum_{s=1}^{N} p_s F(s) = \sum_{j=1}^{n}\sum_{k=1}^{m} \Phi(j,k) p_{jk}. \tag{6}$$

Also one can introduce marginal probability distributions:

$$\Pi_j = \sum_{k=1}^{m} p_{jk} \equiv \sum_{k=1}^{m} p_{s(j,k)}, \qquad j = 1, 2, \ldots, n, \tag{7}$$

$$\mathcal{P}_k = \sum_{j=1}^{n} p_{jk} \equiv \sum_{j=1}^{n} p_{s(j,k)}, \qquad k = 1, 2, \ldots, m. \tag{8}$$

If the numbers p_s determining the joint probability distribution are such that $p_{jk} = \Pi_j \mathcal{P}_k$, where $\sum_{j=1}^{n} \Pi_j = \sum_{k=1}^{m} \mathcal{P}_k = 1$, one has for the marginal distributions (7) and (8) the case of the absence of correlations between the observables associated with the function $\Phi(j,k)$ given by (5).

If the function $F\big(s(j,k)\big)$ has the product form analogous to (5), the mean value of this function $\langle F \rangle$ given by (6) and written as

$$\langle F \rangle = \sum_{j=1}^{n} \sum_{k=1}^{m} \phi(j)\chi(k) p_{jk} \tag{9}$$

can be interpreted as the correlation function, i.e.,

$$\langle F \rangle = \langle \phi(j)\chi(k) \rangle. \tag{10}$$

Thus, for one random variable $F(s)$ we obtain the formula for its mean value in the form of correlation function associated with two observables depending on random variables $\phi(j)$ and $\chi(k)$, using the averaging procedure determined by the joint probability distribution p_{jk}.

In the case of integer $N = n_1 n_2 n_3$, where the factors in the product are integers, one can use the invertible map of the integers s onto the triples of integers j, k, ℓ, where $j = 1, 2, \ldots, n_1$, $k = 1, 2, \ldots, n_2$, and $\ell = 1, 2, \ldots, n_3$. This means that we construct the function of three variables $s(j, k, \ell)$ such that for each three integers we have only one integer s, and for each integer s we have only one triple of integers j, k, ℓ.

In view of this invertible map, the probability distribution p_s used to describe statistical properties of observable $F(s)$ depending on one random variable $F(s)$ may be interpreted as the joint probability distribution $p_{s(j,k,\ell)} \equiv p_{jk\ell}$ of three random variables. For this, we define the function $T(j, k, \ell) \equiv F\big(s(j, k, \ell)\big)$, which can be chosen in the product form

$$T(j, k, \ell) = a(j)b(k)c(\ell). \tag{11}$$

Then one can write the equality

$$\langle F \rangle = \sum_{s=1}^{N} p_s F(s) = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_3} T(j, k, \ell) p_{s(j,k,\ell)} \tag{12}$$

or

$$\langle F \rangle = \sum_{j=1}^{n_1} \sum_{k=1}^{n_2} \sum_{\ell=1}^{n_3} a(j)b(k)c(\ell) p_{s(j,k,\ell)}, \tag{13}$$

which means

$$\langle F \rangle = \langle a(j)b(k)c(\ell) \rangle. \tag{14}$$

Thus, we presented the mean value of the observable depending on one random variable in the form of a correlation function of observables depending on three random variables.

Analogous representations can be developed for highest moments of the observable of one random variable.

3. Examples of $N = 4$ and $N = 8$

We recall that in our approach the integer s is *the random variable*, the numbers p_s (the probability distributions) are the *states*, and the function $F(s)$ is *the observable*, which has a value equal to the number $F(s)$. For $s = 1, 2, \ldots, N$, we have N values of random variable. One can use any other notation for the states and random variables, using an invertible map of the integers $1, 2, \ldots, N$ onto another set of numbers $m_1, m_2, \ldots, m_N$.

3.1. Case of $N = 4$

We study the suggested construction on the example of $N = 4$. As an example, we consider these numbers as numbers associated with a casino roulette (or geographic compass).

This means that we have four different positions of the casino roulette $s = 1, 2, 3, 4$ (or four directions of the compass arrow).

We use the map $1 \leftrightarrow 1,\ 1;\ \ 2 \leftrightarrow 1,\ 2;\ \ 3 \leftrightarrow 2,\ 1;\ \ 4 \leftrightarrow 2,\ 2$ to label the four roulette positions by four pairs of numbers p_{jk} $(j, k = 1, 2)$, i.e., $p_1 \equiv p_{11},\ p_2 \equiv p_{12},\ p_3 \equiv p_{21}$, and $p_4 \equiv p_{22}$.

Now we introduce the observable $F(s)$, which is a function of a random variable s equal to the number $F(s)$ at each value of the variable. In this way, we have four numbers $F(s = 1) = F(1)$, $F(s = 2) = F(2)$, $F(s = 3) = F(3)$, and $F(s = 4) = F(4)$.

The mean value of the observable reads

$$\langle F \rangle = p_1 F(1) + p_2 F(2) + p_3 F(3) + p_4 F(4). \tag{15}$$

The mean value $\langle F \rangle$ is a functional that depends on two functions p_s and $F(s)$, i.e., the state and observable.

Using the mapping procedure developed, we can rewrite Eq. (15) as follows:

$$\langle F \rangle = p_{11} F(1,1) + p_{12} F(1,2) + p_{21} F(2,1) + p_{22} F(2,2) \tag{16}$$

or

$$\langle F \rangle = \sum_{j=1}^{2} \sum_{k=1}^{2} p_{jk} F(j,k). \tag{17}$$

Now we choose the function $F(j,k)$ in the form

$$F(1,1) = \varphi(1)\chi(1), \quad F(1,2) = \varphi(1)\chi(2), \quad F(2,1) = \varphi(2)\chi(1), \quad F(2,2) = \varphi(2)\chi(2). \tag{18}$$

One can introduce two other functions $\widetilde{\varphi}(j,k)$ and $\widetilde{\chi}(j,k)$, which provide the same result of multiplication

$$F(1,1) = \widetilde{\varphi}(1,1)\widetilde{\chi}(1,1), \quad F(1,2) = \widetilde{\varphi}(1,2)\widetilde{\chi}(1,2), \quad F(2,1) = \widetilde{\varphi}(2,1)\widetilde{\chi}(2,1), \quad F(2,2) = \widetilde{\varphi}(2,2)\widetilde{\chi}(2,2). \tag{19}$$

In fact, one should obtain these equalities if

$$\begin{aligned} &\widetilde{\varphi}(1,1)=\varphi(1), \quad \widetilde{\varphi}(1,2)=\varphi(1), \quad \widetilde{\varphi}(2,1)=\varphi(2), \quad \widetilde{\varphi}(2,2)=\varphi(2),\\ &\widetilde{\chi}(1,1)=\chi(1), \quad \widetilde{\chi}(1,2)=\chi(1), \quad \widetilde{\chi}(2,1)=\chi(2), \quad \widetilde{\chi}(2,2)=\chi(2). \end{aligned} \tag{20}$$

We can interpret the functions $\widetilde{\varphi}(j,k)\equiv\widetilde{\varphi}\big(s(j,k)\big)$ and $\widetilde{\chi}(j,k)\equiv\widetilde{\chi}\big(s(j,k)\big)$ as two specific observables or two different kinds of a function of one random variable. The results obtained can be summarized as the equality

$$\langle F\rangle=\langle\widetilde{\varphi}\widetilde{\chi}\rangle; \tag{21}$$

this means that $\langle F\rangle$, being the classical observable mean, can be interpreted as the correlation function of two classical observables $\widetilde{\varphi}$ and $\widetilde{\chi}$. We call the correlations of these two observables $\widetilde{\varphi}$ and $\widetilde{\chi}$ depending on one random variable s *the hidden correlations*.

Analogously, for $N=n_1n_2\cdots n_\ell$ one can obtain the equality

$$\langle F\rangle=\langle\widetilde{\varphi}_1\widetilde{\varphi}_2\cdots\widetilde{\varphi}_\ell\rangle, \tag{22}$$

where the same $\langle F\rangle$ can be considered as the correlation function of ℓ observables $\widetilde{\varphi}_1,\widetilde{\varphi}_2,\ldots,\widetilde{\varphi}_\ell$ (hidden correlations).

3.2. Case of $N=8$

Now we consider the case of $N=8$, where we also have nonnegative numbers $p_1,p_2,\ldots p_8$, with $\sum_{s=1}^{8}p_s=1$. We use the map $s\leftrightarrow s(j,k,\ell)$, i.e.,

$$p_1=\mathcal{P}_{111},\quad p_2=\mathcal{P}_{112},\quad p_3=\mathcal{P}_{121},\quad p_4=\mathcal{P}_{122},\quad p_5=\mathcal{P}_{211},\quad p_6=\mathcal{P}_{212},\quad p_7=\mathcal{P}_{221},\quad p_8=\mathcal{P}_{222}.$$

The nonnegative numbers $\mathcal{P}_{jk\ell}$ satisfy the condition $\sum_{j,k,\ell=1}^{2}\mathcal{P}_{jk\ell}=1$. They can be interpreted as a joint probability distribution of three random variables j, k, and ℓ.

We turn to the observable $F(s)$, $s=1,2,\ldots,8$. In terms of the probability distribution p_s, the mean value $\langle F\rangle$ reads $\langle F\rangle=\sum_{s=1}^{8}F(s)p_s$. In view of the notation $F\big(s(j,k,\ell)\big)\equiv F(j,k,\ell)$, we arrive at

$$\langle F\rangle=\sum_{j,k,\ell}\mathcal{P}_{jk\ell}F(j,k,\ell).$$

If the observable $F(j,k,\ell)$ is taken in the form

$$F(j,k,\ell)=\varphi(j)\chi(k)u(\ell),$$

we obtain

$$\langle F\rangle=\sum_{j,k,\ell}\varphi(j)\chi(k)u(\ell)\mathcal{P}_{jk\ell}=\langle\varphi(j)\chi(k)u(\ell)\rangle.$$

Thus, we obtain the result that the mean of a specific observable $F(s)$ appears in the form of the correlation function of three observables

$$\widetilde{\varphi}(j,k,\ell)=\varphi(j),\quad \widetilde{\chi}(j,k,\ell)=\chi(k),\quad \widetilde{u}(j,k,\ell)=u(\ell),\qquad \text{i.e.,}\qquad \langle F\rangle=\langle\widetilde{\varphi}\widetilde{\chi}\widetilde{u}\rangle.$$

The functions $\widetilde{\varphi}$, $\widetilde{\chi}$, and $\widetilde{u}$ can be interpreted as observables depending on one random variable s. Thus, the mean value of the observable $F(s)$ can be written as the correlation function of three observables $\widetilde{\varphi}(s)$, $\widetilde{\chi}(s)$, and $\widetilde{u}(s)$, i.e.,

$$\sum_{s=1}^{N} p_s F(s) = \sum_{s=1}^{N} p_s \widetilde{\varphi}(s)\widetilde{\chi}(s)\widetilde{u}(s).$$

4. Quantum Qudit States and Observables

In this section, we construct quantum states and observables, extending the approach discussed in the previous sections for classical systems.

Given $N\times N$ matrix $\rho_{ss'}$ $(s, s' = 1, 2, \ldots, N)$. If $\rho = \rho^\dagger$, $\mathrm{Tr}\rho = 1$, and $\rho \geq 0$, this matrix can be interpreted as the density matrix of qudit state with $j = (N-1)/2$.

At $N = nm$, the matrix $\rho_{ss'}$ $(s, s' = 1, 2, \ldots, N)$ can be interpreted as the density matrix of two qudits with $j_1 = (n-1)/2$ and $j_2 = (m-1)/2$, as well as at $N = n_1 n_2 n_3$, it can also be interpreted as the density matrix of three qudits with $j_1 = (n_1 - 1)/2$, $j_2 = (n_2 - 1)/2$, and $j_3 = (n_3 - 1)/2$. An analogous interpretation can be provided for $N = \prod_{k=1}^{\ell} n_k$, and the matrix $\rho_{ss'}$ $(s, s' = 1, 2, \ldots, N)$ can be considered as the density matrix of ℓ qudits with $j_k = (n_k - 1)/2$.

To provide such an interpretation, we use the map of matrix indices $s \leftrightarrow j, k$, $s' \leftrightarrow j', k'$, i.e., $s = s(j, k)$ and $s' = s'(j', k')$, while considering two qudits, and $s = s(j, k, \ell)$ and $s' = s'(j', k', \ell')$, while considering three qudits, etc. We used this tool in [16]. In this paper, we study the possibility to extend this interpretation also for matrices of observables $F_{ss'}$ corresponding to the operators $\hat{F}$ acting in the Hilbert space $\mathcal{H}$.

We can write the matrices of observables either in the form

$$F_{ss'} = F_{s(j,k)\, s'(j',k')} \equiv F_{jk,\, j'k'}, \tag{23}$$

or in the form

$$F_{ss'} = F_{s(j,k,\ell)\, s'(j',k',\ell')} \equiv F_{jk\ell,\, j'k'\ell'}, \tag{24}$$

where indices j, k and j, k, ℓ take the same values as in the density matrix

$$\rho_{ss'} = \rho_{s(j,k)\, s'(j',\, k')} \equiv \rho_{jk,\, j'k'}, \qquad \rho_{ss'} = \rho_{s(j,k,\ell)\, s'(j',\, k',\, \ell')} \equiv \rho_{jkl,\, j'k'\ell'}.$$

Thus, both quantum states and quantum observables described by a density operator $\hat{\rho}$ and an observable operator $\hat{F}$ acting in the $N\times N$-dimensional Hilbert space $\widetilde{H}$ can be associated with the matrices $\rho_{ss'}$ and $F_{ss'}$ given in the basis $\mid s\rangle$, i.e., $\rho_{ss'} = \langle s \mid \hat{\rho} \mid s'\rangle$ or $F_{ss'} = \langle s \mid \hat{F} \mid s'\rangle$.

On the other hand, one can use the basis $\mid s\rangle = \mid s(j,k)\rangle = \mid j\rangle \mid k\rangle$, considering the Hilbert space $\widetilde{H}$ as the tensor product of two Hilbert spaces $\widetilde{H} = \widetilde{H}_1 \otimes \widetilde{H}_2$. In this basis, the matrix of the same density operator $\hat{\rho}$ reads

$$\rho_{jk,j'k'} = \langle s(j,k) \mid \hat{\rho} \mid s'(j',k')\rangle;$$

this is the same numerical $N\times N$ matrix $\rho_{ss'}$ but with the matrix elements labeled by indices jk, $j'k'$.

Analogously, for the observable $\hat{F}$ we can write the matrix $\langle s \mid \hat{F} \mid s'\rangle$ in the form

$$\langle s(j,k) \mid \hat{F} \mid s'(j',k')\rangle \equiv F_{jk,\, j'k'}.$$

Thus, we obtain the same $N\times N$ numerical matrix with matrix elements $F_{ss'}$ $(s,s'=1,2,\ldots,N)$ but the matrix elements are labeled by the indices jk and $j'k'$ $(j,j'=1,2,\ldots,n;\ k,k'=1,2,\ldots,m)$. The map introduced provides a chance to write the mean value of the observable $F_{ss'}=F_{s(j,k)\,s'(j',k')}\equiv F_{jk,\,j'k'}$ as

$$\begin{aligned}\langle\hat F\rangle=\operatorname{Tr}\hat F\hat\rho &= \sum_{s=1}^{N}\sum_{s'=1}^{N}F_{ss'}\rho_{s's}=\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{j'=1}^{n}\sum_{k'=1}^{m}F_{s(j,k)\,s'(j',k')}\,\rho_{s'(j',k')\,s(j,k)}\\ &=\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{j'=1}^{n}\sum_{k'=1}^{m}F_{jk,j'k'}\,\rho_{j'k',jk}.\end{aligned}\tag{25}$$

If one takes the observable $\hat F$ in the form

$$\hat F=\hat F_1\otimes\hat F_2,\tag{26}$$

where $\hat F_1$ is the operator of the observable acting in the Hilbert space $\widetilde{\mathcal H}_1$ and $\hat F_2$ is the operator of the observable acting in the Hilbert space $\widetilde{\mathcal H}_2$, Eq. (25) reads

$$\langle\hat F\rangle=\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{j'=1}^{n}\sum_{k'=1}^{m}(F_1)_{jj'}(F_2)_{kk'}\,\rho_{j'k',\,jk}.\tag{27}$$

In the case where $\hat F=\hat F_1\otimes\hat F_2$, we introduce two commuting observables

$$\widetilde{\hat F}_1=\big(\hat F_1\otimes\hat 1_m\big),\qquad \widetilde{\hat F}_2=\big(\hat 1_n\otimes\hat F_2\big).\tag{28}$$

For these two observables, the mean value of the observable $\hat F$ takes the form of the correlation function of the observables $\widetilde{\hat F}_1$ and $\widetilde{\hat F}_2$, i.e.,

$$\langle\hat F\rangle=\langle\widetilde{\hat F}_1\widetilde{\hat F}_2\rangle.\tag{29}$$

As a result, we obtained a quantum analog of the classical probability relation (21).

For $N=n_1n_2\cdots n_\ell$, we have

$$\langle\hat F\rangle=\langle\widetilde{\hat F}_1\widetilde{\hat F}_2\cdots\widetilde{\hat F}_\ell\rangle;\tag{30}$$

this relation is a generalization of Eq. (29). We showed that the same mean value $\langle\hat F\rangle$ can be considered as the correlation function of ℓ commuting observables $\widetilde{\hat F}_p$ $(p=1,2,\ldots,\ell)$.

5. Entropic and Information Inequalities

In this section, we consider the classical system with one random variable. We recall that there exist inequalities for entropies of joint probability distributions $\mathcal P(j,k)$ of two random variables j and k of the form

$$-\sum_{jk}\mathcal P(j,k)\ln\mathcal P(j,k)\le-\sum_{j}\Big\{\Big[\sum_{k}\mathcal P(j,k)\Big]\ln\Big[\sum_{k}\mathcal P(j,k)\Big]\Big\}-\sum_{k}\Big\{\Big[\sum_{j}\mathcal P(j,k)\Big]\ln\Big[\sum_{j}\mathcal P(j,k)\Big]\Big\}.$$

This inequality (the subadditivity condition) can be interpreted as the subadditivity condition for the probability distribution of one random variable

$$-\sum_s p_s \ln p_s \leq -\sum_j \Big\{ \Big[\sum_k p_{s(j,k)} \Big] \ln \Big[\sum_k p_{s(j,k)} \Big] \Big\} - \sum_k \Big\{ \Big[\sum_j p_{s(j,k)} \Big] \ln \Big[\sum_j p_{s(j,k)} \Big] \Big\}.$$

The other example (for $N = 4$) is the possibility to consider the observable $F(j,k)$ for two random variables j and k, for example, $F(1,1) = 1$, $F(1,2) = -1$, $F(2,1) = -1$, and $F(2,2) = 1$ as an observable for one random variable $F(s)$ such as $F(1) = 1$, $F(2) = -1$, $F(3) = -1$, and $F(4) = 1$. In this case,

$$\langle F \rangle = p_1 F(1) + p_2 F(2) + p_3 F(3) + p_4 F(4) = p_1 - p_2 - p_3 + p_4.$$

Also

$$\langle F \rangle = \mathcal{P}(1,1)F(1,1)+\mathcal{P}(1,2)F(1,2)+\mathcal{P}(2,1)F(2,1)+\mathcal{P}(2,2)F(2,2) = \mathcal{P}(1,1)-\mathcal{P}(1,2)-\mathcal{P}(2,1)+\mathcal{P}(2,2).$$

On the other hand, for two observables

$$\widetilde{F}_1(1,1) = 1, \quad \widetilde{F}_1(1,2) = 1, \quad \widetilde{F}_1(2,1) = -1, \quad \widetilde{F}_1(2,2) = -1$$

and

$$\widetilde{F}_2(1,1) = 1, \quad \widetilde{F}_2(1,2) = -1, \quad \widetilde{F}_2(2,1) = 1, \quad \widetilde{F}_2(2,2) = -1,$$

one has the correlation function of the form

$$\langle \widetilde{F}_1 \widetilde{F}_2 \rangle = \sum_{j,k} \mathcal{P}(j,k) \widetilde{F}_1(j,k) \widetilde{F}_2(j,k),$$

and this correlation function is equal to $\langle F \rangle$. In fact,

$$\langle \widetilde{F}_1 \widetilde{F}_2 \rangle = \mathcal{P}(1,1) - \mathcal{P}(1,2) - \mathcal{P}(2,1) + \mathcal{P}(2,2) = p_1 - p_2 - p_3 + p_4 = \sum_{s=1}^{4} F(s) p_s. \tag{31}$$

In view of the invertibility of the applied map of the indices, we can introduce two observables $F_1'(s)$ and $F_2'(s)$, i.e.,

$$F_1'(1) = 1, \quad F_1'(2) = 1, \quad F_1'(3) = -1, \quad F_1'(4) = 1$$

and

$$F_2'(1) = 1, \quad F_2'(2) = -1, \quad F_2'(3) = 1, \quad F_2'(4) = -1.$$

Then one has the equality $\langle F \rangle = \langle F_1' F_2' \rangle$ or

$$\sum_{s=1}^{4} F(s) p_s = \sum_{1=1}^{4} F_1'(s) F_2'(s) p_s. \tag{32}$$

Thus, we showed that the mean value of observable $F(s)$ can be interpreted as the correlation function of observables $\widetilde{F}_1(j,k)$ and $\widetilde{F}_2(j,k)$. The subadditivity condition for functions p_s of one random variable reflects the correlations of two artificial random variables (j,k) that are connected with two observables $\widetilde{F}_1$ and $\widetilde{F}_2$. Another interpretation of equality (32) reflects the fact that there exist hidden correlations of observables $F_1'(s)$ ad $F_2'(s)$ for the case of a single random variable s.

6. Conclusions

To conclude, we point out the main results of this study.

We showed that for a single qudit with $j = (N-1)/2$ it is possible to find commuting observables (Hermitian matrices), e.g., two observables $\hat{\varphi}$ and $\hat{\chi}$, such that the product of the observables provides the Hermitian matrix $\hat{\varphi}\hat{\chi} = \hat{A}$. Then the mean value of the observable $\hat{A}$ can be interpreted as the correlation function of two observables $\langle \hat{A} \rangle = \langle \hat{\varphi}\hat{\chi} \rangle = \mathrm{Tr}\,(\hat{A}\hat{\rho})$.

In the case $N = n_1 n_2 \cdots n_k$, one can find commuting observables $\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_k$, such that the means of the observable $\hat{A} = \hat{\varphi}_1\hat{\varphi}_2 \cdots \hat{\varphi}_k$ can be treated as the correlation function $\langle \hat{A} \rangle = \langle \hat{\varphi}_1\hat{\varphi}_2 \cdots \hat{\varphi}_k \rangle = \mathrm{Tr}\,(\hat{A}\hat{\rho})$.

In the case of a multiqudit system with the same numerical density matrix $\rho(1, 2, \ldots, k)$, the observables $\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_k$ have the physical meaning of the observables associated with each qudit in the composite system.

Such an observation means that the quantum correlations known for observables associated with the subsystems are also available in single qudit systems like the quantum roulette and the quantum compass. We call these correlations the hidden correlations and they can be used in quantum technology applications analogously to the correlations in composite systems (like, e.g., the entanglement).

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