DOI 10.1007/s10946-015-9472-5
Journal of Russian Laser Research, Volume 36, Number 1, January, 2015

MINKOWSKI-TYPE INEQUALITY FOR ARBITRARY DENSITY MATRICES OF COMPOSITE AND NONCOMPOSITE SYSTEMS

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Abstract

We obtain a new matrix inequality for an arbitrary density matrix of composite/noncomposite qudit systems including a single-qudit state. For bipartite systems, this inequality coincides with a known entropic inequality like the subadditivity condition. The examples of two-qubit system and qudit with $j = 3/2$ are discussed.

Keywords: qudit states, Hermitian matrix, bipartite quantum system, entropic and information inequalities.

1. Introduction

Quantum correlations for multipartite qudit systems are partially characterized by some inequalities for the system density matrices. For example, the system of two qubits has the density matrix, which in the case of the system separable state obeys the Bell inequality [1, 2]. The violation of the inequality provides a characteristic of the entanglement in the two-qubit system, which is related to the degree of correlations between qubits, and the correlations can be associated with a value of the Cirelson bound [3].

On the other hand, there exist the entropic and information inequalities, e.g., the subadditivity condition, which is an inequality for the von Neumann entropies of the bipartite system and two of its subsystem states [4–6]. For three-partite systems, there exists the strong subadditivity condition, which is an inequality for the von Neumann entropies of the composite system and its subsystems [7, 8]. The subadditivity conditions are also valid for the Shannon entropies [9] of the bipartite and three-partite systems. The nonnegativity of the Shannon mutual information, quantum mutual information, and conditional classical and quantum mutual informations follow from the subadditivity and strong subadditivity conditions.

Recently, the portrait qubit and qudit maps were introduced to study the entanglement phenomena [10, 11]. This method is appropriate for studying quantum correlations within the framework of

Manuscript submitted by the authors in English on July 23, 2014.

the tomographic probability representation of quantum states [12–17]. In this representation, which is valid for both discrete and continuous variables [12], the spin states (qudit states) are identified with fair tomographic probability distributions [18–21]. In view of this fact, the standard formulas for classical probability distributions like the formulas for entropy and information can be easily compared with the corresponding quantum relationships [22, 23].

Using the approach based on the portrait method which, in fact, is the positive map approach, many researchers [10, 11, 24–27] show that the entropic inequalities valid for composite systems can be extended to arbitrary systems, including the systems without subsystems. In [28–36], some inequalities associated with positive operators acting in the Hilbert space, which has the structure of tensor product of Hilbert spaces, were studied.

The aim of our work is to obtain new matrix inequalities for density matrices of the qudit states of both composite and noncomposite quantum systems, which do not depend on the tensor product structure of the Hilbert space. We follow the approach of [10, 11, 24–27] based on the portrait positive map method.

This paper is organized as follows.

In Sec. 2, we formulate a new inequality for the Hermitian matrix, which corresponds to the operator inequality for a bipartite quantum system given in [28, 29]. In Sec. 3, we consider the examples of this inequality for 4×4 Hermitian matrix and for the density matrices of the two-qubit state and the single qudit with $j = 3/2$. We give our conclusions and prospectives in Sec. 4.

2. Inequality for $N\times N$ Hermitian Matrix

We present a new inequality for the density $N\times N$ matrix ρ with properties $\rho^\dagger = \rho$, $\mathrm{Tr}\rho = 1$, and $\rho \geq 0$. Let $N = nm$, where n and m are integers. We present the matrix ρ in a block form $\rho = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$, where the blocks a_{jk} $(j, k = 1, 2, \ldots, n)$ are the $m\times m$ matrices. For an arbitrary real number p, we introduce a matrix $\rho^p = \begin{pmatrix} a_{11}(p) & a_{12}(p) & \cdots & a_{1n}(p) \\ a_{21}(p) & a_{22}(p) & \cdots & a_{2n}(p) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(p) & a_{n2}(p) & \cdots & a_{nn}(p) \end{pmatrix}$, which is also presented in a block form, and the blocks $a_{jk}(p)$ are the $m\times m$ matrices depending on the parameter p.

A new inequality valid for an arbitrary $N\times N$ matrix ρ reads

$$\left[\mathrm{Tr}\left(\sum_{j=1}^{n} a_{jj}\right)^{p}\right]^{1/p} \leq \mathrm{Tr}\left[\begin{pmatrix} \mathrm{Tr}\, a_{11}(p) & \mathrm{Tr}\, a_{12}(p) & \cdots & \mathrm{Tr}\, a_{1n}(p) \\ \mathrm{Tr}\, a_{21}(p) & \mathrm{Tr}\, a_{22}(p) & \cdots & \mathrm{Tr}\, a_{2n}(p) \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{Tr}\, a_{n1}(p) & \mathrm{Tr}\, a_{n2}(p) & \cdots & \mathrm{Tr}\, a_{nn}(p) \end{pmatrix}^{1/p}\right]. \qquad (1)$$

This inequality is valid for $p \geq 1$.

The $n\times n$ matrix on the right-hand side of (1) has the matrix elements $\mathrm{Tr}\, a_{jk}(p)$.

If $0 \leq p \leq 1$, inequality (1) reverses.

If $N \neq nm$, we use an integer $N' = N + s$, such that $N' = nm$, and consider the density $N' \times N'$ matrix ρ' of the form

$$\rho' = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, \tag{2}$$

where a_{jk} are the blocks, which provide the block-form representation of the density $N' \times N'$ matrix ρ'. Then inequality (1) holds for the blocks associated with the matrix ρ' by (2).

The new inequality (1) is obtained in view of the inequality reported in [28, 30, 36] for the density operator of a bipartite quantum system. But the new inequality (1) is valid for arbitrary density matrices of multipartite qudit systems, including the single-qudit density matrix. If the density matrix ρ is the diagonal matrix, inequality (1) provides the inequalities for arbitrary probability vectors.

In fact, let us denote the diagonal elements of the matrix ρ as $(a_{jj})_\alpha = P_{j\alpha}, \quad \alpha = 1, 2, \ldots m$. The nonnegative numbers $P_{11}, P_{12}, \ldots, P_{1m}, P_{21}, P_{22}, \ldots, P_{2m}, \ldots, P_{n1}, P_{n2}, \ldots, P_{nm}$ can be considered as components of a probability N-vector $\vec{P}$. Inequality (1) written in terms of the probability vector reads

$$\left[\sum_{\alpha=1}^{m} \left[\left(\sum_{j=1}^{n} P_{j\alpha} \right)^p \right] \right]^{1/p} \leq \sum_{j=1}^{n} \left[\sum_{\alpha=1}^{m} (P_{j\alpha})^p \right]^{1/p}, \quad p \geq 1. \tag{3}$$

The reverse inequality holds for $0 < p \leq 1$.

If $N \neq nm$, we use $N' = N + s = nm$. Inequality (3) for the probability N'-vector holds, and the last s components of the probability N'-vector are equal to zero. In fact, we have inequality (3) for an arbitrary number of nonnegative numbers, which are not necessarily associated with a probability distribution. It is worth pointing out that inequality (1) obtained for the density $N \times N$ matrix ρ is valid for any density matrix obtained from this one by all permutations of the indices $1, 2, \ldots, N \to 1_p, 2_p, \ldots, N_p$. The same statement is true for inequality (3).

More generally, the density matrix $\Phi(\rho)$ obtained from the initial matrix ρ by means of an arbitrary positive map $\rho \to \Phi(\rho)$ satisfies inequality (1). It is clear that one can use different decompositions of the integer $N = nm = n'm'$. This means that there exist different inequalities for the same density matrix ρ corresponding to different product form of the numbers N and N'.

For $N = nm$, we can extend inequality (1) to the case of an arbitrary Hermitian $N \times N$ matrix A. Let $A = A^\dagger$ and the matrix A have the block form corresponding to the decomposition $N = nm$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{4}$$

Let x_0 be the minimum eigenvalue of the matrix A. For an arbitrary $x \geq x_0$, such that $x + x_0 \geq 0$, we introduce the nonnegative Hermitian matrix

$$A(x) = A + x 1_N. \tag{5}$$

The matrix $A(x)$ has the block form

$$A(x)=\begin{pmatrix} a_{11}+x1_m & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22}+x1_m & \cdots & a_{2n}\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}+x1_m \end{pmatrix}. \tag{6}$$

Then the matrix $(A(x))^p$ can also be presented in a block form as follows:

$$(A(x))^p == \begin{pmatrix} a_{11}(x,p) & a_{12}(x,p) & \cdots & a_{1n}(x,p)\\ a_{21}(x,p) & a_{22}(x,p) & \cdots & a_{2n}(x,p)\\ \cdots & \cdots & \cdots & \cdots\\ a_{n1}(x,p) & a_{n2}(x,p) & \cdots & a_{nn}(x,p) \end{pmatrix}. \tag{7}$$

For $N=nm$, the new inequality, which holds for an arbitrary Hermitian $N\times N$ matrix A, reads

$$\left[\operatorname{Tr}\left(\sum_{j=1}^{n} a_{jj}(x)\right)^{p}\right]^{1/p} \leq \operatorname{Tr}\left[\begin{pmatrix} \operatorname{Tr}a_{11}(x,p) & \operatorname{Tr}a_{12}(x,p) & \cdots & \operatorname{Tr}a_{1n}(x,p)\\ \operatorname{Tr}a_{21}(x,p) & \operatorname{Tr}a_{22}(x,p) & \cdots & \operatorname{Tr}a_{2n}(x,p)\\ \cdots & \cdots & \cdots & \cdots\\ \operatorname{Tr}a_{n1}(x,p) & \operatorname{Tr}a_{n2}(x,p) & \cdots & \operatorname{Tr}a_{nn}(x,p) \end{pmatrix}^{1/p}\right]. \tag{8}$$

This inequality is valid for $p\geq 1$. The inequality reverses for $0\leq p\leq 1$.

In the case $N\neq nm$, we use $N'=N+s=nm$, and the new inequality for the Hermitian matrix $A'=\begin{pmatrix} A & 0\\ 0 & 0\end{pmatrix}$ presented in the block form (4) can be given by the above inequality (8).

Using the diagonal Hermitian matrix A, one can write the inequality for an arbitrary finite set of $N=nm$ real numbers $P_{11},P_{12},\ldots,P_{1m}$, P_{21}, $P_{22},\ldots,P_{2m},\ldots,P_{n1}$, $P_{n2},\ldots,P_{nm}$. It has the form of the inequality for two functions $\mathcal{P}_1(x,p)$ and $\mathcal{P}_2(x,p)$, i.e.,

$$\mathcal{P}_1(x,p)\leq\mathcal{P}_2(x,p),\quad p\geq 1\quad\text{and}\quad \mathcal{P}_1(x,p)\geq\mathcal{P}_2(x,p),\quad 0<p\leq 1, \tag{9}$$

where

$$\mathcal{P}_1(x,p)=\left\{\sum_{\alpha=1}^{m}\left[\left(nx+\sum_{j=1}^{n}P_{j\alpha}\right)^{p}\right]\right\}^{1/p},\qquad \mathcal{P}_2(x,p)=\sum_{j=1}^{n}\left\{\left[\left(\sum_{\alpha=1}^{m}P_{j\alpha}\right)+mx\right]^{p}\right\}^{1/p}. \tag{10}$$

For reals, such that $P_{j\alpha}\geq 0$ and $\sum_{j=1}^{n}\sum_{\alpha=1}^{m}P_{j\alpha}=1$, inequality (9) can be interpreted as an inequality for the probability vector, which holds for an arbitrary $x\geq 0$.

Some information on the correlations in the system of qudits, including the case of a single qudit, is available in the difference of terms in inequality (1)

$$\mathcal{J}(p)=\operatorname{Tr}\left[\begin{pmatrix} \operatorname{Tr}a_{11}(p) & \operatorname{Tr}a_{12}(p) & \cdots & \operatorname{Tr}a_{1n}(p)\\ \operatorname{Tr}a_{21}(p) & \operatorname{Tr}a_{22}(p) & \cdots & \operatorname{Tr}a_{2n}(p)\\ \cdots & \cdots & \cdots & \cdots\\ \operatorname{Tr}a_{n1}(p) & \operatorname{Tr}a_{n2}(p) & \cdots & \operatorname{Tr}a_{nn}(p) \end{pmatrix}^{1/p}\right]-\left[\operatorname{Tr}\left(\sum_{j=1}^{n}a_{jj}\right)^{p}\right]^{1/p}\geq 0,\quad p\geq 1. \tag{11}$$

For a bipartite system, the function $\mathcal{J}(p)$ is an additional (to the mutual information) characteristic of correlations given by the subadditivity condition terms.

3. Inequalities for Hermitian 4×4 Matrices

Now we illustrate the inequalities on an example of 4×4 matrices. In this case, the 4×4 matrix A has 2×2 blocks

$$a_{11}=\begin{pmatrix}\rho_{11}&\rho_{12}\\ \rho_{21}&\rho_{22}\end{pmatrix},\quad a_{12}=\begin{pmatrix}\rho_{13}&\rho_{14}\\ \rho_{23}&\rho_{24}\end{pmatrix},\quad a_{21}=\begin{pmatrix}\rho_{31}&\rho_{32}\\ \rho_{41}&\rho_{42}\end{pmatrix},\quad a_{22}=\begin{pmatrix}\rho_{33}&\rho_{34}\\ \rho_{43}&\rho_{44}\end{pmatrix}. \tag{12}$$

The matrix $A(x)$ reads

$$A(x)=\begin{pmatrix}\rho_{11}+x&\rho_{12}&\rho_{13}&\rho_{14}\\ \rho_{21}&\rho_{22}+x&\rho_{23}&\rho_{24}\\ \rho_{31}&\rho_{32}&\rho_{33}+x&\rho_{34}\\ \rho_{41}&\rho_{42}&\rho_{43}&\rho_{44}+x\end{pmatrix}. \tag{13}$$

The matrix $(A(x))^p$ has the block form

$$(A(x))^p=\begin{pmatrix}a_{11}(x,p)&a_{12}(x,p)\\ a_{21}(x,p)&a_{22}(x,p)\end{pmatrix}. \tag{14}$$

Inequality (8) is

$$\left[\mathrm{Tr}\begin{pmatrix}\rho_{11}+\rho_{33}+2x&\rho_{12}+\rho_{34}\\ \rho_{21}+\rho_{43}&\rho_{22}+\rho_{44}+2x\end{pmatrix}^p\right]^{1/p}\leq \mathrm{Tr}\left[\begin{pmatrix}\mathrm{Tr}\,a_{11}(x,p)&\mathrm{Tr}a_{\,12}(x,p)\\ \mathrm{Tr}\,a_{21}(x,p)&\mathrm{Tr}\,a_{22}(x,p)\end{pmatrix}^{1/p}\right],\qquad p\geq 1. \tag{15}$$

If the Hermitian matrix A is nonnegative and $\mathrm{Tr}\,A=1$, one can interpret it as a density matrix either of the two-qubit state or the qudit state with $j=3/2$.

For diagonal density 4×4 matrix with eigenvalues p_{11}, p_{12}, p_{21}, and p_{22}, the inequality reads $(x=0)$

$$\left[\,(p_{11}+p_{21})^p+(p_{12}+p_{22})^p\,\right]^{1/p}\leq (p_{11}^p+p_{12}^p)^{1/p}+(p_{21}^p+p_{22}^p)^{1/p},\qquad p\geq 1. \tag{16}$$

One can check that for $p=2$ the above inequality is equivalent to the inequality $a^2+b^2\geq 2ab$; for this case, the function $\mathcal{J}(p)$ reads

$$\mathcal{J}(p)=(p_{14}^p+p_{12}^p)^{1/p}+(p_{21}^p+p_{22}^p)^{1/p}-\Big[(p_{11}+p_{21})^2+(p_{12}+p_{22})^p\Big]^{1/p}\geq 0,\qquad p\geq 1, \tag{17}$$

and the mutual information is

$$\begin{aligned}I&=p_{11}\ln p_{11}+p_{12}\ln p_{12}+p_{21}\ln p_{21}+p_{22}\ln p_{22}-(p_{11}+p_{12})\ln(p_{11}+p_{12})\\ &-(p_{21}+p_{22})\ln(p_{21}+p_{22})-(p_{11}+p_{21})\ln(p_{11}+p_{21})-(p_{12}+p_{22})\ln(p_{12}+p_{22})\geq 0.\end{aligned} \tag{18}$$

Inequalities (17) and (18) are compatible.

4. Conclusions

To conclude, we point out the main results of our work.

For an arbitrary system of qudits, including the single-qudit case, we obtained the inequalities for the system-state density matrix, which are equivalent to the inequalities known in the case of a bipartite quantum system. We derived a new simple inequality for an arbitrary Hermitian $N \times N$ matrix. The inequality can be used to study the ground-state energy for the Hermitian Hamiltonian. It is worth clarifying the compatibility of this inequality with entropic and information inequalities obtained for the Hamiltonian in [37]. These problems will be discussed in a future publication.

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