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MODELING TESTS BASED ON THE EBERHARD INEQUALITY[†]

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Abstract

Last year, the first experimental tests closing the detection loophole (also referred to as the fair sampling loophole) were performed by two experimental groups, one in Vienna and the other one in Urbana-Champaign. To violate the Bell-type inequalities (the Eberhard inequality in the first test and the Clauser–Horne inequality in the second test), one has to optimize a number of parameters involved in the experiment (angles of polarization beam splitters and quantum state parameters). We study this problem for the Eberhard inequality in detail, using the advanced method of numerical optimization, namely, the Nelder–Mead method.

Keywords: Bell's test, fair sampling loophole, Eberhard inequality, Clauser–Horne inequality, optimization of parameters, fluctuations of angles, coefficient of variation.

1. Introduction

Experimental realization of a loophole-free test for Bell inequalities [1] (see, e.g., [2–4]) will play a crucial role both for quantum foundations [1, 5–13] (see, e.g., [14–20] for recent studies) and quantum technologies, e.g., quantum cryptography and quantum random generators. It is clear that the often used argument that "closing of different loopholes in separate tests can be considered as the solution of the loophole problem" cannot be considered as acceptable. The quantum community put tremendous efforts to perform a loophole-free test, and its final realization (which can be expected rather soon) will be a great event in development of quantum theory and quantum technologies.

Last year, the first experimental tests for photons closing the detection loophole (also referred to as the fair sampling loophole) were performed by two experimental groups [2, 3], see also [21–23]. The problem of detecting the efficiency for photons is very complicated, and its solution is based on the use of advanced photodetectors, i.e., new technology, as well as its testing [24]. The Bell tests with photons [11–13] are promising to close both the detection and locality loopholes, since the latter was closed long ago [25] and recently experiments demonstrating violation of the Bell-type inequalities on large distances [26–33] were performed. However, to violate the Bell-type inequalities, one has to approach the total experimental setup very high efficiency. Hence, although nowadays it is possible to work with photodetectors having

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the efficiency approaching 100%, the losses in the total experimental setup can decrease substantially the total efficiency of the experimental scheme; see [34–38] for theoretical analysis and mathematical modeling.

The experimentalists confront this problem by trying to extend Bell-type tests with sufficiently high efficiency to close the locality loophole. The total efficiency of experimental schemes decreases drastically with the distance. Therefore, it is important to optimize all parameters of the experiment to approach the maximum violation for the minimum possible efficiency. (One has to optimize angles of polarization beam splitters and the initial state parameters.) Although these are technicalities, their optimum determination plays an important role in approaching statistically significant violations of the inequalities.

In this paper, we study this problem for the Eberhard inequality [9] in detail using the advanced method of numerical optimization, namely, the Nelder–Mead method [39]. First of all, we improve the results of optimization for the original Eberhard model [9] and the work [2] (Vienna-13 experiment) employing the model of this experiment presented in [4]. We also take into account the well-known fact that detectors can have different efficiencies and perform the corresponding optimization.

In the previous studies [2, 4, 9], the objective function for optimization of the parameters had the meaning of mathematical expectation. However, it is also useful to investigate a possible level of variability of the results expressed in terms of standard deviation. In this paper, we consider the optimization of parameters for the Eberhard inequality using the coefficient K – the reciprocal of the coefficient of variation, taking into account possible random fluctuations in the setup of angles during the experiment.[‡] It seems that our study is the first contribution to the problem. The study of the problem of sensitivity of the degree of violation of the Eberhard inequality to the precision under the control of angles of polarization beam splitters can be useful for the experimentalists. One of the results of our numerical simulation is an unexpected stability of the degree of violation of the Eberhard inequality to fluctuations of these angles (in neighborhoods of optimum values of the angles).

2. Eberhard Inequality

We follow Eberhard [9]: Photons are emitted in pairs (a, b). Under each measurement setting (α, β), the events in which photon a is detected in the ordinary and extraordinary beams are denoted by symbols (o) and (e), respectively, and the event where photon is not detected is denoted by the symbol (u). The same symbols are used to denote the corresponding events for photon b. Therefore, for the pairs of photons, there are nine types of events: $(o, o), (o, u), (o, e)$, $(u, o), (u, u), (u, e), (e, o), (e, u)$, and (e, e).

Under the conditions of locality, realism, and statistical reproducibility, the following inequality (the Eberhard inequality, E-inequality) was derived:

$$J \equiv n_{oe}(\alpha_1, \beta_2) + n_{ou}(\alpha_1, \beta_2) + n_{eo}(\alpha_2, \beta_1) + n_{uo}(\alpha_2, \beta_1) + n_{oo}(\alpha_2, \beta_2) - n_{oo}(\alpha_1, \beta_1) \geq 0, \quad (1)$$

where $n_{xy}(\alpha_i, \beta_j)$ is the number of pairs detected in a given period of time for settings α_i and β_j with outcomes $x, y = o, e, u$, the outcomes (o) and (e) correspond to detections in the ordinary and extraordinary beams, respectively, and the event where photon is not detected is denoted by the symbol (u).

We point out the main distinguishing features of the E-inequality:

a) derivation without the fair sampling assumption (and without no-enhancement assumption);

[‡]In classical signal analysis, this quantity is known as the signal-to-noise ratio [40, 41].

b) taking into account undetected photons;

c) background events are taken into account;

d) the linear form of presentation (nonnegativity of a linear combination of coincidence and single rates).

The latter feature (which is typically not emphasized in the literature) is crucial to finding a simple procedure of optimization of experimental parameters and, hence, it makes the E-inequality the most promising experimental test to close the detection loophole and reject local realism without the fair sampling assumption.

The Eberhard optimization has two main outputs, which play an important role in the experimental design:

E1). It is possible to perform an experiment without the fair sampling assumption for detection efficiency less than 82,8%. Nevertheless, the detection efficiency must still be very high, at least 66.6% (in the absence of background).

E2). The optimum parameters correspond to non-maximum entangled states.

In 2013, the possibility to proceed with overall efficiencies lower than 82.8% (but larger than 66.6%) was explored for the E-inequality, and the first experimental test (the Vienna test) closing the detection loophole was published [2]; for more detailed presentation of statistical data, see also [4, 21].

2.1. Eberhard Inequality and Quantum-Mechanical Probabilities

Since the use of the Eberhard inequality is not common in quantum-foundational studies, we present here in detail the calculation of quantum-mechanical probabilities that violate the inequality (for specially selected parameters of the experimental test). Here, we follow the original paper of Eberhard but try to adapt the presentation for our purpose of improvement of optimization of the parameters.

Consider two detectors with the same efficiency η, which perform measurements in N experiments. That is, in every experiment, each detector detects a photon in one of the trajectories with the probability η. We construct the density operators for particles in the ordinary and extraordinary beams and use the helicity basis for derivation of these operators.

Two main circular polarization states are described with the vectors $u = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ and $v = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$. They form the transform matrix from the standard basis to a helicity basis: $W = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}$. The inverse transform can be made with the inverse matrix $T = W^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -1 \end{pmatrix}$. Then, we consider a polarization prism, which is rotated by an angle θ. A particle that appears in the ordinary beam has the state $\psi_o = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, and a particle that appears in the extraordinary beam has the state $\psi_e = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$. In the helicity basis, these states are, respectively, described as $\psi'_o = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta} \\ e^{-i\theta} \end{pmatrix}$ and $\psi'_e = \frac{1}{\sqrt{2}}\begin{pmatrix} ie^{i\theta} \\ -ie^{-i\theta} \end{pmatrix}$. Then, the corresponding density operators read $P_o = \psi'_o \cdot \psi'^{\dagger}_o = \frac{1}{2}\begin{pmatrix} 1 & e^{2i\theta} \\ e^{-2i\theta} & 1 \end{pmatrix}$ and

$P_e = \psi'_e \cdot \psi'^{\dagger}_e = \frac{1}{2}\begin{pmatrix} 1 & -e^{2i\theta} \\ -e^{-2i\theta} & 1 \end{pmatrix}$. The density operators of the considered system with two particles are described by the tensor products. If the first particle appears in the ordinary beam, it is described by $P_o \otimes I$, and if the second particle appears in the ordinary beam, the corresponding operator is $I \otimes P_o$. The operators for particles in the extraordinary beam can be constructed in a similar way.

Finally, if we put them all together and use the formula of quantum expectation value, quantum mechanics predicts the following results for the initial state of the system ψ:

$$n_{oo}(\alpha_1, \beta_1) = (N\eta^2/4)\,\psi^\dagger[I + \sigma(\alpha_1)][I + \tau(\beta_1)]\psi, \tag{2}$$
$$n_{oe}(\alpha_1, \beta_2) = (N\eta^2/4)\,\psi^\dagger[I + \sigma(\alpha_1)][I - \tau(\beta_2)]\psi, \tag{3}$$
$$n_{ou}(\alpha_1, \beta_2) = N[\eta(1-\eta)/2]\psi^\dagger[I + \sigma(\alpha_1)]\psi, \tag{4}$$
$$n_{eo}(\alpha_2, \beta_1) = (N\eta^2/4)\,\psi^\dagger[I - \sigma(\alpha_2)][I + \tau(\beta_1)]\psi, \tag{5}$$
$$n_{uo}(\alpha_2, \beta_1) = N[\eta(1-\eta)/2]\psi^\dagger[I + \tau(\beta_1)]\psi, \tag{6}$$
$$n_{oo}(\alpha_2, \beta_2) = (N\eta^2/4)\,\psi^\dagger[I + \sigma(\alpha_2)][I + \tau(\beta_2)]\psi, \tag{7}$$

where

$$\sigma(\alpha) = \begin{vmatrix} 0 & e^{2i(\alpha-\alpha_1)} & 0 & 0 \\ e^{-2i(\alpha-\alpha_1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{2i(\alpha-\alpha_1)} \\ 0 & 0 & e^{-2i(\alpha-\alpha_1)} & 0 \end{vmatrix}$$

and

$$\tau(\beta) = \begin{vmatrix} 0 & 0 & e^{2i(\beta-\beta_1)} & 0 \\ 0 & 0 & 0 & e^{2i(\beta-\beta_1)} \\ e^{-2i(\beta-\beta_1)} & 0 & 0 & 0 \\ 0 & e^{-2i(\beta-\beta_1)} & 0 & 0 \end{vmatrix}.$$

Note that matrices $I + \sigma(\alpha)$, $I - \sigma(\alpha)$, $I + \tau(\beta)$, and $I + \tau(\beta)$ can be simply derived from previously considered density operators with a properly chosen value of θ. Also note that the probability of failing to detect a particle is equal to $1 - \eta$ no matter what particle is considered, the first or the second.

Thus, we obtain the Eberhard inequality for quantum-mechanical quantities:

$$J_{\mathcal{B}}^{\text{ideal}} = n_{uo}(\alpha_2, \beta_1) + n_{eo}(\alpha_2, \beta_1) + n_{ou}(\alpha_1, \beta_2) + n_{oe}(\alpha_1, \beta_2) + n_{oo}(\alpha_2, \beta_2) - n_{oo}(\alpha_1, \beta_1) \geq 0.$$

However, in reality, in addition to correct detections during the experiments, some false positives may arise, which is called the background. In the Eberhard model, it is assumed that the number of false positive detections for events of type (o, o) can be ignored. We assume that the background level does not depend on α and β, so for events $n_{uo}(\alpha_2, \beta_1) + n_{eo}(\alpha_2, \beta_1)$ and $n_{ou}(\alpha_1, \beta_2) + n_{oe}(\alpha_1, \beta_2)$, it has the same value $N\zeta$. The resulting inequality reads $J_{\mathcal{B}} = J_{\mathcal{B}}^{\text{ideal}} + 2N\zeta \geq 0$.

This inequality can be written as $\psi^\dagger \mathcal{B}\psi \geq 0$, where $\mathcal{B}$ is a matrix:

$$\mathcal{B} = N\frac{\eta}{2}\begin{vmatrix} 2-\eta+\xi & 1-\eta & 1-\eta & A^*B^*-\eta \\ 1-\eta & 2-\eta+\xi & AB^*-\eta & 1-\eta \\ 1-\eta & A^*B-\eta & 2-\eta+\xi & 1-\eta \\ AB-\eta & 1-\eta & 1-\eta & 2-\eta+\xi \end{vmatrix},$$

with $A = (\eta/2)(e^{2i(\alpha_1-\alpha_2)} - 1)$, $B = e^{2i(\beta_1-\beta_2)} - 1$, and $\xi = 4\zeta/\eta$.

For implementing the experiment, which could show a violation of this inequality, we should search for the parameters such that $J_{\mathcal{B}} < 0$. Consider the case where $\zeta = 0$, that is, the detectors do not give false positives, and $\alpha_1 - \alpha_2 = \beta_1 - \beta_2 = \theta$. We employ the quantum state $\psi = \frac{1}{2\sqrt{1+r^2}} \begin{vmatrix} (1+r)e^{-i\omega} \\ -(1-r) \\ -(1-r) \\ (1+r)e^{i\omega} \end{vmatrix}$, where $0 \le r \le 1$, $\alpha_1 = \omega/2 - 90°$, and $\beta_1 = \omega/2$.

3. Optimization of the Parameters for Experimental Tests Based on the Eberhard Inequality

Our numerical optimization of the parameters of experimental tests to violate the Eberhard inequality is based on the Nelder–Mead optimization method. The Nelder-Mead method [39] (also known as downhill simplex method) is widely used for the optimization of nonlinear problems. This numerical method is typically applied to the problems for which the derivatives may not be known. Its applications are especially successful in the case of multidimensional spaces of parameters.

In this section, we present a part of the results of our studies, namely, the results of numerical optimization of the parameters to violate the Eberhard inequality as much as possible. We start with the comparison with the original Eberhard model [9], then consider the case of detectors having different efficiencies; thus, in general, $\eta_1 \neq \eta_2$. Finally, we consider the model [4] that was used in the recent Bell's test [2] based on the Eberhard inequality and called the Vienna-13 experiment.

3.1. Optimization of the Parameters for the Eberhard Model

For every η, we find the parameters r, ω, θ that allow the inequality to be violated most strongly. For this, we minimize the function f=$J_{\mathcal{B}}(r,\omega,\theta)/N$, employing the Nelder–Mead method. The values obtained while optimizing $J_{\mathcal{B}}(r,\omega,\theta)/N$ are shown in Table 1.

Table 1. Optimized Parameters for $J_{\mathcal{B}}/N$ from the Eberhard Inequality.

η	r	$\omega\,°$	$\theta\,°$	$J_{\mathcal{B}}/N$
0.7	0.136389	3.40081	21.4266	−0.000453562
0.75	0.310518	9.73143	31.9603	−0.00615095
0.8	0.465228	14.8979	37.9215	−0.02191
0.85	0.607424	18.5808	41.5341	−0.0496902
0.9	0.741202	20.9153	43.6381	−0.0899078
0.95	0.87067	22.141	44.6958	−0.142436
1	0.999997	22.5	45	−0.207107

During the process of optimization, the ζ values were set to zero, because this parameter provides a constant contribution 2ζ to the $J_{\mathcal{B}}/N$ value. It increases the $J_{\mathcal{B}}/N$ value by a constant regardless of other parameters, thus not affecting the result of optimization. Thus, values from Table 1 match with values obtained by Eberhard [9] for nonzero context level. The parameters for which the inequality is violated the most strongly for $\zeta = 0$ match with the parameters for which the inequality is violated, and ζ has the maximum value.

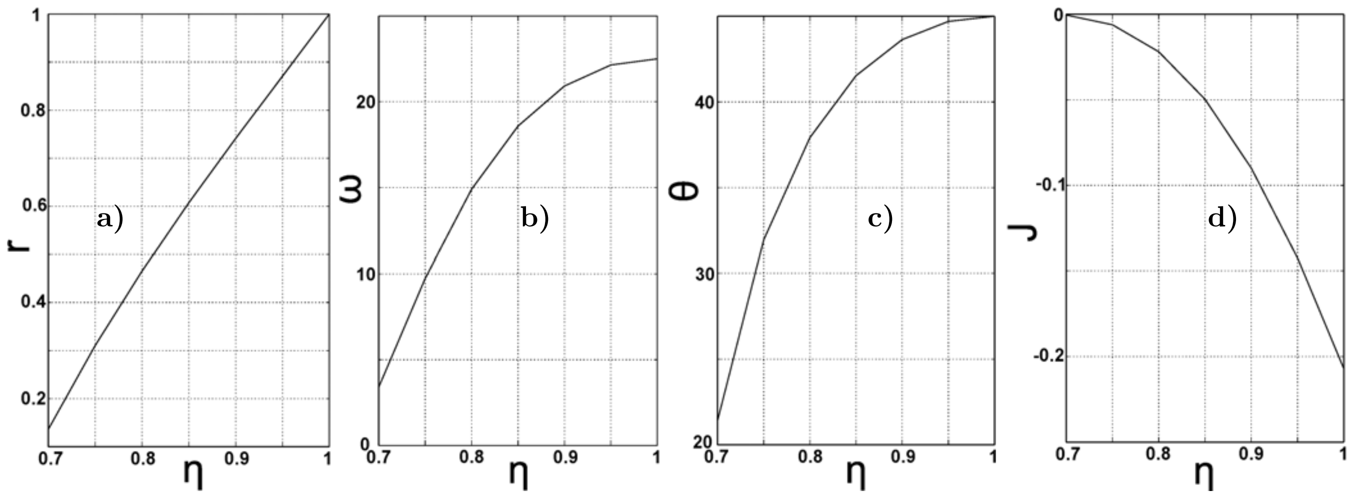

Fig. 1. Optimized values of r (a), ω (b), θ (c), and $J_{\mathcal{B}}/N$ (d) obtained from the Eberhard inequality for different efficiencies of the detectors.

In Fig. 1, the parameters r, ω, θ, and the minimum value of the function $J_{\mathcal{B}}/N$ obtained from the Eberhard inequality are shown versus the efficiency η.

3.2. Optimization of the Parameters for Detectors with Different Efficiencies; Eberhard Model

In real experiments, the detectors have different efficiencies, and formulas (2)–(7) can be easily adapted to this case. We write the Eberhard inequality as $\psi^\dagger \mathcal{B} \psi \geq 0$, with

$$\mathcal{B} = \frac{N}{2} \begin{vmatrix} C+\xi & \eta_1(1-\eta_2) & \eta_2(1-\eta_1) & A^*B^* - \eta_1\eta_2 \\ \eta_1(1-\eta_2) & C+\xi & AB^* - \eta_1\eta_2 & \eta_2(1-\eta_1) \\ \eta_2(1-\eta_1) & A^*B - \eta_1\eta_2 & C+\xi & \eta_1(1-\eta_2) \\ AB - \eta_1\eta_2 & \eta_2(1-\eta_1) & \eta_1(1-\eta_2) & C+\xi \end{vmatrix}, \text{ where } A = (\eta_1/2)\left(e^{2i(\alpha_1-\alpha_2)} - 1\right),$$

$B = \eta_2(e^{2i(\beta_1-\beta_2)} - 1)$, $C = \eta_1 + \eta_2 - \eta_1\eta_2$, and $\xi = 4\zeta$.

In Fig. 2, the optimum parameters together with the minimized function obtained from formulas (2)–(7) are shown versus different efficiencies of the detectors.

In Table 2, we show values of the parameters obtained in the optimization process along with the maximum allowable level of the noise. An increase in each value of the efficiency leads to a stronger violation of the inequality. The minimum values of the efficiency, when the violation is still possible, are close to $\eta_1 = \eta_2 = 0.67$, being in accordance with the Eberhard results.

Generally, for every state ψ that minimizes an expectation value, the following corollary holds.

Theorem. *Quantum state ψ minimizing target function J is an eigenvector of the matrix $\mathcal{B}$, and its dispersion is equal to zero.*

This theorem can be proved using the Courant–Fisher theorem.

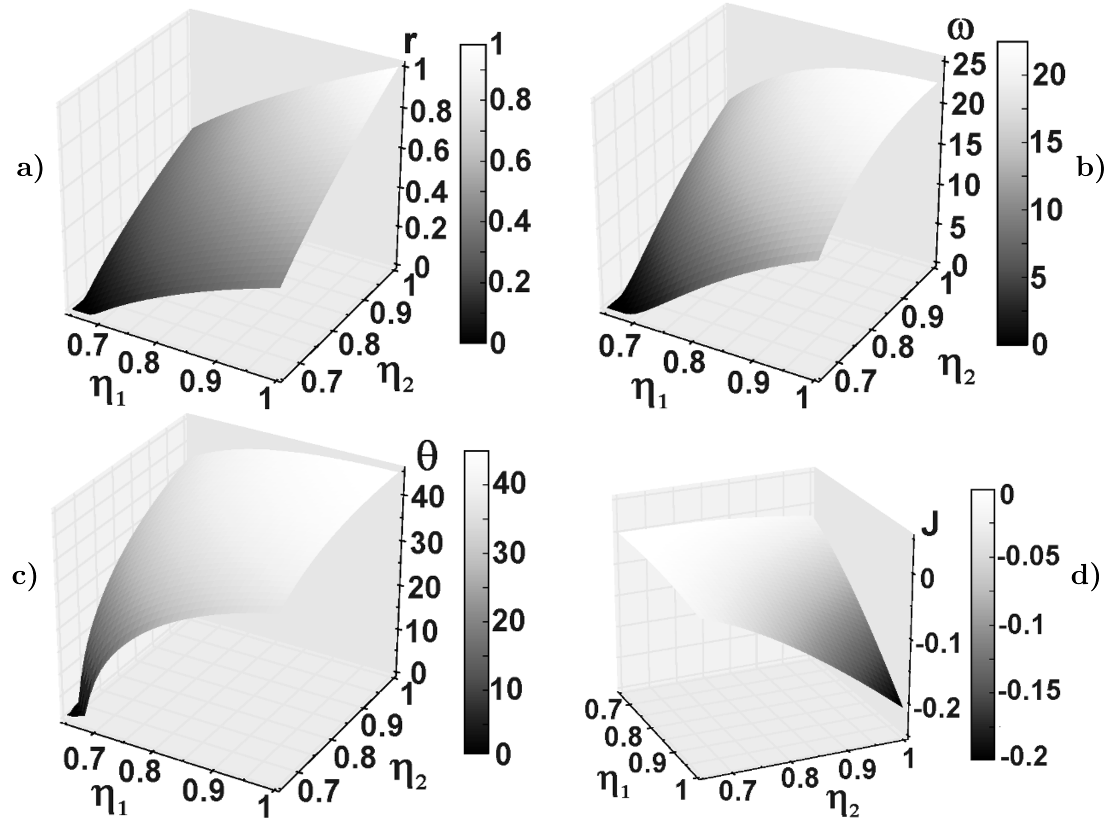

Fig. 2. Optimized values of r (a), ω (b), θ (c), and $J_{\mathcal{B}}/N$ (d) obtained from formulas (2)–(7) for different efficiencies of the detectors.

Proof. Let A be a $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.

The Rayleigh–Ritz quotient for this matrix is defined by $R_A(x) = (Ax, x)/(x, x)$.

For $1 \leq k \leq n$, let S_k denote the span of $v_1, \ldots, v_k$ and S_k denote the orthogonal complement of S_k.

Then $\lambda_1 \leq R_A(x) \leq \lambda_n,\ \forall x \in \mathbb{C}^n \setminus \{0\}, \quad \lambda_k = \max\{\min\{R_A(x) \mid x \neq 0 \in U\} \mid \dim(U) = k\}$, and $\lambda_k = \min\{\max\{R_A(x) \mid x \neq 0 \in U\} \mid \dim(U) = n - k + 1\}$.

This means that the obtained states are optimum not only from the mathematical-expectation point of view, but also from a possible spread of the measurement results expressed in terms of dispersion.

3.3. Optimization of the Parameters in the Model for the Vienna-13 Experiment

To match the real experimental conditions of the Vienna-13 experiment [2], the analysis of the applicability of the Eberhard inequality to this experiment was performed in [4]. The analysis led to the conclusion that data produced in the Vienna-13 experiment [2] are described by a more complicated model (within the standard quantum framework)[§] than the original Eberhard model [9]. This does not decrease the value of the original Eberhard study. Kofler et al. [4] just pointed out that some important additional "technicalities" have to be taken into account.

[§]As one of the aims, the work of Kofler et al. [4] justifies the statistical output of the Vienna-13 experiment using the standard quantum-mechanical tools.

Table 2. Optimized Parameters Obtained from Formulas (2)–(7) for Different Efficiencies of the Detectors.

η_1	η_2	r	ω°	θ°	$J_{\mathcal{B}}/N$	ζ
η_1	η_2	r	ω°	θ°	$J_{\mathcal{B}}/N$	ζ
0.65	0.65	9.73816 e−05	5.32909 e−05	0.609422	4.13366 e−10	–
0.65	0.7	0.0325726	0.4367	10.3918	−5.37452 e−06	2.68726 e−06
0.65	0.75	0.122804	2.94407	20.3249	−0.000327839	0.00016392
0.65	0.8	0.199412	5.64399	25.8864	−0.00152446	0.000762231
0.65	0.85	0.266448	8.1196	29.7694	−0.00385363	0.00192682
0.65	0.9	0.326193	10.2951	32.6807	−0.00737942	0.00368971
0.65	0.95	0.380046	12.1725	34.9409	−0.0120653	0.00603265
0.65	1	0.42905	13.7822	36.7384	−0.0178271	0.00891353
0.7	0.65	0.0325721	0.436691	10.3917	−5.37452 e−06	2.68726 e−06
0.7	0.7	0.136389	3.40081	21.4266	−0.000453562	0.000226781
0.7	0.75	0.223629	6.53572	27.3754	−0.00217282	0.00108641
0.7	0.8	0.299639	9.33741	31.4432	−0.00551773	0.00275887
0.7	0.85	0.367155	11.7325	34.4284	−0.0105412	0.00527059
0.7	0.9	0.427895	13.7454	36.6985	−0.0171516	0.0085758
0.7	0.95	0.48304	15.4224	38.4617	−0.0251977	0.0125989
0.7	1	0.533433	16.8118	39.8493	−0.034513	0.0172565
0.75	0.65	0.122804	2.94407	20.3249	−0.000327839	0.00016392
0.75	0.7	0.223629	6.53572	27.3754	−0.00217282	0.00108641
0.75	0.75	0.310518	9.73143	31.9603	−0.00615095	0.00307547
0.75	0.8	0.387235	12.4151	35.2194	−0.0123604	0.0061802
0.75	0.85	0.455977	14.6188	37.6299	−0.0206635	0.0103318
0.75	0.9	0.518149	16.4057	39.45	−0.0308332	0.0154166
0.75	0.95	0.574864	17.8435	40.8423	−0.0426257	0.0213128
0.75	1	0.626962	18.9919	41.9138	−0.0558135	0.0279068
0.8	0.65	0.199412	5.64399	25.8864	−0.00152446	0.000762231
0.8	0.7	0.299639	9.33741	31.4432	−0.00551773	0.00275887
0.8	0.75	0.387235	12.4151	35.2194	−0.0123604	0.0061802
0.8	0.8	0.465228	14.8979	37.9215	−0.02191	0.010955
0.8	0.85	0.535462	16.8648	39.9011	−0.0338613	0.0169307
0.8	0.9	0.59933	18.4036	41.3692	−0.0478775	0.0239387
0.8	0.95	0.657889	19.5935	42.4623	−0.063646	0.031823
0.8	1	0.712001	20.5014	43.274	−0.0808982	0.0404491
0.85	0.65	0.266448	8.1196	29.7694	−0.00385363	0.00192682
0.85	0.7	0.367155	11.7325	34.4284	−0.0105412	0.00527059
0.85	0.75	0.455977	14.6188	37.6299	−0.0206635	0.0103318
0.85	0.8	0.535462	16.8648	39.9011	−0.0338613	0.0169307
0.85	0.85	0.607424	18.5808	41.5341	−0.0496902	0.0248451
0.85	0.9	0.673214	19.8694	42.7109	−0.0677295	0.0338647
0.85	0.95	0.733924	20.817	43.552	−0.087619	0.0438095
0.85	1	0.790464	21.4958	44.1427	−0.109064	0.0545318
0.9	0.65	0.326193	10.2951	32.6807	−0.00737942	0.00368971
0.9	0.7	0.427895	13.7454	36.6985	−0.0171516	0.0085758
0.9	0.75	0.518149	16.4057	39.45	−0.0308332	0.0154166
0.9	0.8	0.59933	18.4036	41.3692	−0.0478775	0.0239387
0.9	0.85	0.673214	19.8694	42.7109	−0.0677295	0.0338647
0.9	0.9	0.741202	20.9153	43.6381	−0.0899078	0.0449539
0.9	0.95	0.804477	21.6349	44.2627	−0.11402	0.0570101
0.9	1	0.863896	22.1002	44.661	−0.139755	0.0698776
0.95	0.65	0.380046	12.1725	34.9409	−0.0120653	0.00603265
0.95	0.7	0.48304	15.4224	38.4617	−0.0251977	0.0125989
0.95	0.75	0.574864	17.8435	40.8423	−0.0426257	0.0213128
0.95	0.8	0.657889	19.5935	42.4623	−0.063646	0.031823
0.95	0.85	0.733924	20.817	43.552	−0.087619	0.0438095
0.95	0.9	0.804477	21.6349	44.2627	−0.11402	0.0570101
0.95	0.95	0.87067	22.141	44.6958	−0.142436	0.0712182
0.95	1	0.933431	22.4101	44.924	−0.172546	0.086273
1	0.65	0.42905	13.7822	36.7384	−0.0178271	0.00891353
1	0.7	0.533433	16.8118	39.8493	−0.034513	0.0172565
1	0.75	0.626962	18.9919	41.9138	−0.0558135	0.0279068
1	0.8	0.712001	20.5014	43.274	−0.0808982	0.0404491
1	0.85	0.790464	21.4958	44.1427	−0.109064	0.0545318
1	0.9	0.863896	22.1002	44.661	−0.139755	0.0698776
1	0.95	0.933431	22.4101	44.924	−0.172546	0.086273
1	1	0.999997	22.5	45	−0.207107	0.103553

In the experiments performed in [2], the values of the quantities n_{ou} and n_{uo} were found using the following formulas:

$$n_{ou}(\alpha_1,\beta_2)=S_o^A(\alpha_1)-n_{oo}(\alpha_1,\beta_2)-n_{oe}(\alpha_1,\beta_2),\quad n_{uo}(\alpha_2,\beta_1)=S_o^B(\beta_1)-n_{oo}(\alpha_2,\beta_1)-n_{eo}(\alpha_2,\beta_1),$$

where S_o^A and S_o^B are amounts of clicks in the ordinary beam for the first and second systems, respectively.

In this case, the Eberhard inequality reads

$$J=-n_{oo}(\alpha_1,\beta_1)+S_o^A(\alpha_1)-n_{oo}(\alpha_1,\beta_2)+S_o^B(\beta_1)-n_{oo}(\alpha_2,\beta_1)+n_{oo}(\alpha_2,\beta_2)\geq 0. \tag{8}$$

To model the output of the Vienna-13 experiment, one cannot proceed, as Eberhard did, with pure states. Consider a density operator of the system quantum state

$$\rho=\frac{1}{\sqrt{1+r^2}}\begin{vmatrix}0&0&0&0\\0&1&Vr&0\\0&Vr&r^2&0\\0&0&0&0\end{vmatrix},$$

where $0\leq r\leq 1$ and $0\leq V\leq 1$. In this case, predictions of quantum mechanics for values included into the inequality become

$$\tilde{S}_o^A(\alpha_i)=\eta_1 N\mathrm{Tr}[\rho(\hat{P}_A(\alpha_i)\otimes I)],\quad \tilde{S}_o^B(\beta_i)=\eta_1 N\mathrm{Tr}[\rho(I\otimes\hat{P}_B(\beta_i))],$$
$$\tilde{n}_{oo}(\alpha_i,\beta_i)=\eta_1\eta_2 N\mathrm{Tr}[\rho(\hat{P}_A(\alpha_i)\otimes\hat{P}_B(\beta_i))],$$

where $\hat{P}_A$ and $\hat{P}_B$ are the projection operators on the ordinary beam direction for the first and second prisms,

$$\hat{P}(\gamma)=\begin{pmatrix}\cos^2\gamma&\cos\gamma\sin\gamma\\\cos\gamma\sin\gamma&\sin^2\gamma\end{pmatrix}.$$

With regard to false clicks during time T, the values S_o^A and S_o^B are

$$S_o^A(\alpha_i)=\tilde{S}_o^A(\alpha_i)+\zeta T,\qquad S_o^B(\beta_i)=\tilde{S}_o^B(\beta_i)+\zeta T.$$

Besides noise, the model of the Vienna-13 experiment (see Kofler et al. [4]) considers also inconsistencies in time when pairs from different launches are detected as a conjugate event.

We introduce a temporary window value τ_c, within which conjugate events must be detected. In this case, $n_{oo}(\alpha_i,\beta_i)$ can be found using the following formulas:

$$n_{oo}(\alpha_i,\beta_i)=\tilde{n}_{oo}(\alpha_i,\beta_i)+n_{oo}^{acc}(\alpha_i,\beta_i),$$
$$n_{oo}^{acc}(\alpha_i,\beta_i)=S_o^A(\alpha_i)S_o^B(\beta_i)\frac{\tau_c}{T}\left(1-\frac{\tilde{n}_{oo}(\alpha_i,\beta_i)}{S_o^A(\alpha_i)}\right)\left(1-\frac{\tilde{n}_{oo}(\alpha_i,\beta_i)}{S_o^B(\beta_i)}\right).$$

For this model, we performed the optimization for the quantity J given by expression (8) for selected values of the experimental parameters $(\eta_1,\eta_2,r,V,T,\tau_c,N,\zeta)$ and found the values of the angles that minimize the target function. In particular, we remark that in [2] the levels of detector efficiencies $\eta_1=73.77$ and $\eta_2=78.59$ were approached.

Table 3. Comparison of the Results of Optimization of the Parameters with the Results of [2].

	$\alpha_1{}^\circ$	$\alpha_2{}^\circ$	$\beta_1{}^\circ$	$\beta_2{}^\circ$	J
Results of [2]	85.6	118.0	−5.4	25.9	−120191
Results of optimization	85.0	115.1	−4.0	27.4	−126060

Optimization results are shown in Table 3. According to the values obtained, the optimum angle values for prism installation differ from the ones given in [2], and they provide the possibility to violate the inequality more strongly. We also point out that asymmetry in the detector contributions provides one with a possibility to play with this asymmetry. In particular, we found that, if the experimentalists who performed the Vienna-13 experiment simply permuted the detectors, they would get a stronger violation: $J = -123050$.

4. Optimization of the Parameters for Randomly Fluctuating Angles of the Polarization Beam Splitters

In the Eberhard model [9] and the model used for the Vienna-13 experiment [2], the optimization of experimental parameters was performed under the assumption that the angles of polarization beam splitters can be chosen exactly. The optimization provides some concrete values, and it was assumed that experimentalists can set up the experimental design with precisely these angles. However, this assumption does not match the real experimental situation. Although the precision of selecting the angles of polarization beam splitters is very high (e.g., in the Vienna-13 experiment [2], private communication), nevertheless, there are errors that can lead to deviations from the expected value of $f = J_{\mathcal{B}}(r, \omega, \theta)$. Therefore, it is important to study the problem of statistical stability of optimization with respect to random fluctuations of the angles. In the following, we present the corresponding theoretical considerations and the results of numerical optimization (again with the aid of the Nelder–Mead method).

Taking into account possible random fluctuations of the angles makes the question of optimization more complicated. As the result of such fluctuations, in the optimal point for the mathematical expectation, the dispersion is nontrivial. In principle, one can get a large magnitude of the absolute value of the mathematical expectation but, at the same time, also a large magnitude of the standard deviation. Therefore, it is natural to optimize not simply the mathematical expectation given by the function $J_{\mathcal{B}}$ but the quantity $J_{\mathcal{B}}/\sigma$.

In signal processing, the quantity $K = \mu/\sigma$, where μ is the average and σ is the standard deviation, is widely used and known as the signal-to-noise ratio (SNR); see [40, 41]. This interpretation can be used even within our framework (if we interpret random fluctuations of angles as generated by a kind of noise), although we operate not with continuous signals but with the discrete clicks of detectors. We also remark that the SNR is considered as the reciprocal of the coefficient of variation σ/μ, and it shows the extent of variability with respect to the mean of the sample.

A feature of employing the SNR or the coefficient of variation consists in the fact that, in the standard situations, they both are used only for measurements with nonnegative values. In our case, the values are negative. However, we can simply change the sign of the measurement quantity. Therefore, we proceed with negative K by taking into account that statistical meaning has to be assigned to its absolute value

– the reciprocal of the relative standard deviation (RSD), which is the absolute value of the coefficient of variation $|\sigma/\mu|$.

Now we move to the theoretical modeling of randomly fluctuating angles of polarization beam splitters. Generally, any self-adjoint quantum operator A can be represented using the spectral decomposition as $A = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda$. Then its mathematical expectation value for a state ψ can be expressed as $\overline{A}_\psi = \int_{-\infty}^{+\infty} \lambda dp_\psi(\lambda)$, where $dp_\psi(\lambda) = d\langle E_\lambda \psi, \psi\rangle$ is the probability distribution for the corresponding spectral decomposition and quantum state. Therefore, for a fixed ψ, the quantum observable can be regarded as a classical random variable with the probability distribution $p_\psi(O) = \int_O d\langle E_\lambda \psi, \psi\rangle, O \subset \mathbb{R}$.

Consider the following problem.

Let the observable A depend on some classical random variable $\omega : A = A(\omega)$ corresponding to the case where the angle values for the positions of the prisms cannot be set without an error during the experiments. In this case, the spectral decomposition $A(\omega) = \int_{-\infty}^{+\infty} \lambda dE_\lambda(\omega)$ and the density distribution function $dp_\psi(\lambda|\omega) = d\langle E_\lambda(\omega)\psi, \psi\rangle$ also depend on this random variable. For every fixed ω, the condition $\int_{-\infty}^{+\infty} dp_\psi(\lambda|\omega) = 1$ also holds.

Let the random variable ω be described using the Kolmogorov probability space $(\Omega, \mathcal{F}, P)$, where Ω is a set of elementary events, $\mathcal{F}$ is the σ-algebra of the events, and P is the probability measure. The mathematical expectation value of $A(\omega)$ reads

$$\overline{A}_\psi(\omega) = \int_\Omega \left[\int_{-\infty}^{+\infty} \lambda dp_\psi(\lambda|\omega)\right] dP(\omega) = \int_\Omega \langle A(\omega)\psi, \psi\rangle dP(\omega) = \tilde{\mathbf{E}}[\langle A(\omega)\psi, \psi\rangle],$$

where $\tilde{\mathbf{E}}[\cdot]$ is the classical mathematical expectation.

In a similar way, we obtain the expression for the dispersion $A(\omega)$,

$$\sigma^2(A) = \tilde{\mathbf{E}}[\langle A^2(\omega)\psi, \psi\rangle - \langle A(\omega)\psi, \psi\rangle^2].$$

4.1. The Model with Uniform Random Fluctuations of Four Angles of Polarization Beam Splitters

We consider a model in which the value of each angle in the experiment is uniformly distributed at a section around the desired value. In this case, the mathematical expectation and dispersion values are

$$J_\mathcal{B} = \frac{1}{16\,\delta^4} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \langle \mathcal{B}(x_1, x_2, x_3, x_4)\psi, \psi\rangle \, dx_1 \, dx_2 \, dx_3 \, dx_4,$$
$$\sigma^2 = \frac{1}{16\,\delta^4} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} \int_{-\delta}^{\delta} (\langle \mathcal{B}^2\psi, \psi\rangle - \langle \mathcal{B}\psi, \psi\rangle^2) \, dx_1 \, dx_2 \, dx_3 \, dx_4.$$

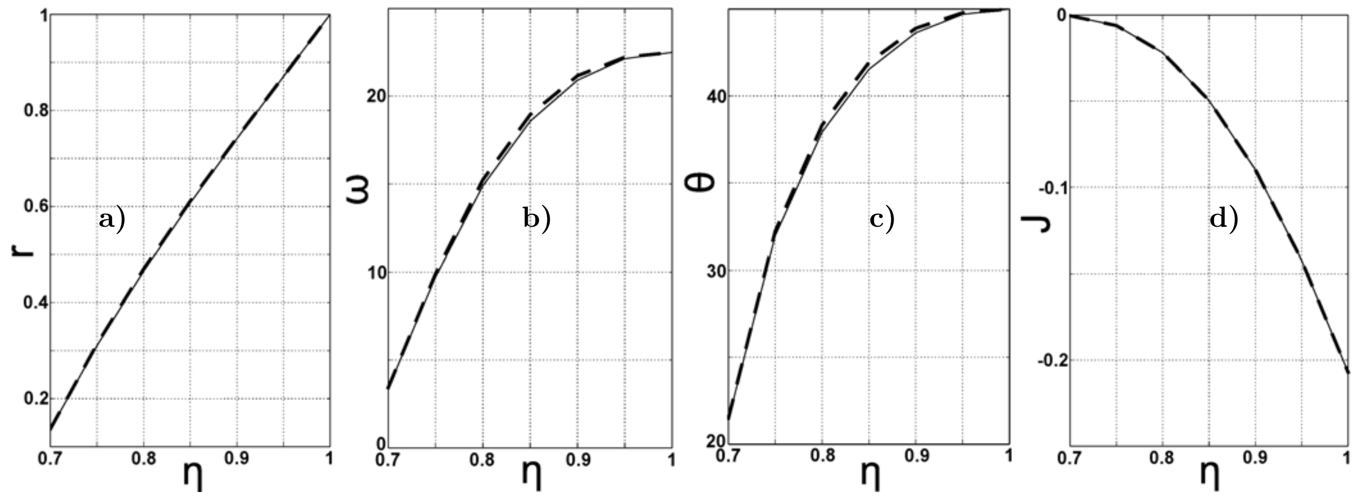

Fig. 3. Optimized values of r (a), ω (b), θ (c), and $J_{\mathcal{B}}/N$ (d) versus η for $\delta = 0$ (solid curve) and 0.25 (dashed curve).

Table 4. Optimized Values of the Parameters for the Errors in Four Angles Separately at $\delta = 0.25°$.

η	$\delta°$	r	$\omega°$	$\theta°$	J	J_δ	σ_δ	$K = J_\delta/\sigma_\delta$
0.7	0.25	0.136389	3.40081	21.4266	−0.000453562	−0.000444565	0.00241554	−0.184044
		0.136389	3.40081	21.4266	−0.000453562	−0.000444565	0.00241554	−0.184044
		0.137124	3.42997	21.496	−0.000453514	−0.000444515	0.00241503	−0.184062
0.75	0.25	0.310518	9.73143	31.9603	−0.00615095	−0.00614082	0.00248895	−2.46724
		0.310518	9.73143	31.9603	−0.00615095	−0.00614082	0.00248895	−2.46724
		0.313658	9.91344	32.2158	−0.00614786	−0.00613773	0.00248642	−2.4685
0.8	0.25	0.465228	14.8979	37.9215	−0.02191	−0.0218985	0.002596	−8.43546
		0.465228	14.8979	37.9215	−0.02191	−0.0218985	0.002596	−8.43546
		0.469841	15.2419	38.3231	−0.0218953	−0.0218838	0.00259252	−8.44116
0.85	0.25	0.607424	18.5808	41.5341	−0.0496902	−0.0496772	0.00275221	−18.0499
		0.607424	18.5808	41.5341	−0.0496902	−0.0496772	0.00275221	−18.0499
		0.61123	18.9498	41.9292	−0.0496699	−0.0496569	0.00274992	−18.0576
0.9	0.25	0.741202	20.9153	43.6381	−0.0899078	−0.0898932	0.00296469	−30.3213
		0.741202	20.9153	43.6381	−0.0899078	−0.0898932	0.00296469	−30.3213
		0.743038	21.167	43.8961	−0.0898969	−0.0898824	0.00296395	−30.3252
0.95	0.25	0.87067	22.141	44.6958	−0.142436	−0.14242	0.00323493	−44.0258
		0.87067	22.141	44.6958	−0.142436	−0.14242	0.00323493	−44.0258
		0.871004	22.2272	44.7823	−0.142435	−0.142419	0.00323486	−44.0263
1.0	0.25	0.999997	22.5	45	−0.207107	−0.207089	0.0035626	−58.1286
		0.999997	22.5	45	−0.207107	−0.207089	0.0035626	−58.1286
		0.999999	22.4981	44.998	−0.207107	−0.207089	0.0035626	−58.1286

The results of the optimization are shown in Table 4 (with rows grouped in triads) and Fig. 3. We see that the addition of random fluctuations of the angles almost does not change the optimized values of the parameters. Therefore, we can suggest that the control of the angle values can be reduced.

5. Conclusions

In this paper, we analyzed the Eberhard inequality [9].[¶]

Our goal was to find angles of polarization beam splitters and a quantum state (which is entangled but not maximum entangled) that allow one to violate the inequality as much as possible. The required parameters were found using the optimization procedure based on the Nelder–Mead optimization method [39]. We considered two models: the Eberhard model [9] and the model used for the Vienna-13 experiment; see [2, 4]. In the first case, we obtained values consistent with the values of [9]. Note that the Eberhard model describes only the case of equal detector efficiencies. However, in real experiments, the detector efficiencies may differ substantially. Therefore, it was important to perform a study similar to [9] for detectors of different efficiencies, and we did this. In the second case (Vienna-13), we obtained values of the parameters that differ slightly from the values used for the Vienna-13 experiment [2,4]. The optimum values of the angles and state parameters provide a possibility to obtain a stronger violation of the Eberhard inequality than in [2,4]. Also we point out that the model of Kofler et al. [4] is asymmetric with respect to the detector efficiencies. We explored this feature of the model and found (curious fact) that experimentalists from Vienna would be able to obtain a stronger violation of the Eberhard inequality simply by permutation of the detectors, which they used for the experiment.

In both aforementioned models, it was assumed that, in real experiments, the optimum values of the angles of polarization beam splitters (obtained as the results of optimization) can be fixed in perfect accordance with the theoretical prediction. Although this assumption is justified to a high degree, in real experiments the errors in fixing these angles are always present. Such random errors have to be taken into account, and this fact is an important part of this paper. We performed the corresponding theoretical modeling completed with numerical simulation. In this model, the magnitude of a possible spread of experimental data, which can be expressed using the reciprocal of the coefficient of variation σ_J/J (also known as the signal-to-noise ratio), was studied. The obtained parameters differ from the results of Eberhard [9], Gustina et al. [2], and Kofler et al. [4]. The simulation results can be interesting for experimenters since they allow one to weaken control over the precision of orientation of the axes of the polarization beam splitters.

The obtained results allow us to expect that, in the experimental test of a Bell-type inequality (in the Eberhard form) with the optimum values of physical parameters from this paper, the inequality will be significantly violated even for different detector efficiencies, inaccuracy in the installation angles, and without the assumption of the purity of the initial state. We hope that our study may be useful for experimentalists trying to perform a loophole-free Bell test, i.e., trying to combine closing of the detection loophole with closing of the locality loophole.

[¶]This inequality obtained in 1993 was practically forgotten. Both experimentalists and theoreticians were interested mainly in the CHSH inequality.

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