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GENERAL APPROACH TO THE CONSTRUCTION OF PHOTON-ADDED $SU(1,1)$ COHERENT STATES

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Abstract

We develop a general approach to building photon-added generalized Peremolov coherent states (PA-GPCSs) and photon-added generalized Barut–Girardello coherent states (PA-GBGCSs) associated to generalized $su(1,1)$ algebra. We study the problem of completeness of these coherent states for some particular cases and investigate the physical properties of these states through the evaluation of the Mandel parameter using an alteration of the Holstein–Primakoff realization of the $su(1,1)$ algebra. We show that these states exhibit sub-Poissonian, Poissonian, or super-Poissonian statistics. These features make the photon-added approach a good candidate for implementation of quantum optics schemes and coherent information processing.

Keywords: $SU(1,1)$ coherent states, quantum groups, generalized $su(1,1)$ algebra, Mandel parameter.

1. Introduction

Over the past four decades, various works have been devoted to the study of coherent states and their applications in different branches of physics. The concept of coherent states was first introduced by Schrödinger in the context of the harmonic oscillator in the attempt to find quantum-mechanical states, which prove a close connection between quantum and classical formulations of a given physical system [1]. The importance of coherent states was put forward by Glauber [2, 3] in quantum optics as eigenstates of the annihilation operator $\hat{a}|z\rangle = z|z\rangle$, while he realized that these states have the interesting property of minimizing the Heisenberg uncertainty relation. The same states were also reintroduced by Klauder [4, 5]. The common point here was that all these coherent states were associated to the quantum harmonic oscillator. Because of their important properties, these states were then generalized to other systems either from a physical or mathematical point of view. For a review of all these generalizations, see [6–8].

Perelomov [9] and Gilmore [10] have independently developed a set of coherent states associated to any Lie groups (not only the Heisenberg–Weyl group related to the quantum harmonic oscillator). A particular case of these states are the $SU(1,1)$ coherent states that are associated with the $SU(1,1)$ group. These states describe several systems and have interesting applications in quantum optics, statistical mechanics and condensed matter physics [6–8, 11–13].

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There are two forms of $SU(1,1)$ coherent states. The first form, often called the Peremolov $SU(1,1)$ coherent states, which are close in spirit to the Radcliffe $SU(2)$ (or spin) coherent states, is introduced by action of a unitary z−displacement ($z \in \mathcal{C}$) operator on the ground state, i.e., using an element of the corresponding group. The second form is defined as eigenstates of the lowering operator of the $SU(1,1)$ group (the $su(1,1)$ lowering operator); they are called the Barut–Girardello $SU(1,1)$ coherent states.

On the other hand, the quantum groups were introduced as a mathematical description of deformed Lie algebras [14,15] that gave the possibility to construct deformed coherent states. They were introduced as a natural extension of the notion of coherent states. Generalized deformation of coherent states were then constructed (see [16,17]) as being related to deformed harmonic oscillators. Deformed $SU(1,1)$ coherent states were also constructed as coherent states related to the quantum algebra $su_q(1,1)$ [18,19].

Mathematically, it has been noted by Klauder [4,5] that the minimum set of conditions required to construct coherent states are

- normalizability;
- continuity in the label z;
- existence of a resolution of unity with a positive definite weight function.

The last condition is certainly the most important and most restrictive, as we shall see in this paper. Determination of the existence of a resolution relation is a difficult task, which has not been solved for a large class of coherent states. In this paper, we shall give a general scheme for building coherent states associated to generalized $su(1,1)$ Lie algebra by discussing the minimum set of conditions required to construct the Klauder coherent states. Then we study the statistical properties of these coherent states through the Mandel parameter using an alteration of the Holstein–Primakoff realization of the $su(1,1)$ algebra.

This paper is organized as follows.

In Sec. 2, we define what the generalized $su(1,1)$ algebra means. In Sec. 3, we give a general scheme for building the corresponding generalized photon-added $SU(1,1)$ coherent states for both forms and study the physical properties (classicality) of these states through evaluating the Mandel parameter, using an alteration of the Holstein–Primokoff realization of the $su(1,1)$ algebra. We give our summary in the last section.

2. Generalized *su*(1,1) Algebra

The generalized $su(1,1)$ algebra is defined by the commutation relations

$$[K_-, K_+] = [2K_z], \qquad [K_z, K_\pm] = \pm K_\pm \ , \tag{1}$$

where its generators $(K_\pm, K_z)$ also obey the hermiticity properties, that is,

$$(K_+)^\dagger = K_-, \qquad (K_-)^\dagger = K_+, \qquad (K_z)^\dagger = K_z. \tag{2}$$

The box function [ ] given in Eq. (1) determines the deformation of the $su(1,1)$ algebra, i.e., choosing particular forms of this function, we obtain a specific deformation of the $su(1,1)$ algebra. For example, for $[\mathcal{X}] = \mathcal{X}$ we obtain the nondeformed classical $su(1,1)$ algebra, while the box function [20,21]

$$[\mathcal{X}] = \frac{q^{\mathcal{X}} - q^{-\mathcal{X}}}{q - q^{-1}} \, , \qquad q \in \mathbb{R}, \tag{3}$$

defines the standard deformation of the $su(1,1)$ algebra. However, there are other widely-known deformations, which can be achieved by employing the following box function [22–24]:

$$[\mathcal{X}] = \frac{q^{\mathcal{X}} - 1}{q - 1}, \qquad q \in \mathbb{R}, \tag{4}$$

$$[\mathcal{X}] = \frac{q^{\mathcal{X}} - p^{-\mathcal{X}}}{q - p^{-1}}, \qquad (p,q) \in \mathbb{R}. \tag{5}$$

Here, we consider q and p as real parameters, although they can be considered to be complex.

These deformations can be used for characterization of the photon statistics of laser outputs reasonably close to the threshold, single-atom resonance fluorescence, the micromaser field, and absorption by two-level atoms. Specifying certain values of the deformation parameter, we obtain some new algebras with interesting properties. The quantum algebras with these deformation function has been considered to be a mere mathematical curiosity that may be useful for developing exactly solvable toy models, especially for statistical and many-body physics, where exactly solvable model are rare. In fact, a wide literature concerning deformed algebras with these deformations has been developed in the last few decades in different fields of physics.

These deformations (as well as any other deformations) share the important property that one recovers the nondeformed classical $su(1,1)$ algebra for particular values of the parameters of deformation. For example, taking the limit $q \to 1$ and $p \to 1$ in Eqs. (4) and (5), we recover the nondeformed $su(1,1)$ algebra.

The unitary irreducible representation of the generalized $su(1,1)$ algebra may be obtained via the unitary representation [25] of the nondeformed classical $su(1,1)$ algebra. It is described as follows:

$$K_z|k,m\rangle = m|k,m\rangle, \qquad K_{\pm}|k,m\rangle = \left([m \pm k][m \mp k \pm 1]\right)^{1/2}|k,m \pm 1\rangle, \tag{6}$$

where $|k,m\rangle$ is the complete orthonormal basis of the space of irreducible representation, with $m \in \{k, k+1, k+2, \ldots\}$ and $k \in \{1/2, 1, 3/2, \ldots\}$ being the Bargman indices labeling the irreducible representation.

$SU(1,1)$ is a noncompact group whose unitary representations must necessarily be infinite-dimensional, and there are two principal kinds of generalized coherent states to consider within the positive discrete series – generalized Peremolov $SU(1,1)$ coherent states defined in a manner analogous to the $SU(2)$ coherent states, and generalized Barut–Girardello $SU(1,1)$ coherent states defined as eigenstates of the generalized $su(1,1)$ lowering K_-.

3. Generalized Photon-Added $SU(1,1)$ Coherent States

Following Klauder [4, 5], the minimum set of conditions to obtain coherent states are:

- Normalizability

$$\langle z|z\rangle = 1. \tag{7}$$

- Continuity in the label

$$\Big| |z\rangle - |z'\rangle \Big| \longrightarrow 0 \quad \text{when} \quad |z - z'|^2 \longrightarrow 0. \tag{8}$$

- Resolution of unity

$$\iint d\mu(z)|z\rangle\langle z| = I, \tag{9}$$

where $d\mu(z)$ is a measure in the label space.

Now we analyze the above minimum set of conditions to obtain the Klauder coherent states associated to the generalized photon-added $SU(1,1)$ coherent states. We start with a generalization of the Perelomov approach.

3.1. Photon-Added Generalized Peremolov $SU(1,1)$ Coherent States

The generalized Peremolov $SU(1,1)$ coherent states related to representations characterized by $k = 1/2, 1, 3/2, \ldots$ are defined by

$$|z,k\rangle = \frac{1}{\sqrt{\mathcal{N}(|z|^2)}}\mathbf{E}(zK_+)|k,k\rangle, \tag{10}$$

where the function $\mathbf{E}(x)$ is some sort of a generalization of the ordinary exponential function and is defined by the formula

$$\mathbf{E}(x) = \sum_{n=0}^{\infty}\frac{x^n}{[n]!} \quad \text{with} \quad [n]! = [n][n-1]\cdots[1] \quad \text{and} \quad [0]! = 1. \tag{11}$$

In general, one should pay attention to the convergence of the series defining the generalized exponential function. For example, in the standard deformation, $[n] = (q^n - q^{-n})/(q - q^{-1})$, it is easy to prove that the series converges for $|x| < (q - q^{-1})^{-1}$, while for $[n] = (q^n - 1)/(q - 1)$ and $[n] = (q^n - p^{-n})/(q - p^{-1})$ the convergence is ensured for $|x| < (q-1)^{-1}$ and $|x| < |q - p^{-1}|^{-1}$, respectively.

The normalization $\langle z,k|z,k\rangle = 1$ requires the following value for the normalization function:

$$\mathcal{N}(|z|^2) = \left(1 - |z|^2\right)_Q^{(-2k)}, \tag{12}$$

where the generalized binomial formula (generalized version of the Newton formula) is used:

$$(x+y)_Q^{(n)} := \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_Q x^{n-m}y^m\ . \tag{13}$$

Here, the generalized binomial function reads

$$\begin{bmatrix} n \\ m \end{bmatrix}_Q = \frac{[n]!}{[n]![n-m]!} \qquad \text{for} \qquad n \geq m\ . \tag{14}$$

We have used the subscript Q in the notation above to mean that the notation is generalized versions (being one parametric or multi-parametric [26]) of widely-known expressions. A particular form of the generalized binomial formula, corresponding to the deformation (14), was already known by mathematicians (see [27]) and was used in [19].

Using these definitions, we may write the GPCSs as

$$|z,k\rangle = \left(\left(1 - |z|^2\right)_Q^{-(2k)}\right)^{-1/2}\sum_{m=k}^{\infty}\left(\begin{bmatrix} m+k-1 \\ m-k \end{bmatrix}_Q\right)^{1/2} z^{m-k}|k,m\rangle. \tag{15}$$

In order to proceed with the study of statistical properties of these generalized coherent states, we need to express the basis vectors $|j,m\rangle$ in terms of the Fock states $|n\rangle$ ($|k,m\rangle \sim |n\rangle$). One way of achieving this goal is to employ an alteration of the Holstein–Primakoff realization of $su(1,1)$ algebra given by

$$K_+ = a^\dagger \sqrt{[2k+N]}, \qquad K_- = \sqrt{[2k+N]}\, a, \qquad K_z = N + k, \tag{16}$$

where a and $a^\dagger$ are generalized annihilation and creation operators acting on the Fock states $|n\rangle$, such that

$$a|n\rangle = \sqrt{[n]}\,|n-1\rangle\ ,\ a^\dagger|n\rangle = \sqrt{[n+1]}\,|n+1\rangle\ ,\ N|n\rangle = n|n\rangle\ . \tag{17}$$

They obey the following generalized commutation relations:

$$\left[a,a^+\right] = aa^+ - a^+a = \ \Delta', \quad [a,\Delta] = a\Delta - \Delta a = \Delta' a, \quad [a^+,\Delta] = a^+\Delta - \Delta a^+ = -a^+\Delta',\ \ldots\ , \tag{18}$$

where $\Delta = a^+a$, and Δ' is to be interpreted as a "generalized or deformed derivative" of Δ.

This generalized harmonic oscillator (or generalized Heisenberg algebra) was already introduced with the related coherent states constructed in [16, 26, 28].

Using this realization, one obtains the following change of variables $m = n + k$ or $n = m - k$, which applied to Eq. (15) yields the following expression of the GPCSs in terms of the Fock states:

$$|z,k\rangle = \left(\left(1-|z|^2\right)_Q^{(-2k)}\right)^{-1/2} \sum_{n=0}^{\infty} \left(\begin{bmatrix} 2k+n-1 \\ n \end{bmatrix}_Q\right)^{1/2} z^n |n\rangle, \tag{19}$$

where $z \in D^k = \{|z|^2 < Q^{k-1}\}$.

The photon-added states, first introduced by Agarwal and Tara [29] (their properties were studied in [30]), is the result of successive elementary one-photon excitations of a coherent state, and it is an intermediate state between the Fock state and the coherent state. The purpose of this paper is to construct an important special family of the states called PA-GPCSs obtained by adding photons to a conventional GPCSs; $|z.k,l\rangle$, one-photon excitations of the mode in GPCSs. These excited states can be obtained by repeated application of the raising operator K_+ to the conventional GPCSs $|z,k\rangle$

$$||z,k,l\rangle = (K_+)^l\, |z,k\rangle. \tag{20}$$

Making use of the expression

$$(K_+)^l\, |n\rangle = \sqrt{\frac{[n+l]![n+2k+l-1]!}{[n]![n+2k-1]!}}|n+l\rangle, \tag{21}$$

we arrive at the PA-GPCSs given by

$$|z,k,l\rangle = N_l\left(|z|^2\right) \sum_{n=0}^{\infty} \sqrt{\frac{[n+l]![n+2k+l-1]!}{([n]!)^2\,[2k-1]!}} z^n |n+l\rangle, \tag{22}$$

where the normalization constant N_l is

$$N_l\left(|z|^2\right) = \left(\frac{[n+l]![n+2k+l-1]!}{([n]!)^2\,[2k-1]!}|z|^{2n}\right)^{-1/2}, \tag{23}$$

and l is a positive integer, being the number of added quanta (or added photons).

In general, the first two conditions (7) and (8) are easily satisfied. This is not the case for the third condition (9). In fact, this condition restricts considerably the choice of the states to be considered. To this end, we assume the existence of a positive weight function $W_l(|z|^2)$ so that an integral over the complex plane exists and provides the result

$$\iint_{\mathcal{C}} d^2z|z,k,l\rangle\langle z,k,l|W_l\left(|z|^2\right) = \sum_n |n+l\rangle\langle n+l| = I. \tag{24}$$

Thus, in order to resolve the identity operator in (24), one should find an adequate weight function $W_l\left(|z|^2\right)$. By means of a change of the complex variables in terms of polar coordinates ($z = re^{i\theta}$), using the completeness of the states $|n+l\rangle$, and integrating on the angular variable θ, we can use the resolution of the identity operator (24) to solve the following integral equation:

$$\int_0^{+\infty} x^n N_l^2(x) W_l(x) dx = \frac{1}{\pi}\left(\frac{[n+l]![n+2k+l-1]!}{\left([n]!\right)^2 [2k-1]!}\right)^{-1}, \tag{25}$$

where $x = r^2$. Instead of solving Eq. (25) for $W(x)$, we shall study the existence of solutions for the following integrable equation:

$$\int_0^{+\infty} x^n \hat{W}_l(x) dx = \rho_l(n) \tag{26}$$

where

$$\hat{W}_l(x) = N_l^2(x) W_l(x), \qquad \rho_l(n) = \frac{1}{\pi}\frac{\left([n]!\right)^2 [2k-1]!}{[n+l]![n+2k+l-1]!}. \tag{27}$$

Equation (26) is the well-known Stieltjes moment problem, and this integral equation does not admit a general solution. In fact, whether a solution exists or not and the form of this solution in the positive case depends on the form of the box function. So, at this stage, the only thing that can be affirmed is that Eq. (15), or equivalently Eq. (19), defines PA-GPCSs iff Eq. (25), or equivalently Eq. (26), admits a solution.

For solving this integral equation, we use the Fourier transforms method — multiplying Eq. (26) by $(iy)^n/n!$ and summing over n yields

$$\int_0^{+\infty} dx\, e^{ixy}\, \hat{W}_l\left(x\right) = \sum_{n=0}^{\infty}\left(\frac{(iy)^n}{n!}\right)\rho_l(n) := \bar{W}_l(y). \tag{28}$$

In the case where the series defining the function $\bar{W}(y)$ above converges, the inverse Fourier transforms reads

$$\hat{W}_l\left(x\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixy}\, \bar{W}_l(y)\, dy. \tag{29}$$

Finally, the weight function $W(x)$ allowing for a resolution of the identity operator, in the form given in Eq. (25), is written as

$$W\left(x\right) = \frac{1}{2\pi N_l^2(x)}\int_{-\infty}^{+\infty} e^{-ixy}\, \bar{W}_l(y)\, dy. \tag{30}$$

This completes the Klauder criteria to be imposed on the PA-GPCSs introduced in this paper in order to be worth the nomenclature (suffix) coherent states.

It is important to notice that the resolution of the identity operator has been carried out for the particular forms of box function in [23] using a different approach. In fact, this is done by using the same weight function as in the nondeformed case but by altering the definition of integration and using the Jackson q-integral [27].

To examine what is going on and to get a handle on the nature of the PA-GPCSs, we need to make a comparison with the Glauber coherent states using the Holstein–Primakoff realization of the generalized $su(1,1)$ algebra. To realize this, we use the Mandel $\mathbf{Q}$-parameter, which is a good measure to decide whether the weighting distribution is Poissonian, super-Poissonian, or sub-Poissonian. This parameter is defined as [31]

$$\mathbf{Q} = \frac{\langle(\Delta N)^2\rangle - \langle N\rangle}{\langle N\rangle}, \tag{31}$$

where $\langle N\rangle$ is the average number of particles in the state $|z,k,l\rangle$, and $\langle(\Delta N)^2\rangle = \langle N^2\rangle - \langle N\rangle^2$ is the variance of the distribution. We say that the weighting distribution is Poissonian if $\mathbf{Q} = 0$, super-Poissonian if $\mathbf{Q} > 0$, and sub-Poissonian if $-1 < \mathbf{Q} < 0$.

The Mandel parameter is a useful tool to investigate the different physical properties of quantum systems that contain all information on the optical field, being completely equivalent to the density operator, and it can be interpreted as an information measure for such joint measurement. The Mandel parameter clearly distinguishes coherent states, and it can be used as a measure of the statistical properties of optical fields. It conjectured that the zero value is obtained for ordinary coherent states, which exhibits the Heisenberg uncertainty.

In order to study the influence of different parameters on the statistical properties of the PA-GPCSs, we show in Figs. 1 and 2 variations in the Mandel parameter $\mathbf{Q} = 0$ in terms of the different parameters. The dashed curve presents the variation in $\mathbf{Q}$ for $l = 0$, the dash-dotted curve is for $l = 1$, the dotted curve is for $l = 2$, and the solid curve is for $l = 4$. Figure 1 a shows the Mandel parameter of the PA-GPCSs in terms of the parameter q for various values of the number of added quanta l with $|z| = 0.6$ for small values of k. It can be seen that the Mandel parameter satisfies the inequalities $-1 \le \mathbf{Q} \le 0$ and $\mathbf{Q} > 0$, indicating sub-Poissonian, Poissonian, and super-Poissonian distributions. Interestingly, $\mathbf{Q}$ approaches zero as q approaches unity, showing that the PA-GPCSs approach and become equal to Glauber states in this limit. This means that the PA-GPCSs become more quantum-mechanical as q gets farther from the nondeformed case.

Furthermore, the Mandel parameter stabilizes at the minimum values $\mathbf{Q} = -1$ as q becomes significantly large. The dependence on the excitation parameter l shows that $\mathbf{Q}$ goes rapidly to the value -1 as l increases. Figure 1 b shows the Mandel parameter $\mathbf{Q}$ in terms of q for $k = 4$. From the figure, we can see that increasing parameter k is accompanied by quick tendency of $\mathbf{Q}$ to -1, showing that the field becomes less classical in this limit. From these results, we find that parameter l and k may enhance the quantum character in the coherent field.

In Fig. 2, we plot the Mandel parameter as a function of the amplitude $|z|$ for various values of the added number l for the case $q = 1$. From the figure we see that the parameter $\mathbf{Q}$ increases with increasing $|z|$ and attains its maximum value as $|z|$ is close to unity.

3.2. Photon-Added Generalized Barut–Girardello $SU(1,1)$ Coherent States

In this section, we discuss the generalization of the $su(1;1)$ lowering operator coherent states introduced by Barut and Girardello and their statistical properties.

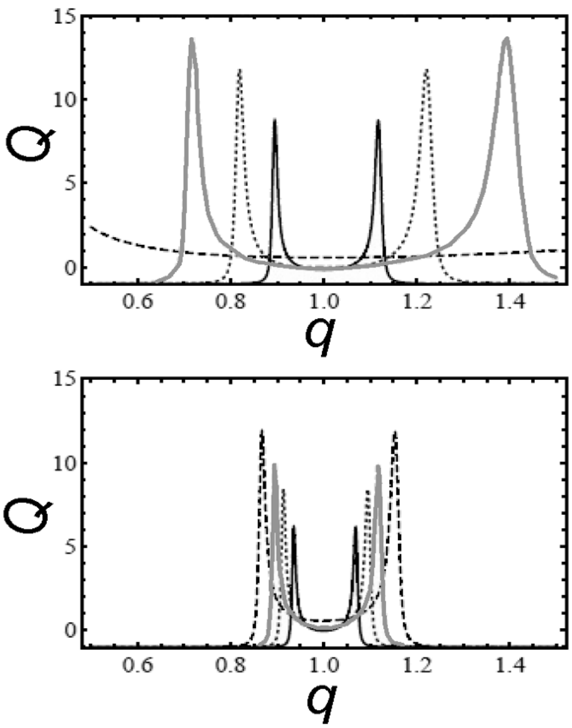

Fig. 1. The Mandel parameter of the PA-GPCS as a function of q for various values of the added number l with $|z| = 0.6$ for the box function $\left(q^n - q^{-n}\right) \Big/ \left(q - q^{-1}\right)$ at $k = 1$ (a) and $k = 4$ (b). The Mandel parameter for $l = 0$ (dashed curve), $l = 1$ (gray curve), $l = 2$ (dotted curve), and $l = 4$ (solid curve).

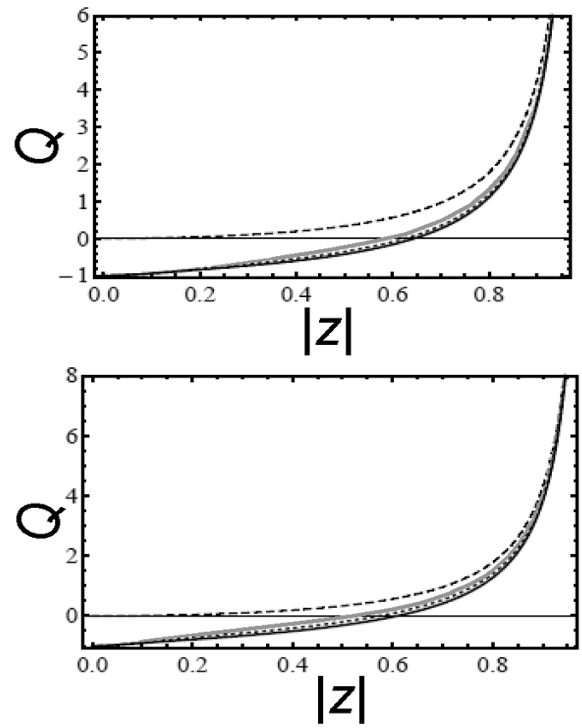

Fig. 2. The Mandel parameter of the PA-GPCS as a function of $|z|$ for various values of the number of added photons l with $q = 1$ for the box function $\left(q^n - q^{-n}\right) \Big/ \left(q - q^{-1}\right)$ at $k = 1$ (a) and $k = 4$ (b). The Mandel parameter for $l = 0$ (dashed line), $l = 1$ (gray curve), $l = 2$ (dotted curve), and $l = 4$ (solid curve).

The Barut–Girardello coherent states of the generalized $su(1,1)$ algebra are defined as the eigenstates of the lowering operator K_-,

$$K_- \left|Z,k\right\rangle = Z \left|Z,k\right\rangle \qquad\qquad Z \in C. \tag{32}$$

They can be expressed as

$$\left|Z,k\right\rangle = \mathcal{N}\left(|Z|^2\right) \sum_{m=k}^{\infty} \frac{Z^{(m-k)}}{\sqrt{[m-k]![m+k-1]!}} |k,m\rangle. \tag{33}$$

The normalization factor is given by

$$\mathcal{N}\left(|Z|^2\right) = \left(\frac{\mathcal{I}^{(Q)}_{2k-1}\left(2|Z|\right)}{|Z|^{2k-1}}\right)^{-1/2}, \tag{34}$$

where we have used the q-deformed modified Bessel function of integer order l

$$\mathcal{I}^{(Q)}_{l}\left(2Z\right) = \sum_{m=k}^{\infty} \frac{Z^{l+2n}}{[n,l]_Q!}, \qquad \mathcal{I}^{(Q)}_{-l}\left(2Z\right) = \mathcal{I}^{(Q)}_{l}\left(2Z\right), \tag{35}$$

with $[n, l]_Q! = [n]![l+n]!$.

In the nondeformed limit, the definition (35) reduces to the standard representation of a modified Bessel function. Using these definition, we may write the GBGCSs as

$$|Z,k\rangle_q = \frac{|Z|^{k-\frac{1}{2}}}{\sqrt{\mathcal{I}^{(Q)}_{2k-1}\left(2|Z|\right)}} \sum_{m=k}^{\infty} \frac{Z^{(m-k)}}{\sqrt{[m-k]![m+k-1]!}} |k,m\rangle. \tag{36}$$

Employing the Holstein–Primakoff form of the operator K_-, we write the corresponding Barut–Girardello coherent states in terms of Fock states as

$$|Z,k\rangle_q = \frac{|Z|^{k-\frac{1}{2}}}{\sqrt{\mathcal{I}^{(Q)}_{2k-1}\left(2|Z|\right)}} \sum_{n=0}^{\infty} \frac{Z^{n}}{\sqrt{[n]![2k+n-1]!}} |n\rangle. \tag{37}$$

Now we construct an important special family of states obtained by adding photons to a conventional GBGCSs $|Z,k,l\rangle$, l-photon excitations of the mode in the GBGCSs. These excited states can be obtained by repeated application of the raising operator K_+ to the conventional GBGCS $|Z,k\rangle$

$$||Z,k,l\rangle = \left(K_+\right)^l |Z,k\rangle. \tag{38}$$

In view of Eq. (21), the PA-GBGCSs are given by

$$|Z,k,l\rangle = N_l\left(|Z|^2\right) \sum_{n=0}^{\infty} \sqrt{\frac{[n+l]![n+2k+l-1]!}{\left([n]![n+2k-1]_q!\right)^2}} Z^n |n+l\rangle, \tag{39}$$

where the normalization constant N_l is

$$N_l\left(|Z|^2\right) = \left(\sum_{n=0}^{\infty} \frac{[n+l]![n+2k+l-1]!}{\left([n]![n+2k-1]_q!\right)^2} |Z|^{2n}\right)^{-1/2}, \tag{40}$$

and l is a positive integer, being the number of added quanta (or added photons).

Similarly to PA-GPCSs, the set of PA-GBGCSs $|Z,k,l\rangle$ for the generalized $su(1,1)$ algebra possesses the important property of completeness, which may be expressed as a resolution of the identity operator in the form

$$\iint_{\mathcal{C}} d^2z |Z,k,l\rangle\langle Z,k,l| W_l\left(|z|^2\right) = \sum_{n} |n+l\rangle\langle n+l| = I. \tag{41}$$

Thus, in order to resolve the identity operator in (41), one should find an adequate weight function $W_l\left(|z|^2\right)$. By means of a change of the complex variables in terms of polar coordinates ($Z = re^{i\theta}$), using the completeness of the states $|n+l\rangle$, and integrating on the angular variable θ, we can use the resolution of the identity operator (41) to solve the following integral equation:

$$\int_0^{+\infty} x^n \hat{W}_l(x) dx = \rho_l(n), \tag{42}$$

where

$$\hat{W}_l(x) = N_l^2(x) W_l(x) \text{ and } \rho_l(n) = \frac{1}{\pi} \frac{\left([n]![n+2k-1]!\right)^2}{[n+l]![n+2k+l-1]!}. \tag{43}$$

The series given in the above equation should converge. This imposes a severe restriction on the [ ]'s to be chosen, i.e., on the corresponding particular deformed $su(1,1)$ algebra.

Using the Fourier transforms method, we find that

$$\hat{W}_l\,(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixy}\ \bar{W}_l(y)\ dy. \tag{44}$$

Finally, the weight function $W_l(x)$ allowing for a resolution of the identity operator, in the form given by Eq. (41), reads

$$W\,(x) = \frac{1}{2\pi N_l^2(x)}\int_{-\infty}^{+\infty} e^{-ixy}\ \bar{W}_l(y)\ dy. \tag{45}$$

In the $Q \to 1$ and $l = 0$ limits, i.e., the nonndeformed situation with no-photon excitation, the PA-GBGCSs reduce to the expression of the ordinary Barut–Girardello coherent states with the normalization $N_0\left(|Z|^2\right) = |Z|^{k-1/2}\Big/\sqrt{I_{2k-1}\left(2|Z|\right)}$, and the weight function is given in [25].

In order to explore the influence of different parameters on the statistical properties of the PA-GBGCSs, we have displayed the variation of the Mandel parameter $\mathbf{Q}$ in terms of the different parameters in Figs. 3 and 4. The dashed curve presents the variation of $\mathbf{Q}$ for $l = 0$, the dashed-dotted curve is for $l = 1$, the dotted curve is for $l = 2$, and the solid curve is for $l = 4$.

In Fig. 3, we show the Mandel parameter of the PA-GBGCSs in terms of the parameter q for various values of the number of added quanta with $|Z| = 2$. The Mandel parameter satisfies the inequality $-1 \le \mathbf{Q} \le 0$, indicating sub-Poissonian statistics, apart from the trivial case of the vacuum field.

Interestingly, $\mathbf{Q}$ approaches zero for significantly large values of k in the case of the no-photon excitation limit $l = 0$, showing that the PA-GBGCSs approach the Glauber states in this limit. On the other hand, the parameter $\mathbf{Q}$ decreases as the deformation parameter moves away from unity. Furthermore, the Mandel parameter decreases with increasing excitation number l and stabilizes at minimum values $\mathbf{Q} = -1$ for high values of l.

Figure 4 shows the Mandel parameter of the PA-GBGCS in terms of $|Z|$ for different values of l with $q = 1$. From the figure, we see that the Mandel parameter also satisfies the inequality $-1 \le \mathbf{Q} \le 0$ for different ranges of $|Z|$, indicating the sub-Poissonian distribution. Interestingly, the Mandel parameter increases with increase in $|Z|$, showing that the field becomes more classical for $l \neq 0$.

It is well known that the study of the statistical properties is an important topic in quantum optics. Our results show that the $SU(1,1)$ coherent states provide a much richer structure than the nondeformed ones. These coherent states may be helpful to describe the states of real and ideal lasers by the proper choice of the Q-deformation parameter and Bargman index k. Moreover, these states are important in deformed quantum optics, and their use is not only of theoretical but also of some practical importance, having in mind their experimental accessibility [32].

4. Conclusions

Coherent states are one of the most important concepts used in all branches of physics such as quantum optics, quantum electrodynamics, quantization problems, quantum information, nuclear physics, atomic physics, and metrology. In this paper, we developed a general approach for constructing generalized $su(1,1)$ algebras and their associated coherent states. We investigated the minimum set of conditions required to construct the Klauder coherent states. We studied the classicality of these states in view

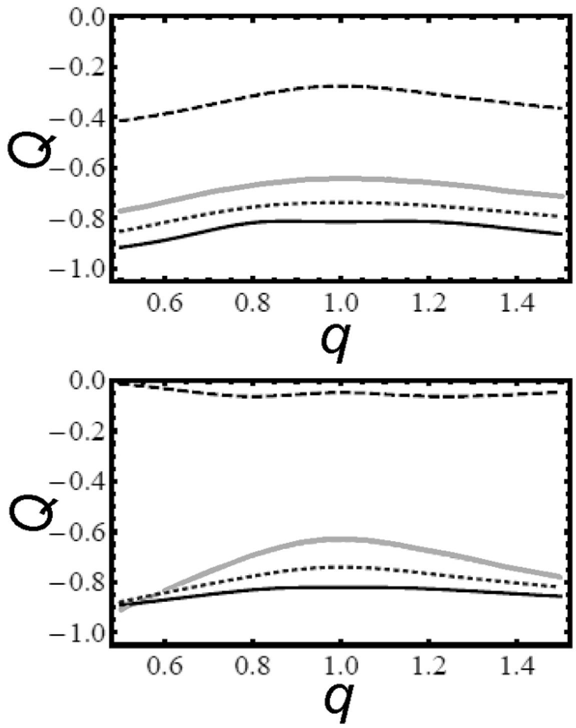

Fig. 3. The Mandel parameter of the PA-GBGCS as a function of q for various values of the number of added photons l with $|Z| = 2$ at $k = 1$ (a) and $k = 4$ (b). The Mandel parameter for $l = 0$ (dashed line), $l = 1$ (gray curve), $l = 2$ (dotted curve), and $l = 4$ (solid curve).

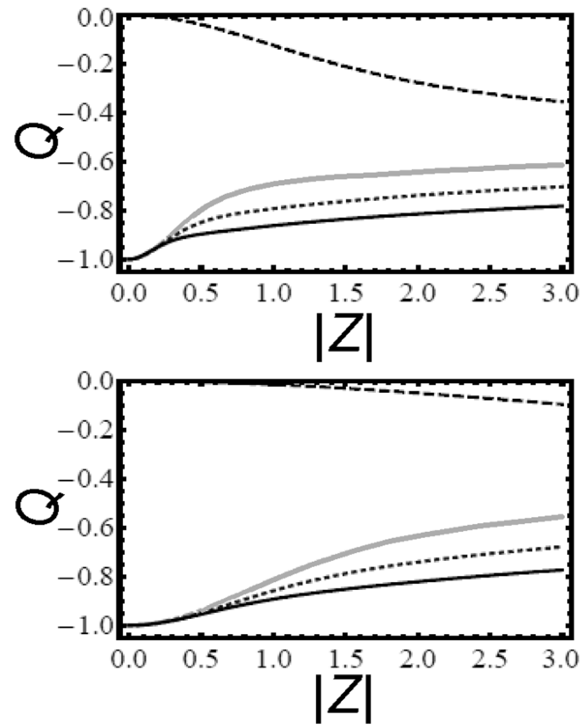

Fig. 4. The Mandel parameter of the PA-GBGCS as a function of $|Z|$ for various values of the number of added photons l with $q = 1$ at $k = 1$ (a) and $k = 4$ (b). The Mandel parameter for $l = 0$ (dashed line), $l = 1$ (gray curve), $l = 2$ (dotted curve), and $l = 4$ (solid curve).

of the Mandel parameter. In particular, we have shown that they can exhibit all types of statistics — sub-Poissonian, super-Poissonian, or even Poissonian statistics. This study is important to show the pertinence and usability of the introduced states in the domain of quantum information. Let us recall that in many applications one is interested in using highly quantum states (less classical) in order to be able to use, among other things, the entanglement of such states. In the previous works (see, for example, [33]), it was demonstrated that less classical states exhibit stronger entanglement. The results obtained can stimulate further studies and applications of the generalized $su(1,1)$ coherent states.

The PA-GPCSs and PA-GBCSs provide a much richer structure for developing different tasks of quantum information processing. They are useful to generate and measure the entanglement, and their use is not only of theoretical but also of some practical importance due to their experimental accessibility [34]. In the future, it will be important to study the entanglement generated via a beam splitter when the input state is defined in a mixed state, which makes a useful contribution for further understanding the process of correlations related to the beam-splitter device. Another interesting line is to study the entanglement dynamics of the beam output states considering the environment effects.

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