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NEW INEQUALITIES FOR QUANTUM VON NEUMANN AND TOMOGRAPHIC MUTUAL INFORMATION

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Abstract

We study the entropic inequalities related to the quantum mutual information for bipartite system and tomographic mutual information for the Werner state of two qubits. We discuss quantum correlations corresponding to the entanglement properties of the qubits in the Werner state.

Keywords: Entropic inequalities, quantum information, Werner state, tomographic probability, qudit.

1. Introduction

The two-qubit systems can demonstrate quantum correlations, and these correlations correspond to the entanglement phenomenon [1–3] or to the violation of Bell inequalities [4]. Also the correlations can be associated to quantum discord [5, 6]. The quantum discord is related to the difference in the classical Shannon information [7] and quantum information determined by the von Neumann entropy of a composite bipartite systems and the entropies of its subsystems. Recently, the tomographic probability representation of spin (qudit) states was introduced [8, 9]. In this representation, the qudit states are identified with spin tomograms that are fair probability-distribution functions determined by the density operator of the states. The relation of the density operator to the spin tomogram is invertible. Due to this, the tomogram contains complete information on the qudit state. For several qudits, the spin tomogram is also determined through the state density operator, and it is a joint probability distribution that provides the possibility to reconstruct the density operator. Since the qudit state in the tomographic probability representation is identified with the standard probability distribution, one can use all the characteristics of the distributions like the Shannon entropy and information, as well as other entropies [10, 11].

The von Neumann entropy was shown [12] to be the minimum of the spin-tomographic Shannon entropy with respect to all the unitary transforms in the Hilbert space of the qudit system. There exist different kinds of entropic inequalities for both classical and quantum systems [13–17]. The inequalities relating the spin-tomographic and von Neumann entropies were used for composite and noncomposite systems in [18–21]. A particular quantum state, which has properties to be either separable or entangled depending on the parameter values p of its density matrix, is the Werner state [22] of two qubits.

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The aim of our work is to study the tomographic Shannon and von Neumann entropies and information discussed in [21] on the example of the Werner state. We discuss quantum correlations in the state, in view of a specific characteristic of the two-qubit density matrix. This characteristic is the difference in the quantum von Neumann information and the maximum of the Shannon tomographic information taken with respect to all the local unitary transforms in the Hilbert space of the bipartite qubit systems. We calculate explicitly this characteristic and analyze this parameter behavior as a function of the Werner-state parameters.

This paper is organized as follows.

In Sec. 2, we review the tomographic probability representation of the Werner state and introduce the tomographic Shannon information and entropy for this two-qubit state. In Sec. 3, we discuss the maximum of the spin-tomographic entropy of the composite two-qubit system with respect to the local unitary transforms in the Hilbert space.

2. Entropy and Information for the Werner State

The tomographic-probability distribution for spin states provides the possibility to describe the states with the density matrix ρ of two qubits by means of tomograms. By definition, the spin tomogram reads

$$\omega(m_1, m_2, \overline{n}_1, \overline{n}_2) = \langle m_1, m_2 | U \cdot \rho \cdot U^\dagger | m_1, m_2 \rangle, \tag{1}$$

where $m_{1,2} = -j, -j+1, \ldots, j$ $(j = 0, 1/2, 1 \ldots)$ are the spin projections and U is the rotation matrix:

$$\begin{aligned} U = & \begin{pmatrix} \cos(\theta_1/2)e^{i(\varphi_1+\psi_1)/2} & \sin(\theta_1/2)e^{i(\varphi_1-\psi_1)/2} \\ -\sin(\theta_1/2)e^{i(\psi_1-\varphi_1)/2} & \cos(\theta_1/2)e^{-i(\varphi_1+\psi_1)/2} \end{pmatrix} \\ & \otimes \begin{pmatrix} \cos(\theta_2/2)e^{i(\varphi_2+\psi_2)/2} & \sin(\theta_2/2)e^{i(\varphi_2-\psi_2)/2} \\ -\sin(\theta_2/2)e^{i(\psi_2-\varphi_2)/2} & \cos(\theta_2/2)e^{-i(\varphi_2+\psi_2)/2} \end{pmatrix}. \end{aligned} \tag{2}$$

The matrix (2) is considered as the direct product of two matrices of irreducible representations of the $SU(2)$ group [23]. The Werner state of two qubits is determined by the density matrix [22] of the form

$$\rho_W(p) = \begin{pmatrix} \rho_{1111} & \rho_{1112} & \rho_{1121} & \rho_{1122} \\ \rho_{1211} & \rho_{1212} & \rho_{1221} & \rho_{1222} \\ \rho_{2111} & \rho_{2112} & \rho_{2121} & \rho_{2122} \\ \rho_{2211} & \rho_{2212} & \rho_{2221} & \rho_{2222} \end{pmatrix} = \begin{pmatrix} (1+p)/4 & 0 & 0 & p/2 \\ 0 & (1-p)/4 & 0 & 0 \\ 0 & 0 & (1-p)/4 & 0 \\ p/2 & 0 & 0 & (1+p)/4 \end{pmatrix}, \tag{3}$$

where $-1/3 \leq p \leq 1$. The parameter domain $1/3 < p \leq 1$ corresponds to the entangled state.

The eigenvalues of (3) are $\lambda_1 = (1+3p)/4$ and $\lambda_{2,3,4} = (1-p)/4$. The reduced density matrices of the first and second qubits read

$$\rho_1 = \begin{pmatrix} \rho_{1111} + \rho_{1212} & \rho_{1121} + \rho_{1222} \\ \rho_{2111} + \rho_{2212} & \rho_{2121} + \rho_{2222} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix},$$

$$\rho_2 = \begin{pmatrix} \rho_{1111} + \rho_{2121} & \rho_{1112} + \rho_{2122} \\ \rho_{1211} + \rho_{2221} & \rho_{1212} + \rho_{2222} \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}.$$

Hence the von Neumann entropies of both qubit states and the entropy of the whole system are

$$\begin{gathered} S_1 = -\mathrm{Tr}\,\rho_1 \ln \rho_1 = \ln 2, \qquad S_2 = -\mathrm{Tr}\,\rho_2 \ln \rho_2 = \ln 2, \\ S_{12} = -\mathrm{Tr}\,\rho(p) \ln \rho(p) = -\big((1+3p)/4\big) \ln \big((1+3p)/4\big) - \big(3(1-p)/4\big) \ln \big((1-p)/4\big). \end{gathered} \tag{4}$$

The quantum information is defined as $I_q = S_1 + S_2 - S_{12}$, and, obviously, it satisfies the inequality $I_q \geq 0$. To construct the state tomogram, we calculate the diagonal matrix elements of the density matrix in the unitarily rotated basis in the system's Hilbert space. The diagonal matrix elements of the matrix $U \cdot \rho \cdot U^\dagger$ are

$$\begin{aligned} \omega_{11}(\uparrow,\uparrow) &= 4^{-1}\left[p\left(\cos\theta_1\cos\theta_2 + \cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2\right) + 1\right], \\ \omega_{22}(\uparrow,\downarrow) &= 4^{-1}\left(1 - p\left(\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\right)\right), \\ \omega_{33}(\downarrow,\uparrow) &= 4^{-1}\left[1 - p\left(\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2\right)\right], \\ \omega_{44}(\downarrow,\downarrow) &= 4^{-1}\left[p\left(\cos\theta_1\cos\theta_2 + \cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2\right) + 1\right], \end{aligned} \tag{5}$$

where we introduced the notation for the tomographic probabilities given by Eq. (1), for example, such as $\omega_{11}(\uparrow,\uparrow) \equiv \omega\left(+1/2, +1/2, \overline{n}_1, \overline{n}_2\right)$. The trace of the rotated density matrix satisfies the normalization condition $\mathrm{Tr}\left(U \cdot \rho \cdot U^\dagger\right) = \omega_{11}(\uparrow,\uparrow) + \omega_{22}(\uparrow,\downarrow) + \omega_{33}(\downarrow,\uparrow) + \omega_{44}(\downarrow,\downarrow) = 1$.

The marginal distributions corresponding to the first and second qubit density matrix are

$$\begin{aligned} W_1(\uparrow,\overline{n}_1) &= \omega_{11}(\uparrow,\uparrow) + \omega_{22}(\uparrow,\downarrow), \qquad W_1(\downarrow,\overline{n}_1) = \omega_{33}(\downarrow,\uparrow) + \omega_{44}(\downarrow,\downarrow), \\ W_2(\uparrow,\overline{n}_2) &= \omega_{11}(\uparrow,\uparrow) + \omega_{33}(\downarrow,\uparrow), \qquad W_2(\downarrow,\overline{n}_2) = \omega_{22}(\uparrow,\downarrow) + \omega_{44}(\downarrow,\downarrow). \end{aligned}$$

According to the definition of Shannon entropy [7], we construct the following tomographic entropies of the qubit subsystems:

$$\begin{aligned} H_1 &= -W_1(\uparrow,\overline{n}_1)\ln W_1(\uparrow,\overline{n}_1) - W_1(\downarrow,\overline{n}_1)\ln W_1(\downarrow,\overline{n}_1) = \ln 2, \\ H_2 &= -W_2(\uparrow,\overline{n}_2)\ln W_2(\uparrow,\overline{n}_2) - W_2(\downarrow,\overline{n}_2)\ln W_2(\downarrow,\overline{n}_2) = \ln 2. \end{aligned} \tag{6}$$

The tomographic Shannon entropy of the bipartite system reads

$$H_{12} = -\omega_{11}(\uparrow,\uparrow)\ln\omega_{11}(\uparrow,\uparrow) - \omega_{22}(\uparrow,\downarrow)\ln\omega_{22}(\uparrow,\downarrow) - \omega_{33}(\downarrow,\uparrow)\ln\omega_{33}(\downarrow,\uparrow) - \omega_{44}(\downarrow,\downarrow)\ln\omega_{44}(\downarrow,\downarrow). \tag{7}$$

We define information I_t as the maximum of the sum of the difference between the sum of entropies of subsystems (6) and the entropy of the whole system (7), namely,

$$I_t = \max_{\psi_1,\psi_2,\theta_1,\theta_2}(H_1 + H_2 - H_{12}); \tag{8}$$

it satisfies the inequality $I_t \geq 0$.

3. Maximum of the Shannon Information

Introducing the notation $\widetilde{H} \equiv \widetilde{H}(\psi_1,\psi_2,\theta_1,\theta_2,p) = H_1 + H_2 - H_{12}$, in view of (5)–(7), we obtain

$$\begin{aligned} \widetilde{H} = {} & \ln 4 - 2^{-1}\ln\left[4^{-1}\big(1 - p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2 - p\cos\theta_1\cos\theta_2\big)\right] \\ & \times\big[p\cos\theta_1\cos\theta_2 + p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2 - 1\big] \\ & + 2^{-1}\ln\left[4^{-1}\big(p\cos\theta_1\cos\theta_2 + p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2 + 1\big)\right] \\ & \times\big(p\cos\theta_1\cos\theta_2 + p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2 + 1\big). \end{aligned} \tag{9}$$

To find the maximum of $\widetilde{H}$ with respect to angles ψ_1, ψ_2, θ_1, and θ_2, first we find its stationary points. Taking the first derivatives

$$\begin{aligned}
\frac{\partial(\widetilde{H})}{\partial\theta_1} &= (p/2)[\cos\theta_2\sin\theta_1-\cos(\psi_1+\psi_2)\cos\theta_1\sin\theta_2]\\
&\times\Big\{\ln\big[4^{-1}\big(1-p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2-p\cos\theta_1\cos\theta_2\big)\big]\\
&-\ln\big[4^{-1}\big(p\cos\theta_1\cos\theta_2+p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2+1\big)\big]\Big\},\\
\frac{\partial(\widetilde{H})}{\partial\theta_2} &= (p/2)\big(\cos\theta_1\sin\theta_2-\cos(\psi_1+\psi_2)\cos\theta_2\sin\theta_1\big)\\
&\times\Big\{\ln\big[4^{-1}\big(1-p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2-p\cos\theta_1\cos\theta_2\big)\big]\\
&-\ln\big[4^{-1}\big(p\cos\theta_1\cos\theta_2+p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2+1\big)\big]\Big\},\\
\frac{\partial(\widetilde{H})}{\partial\psi_1} &= \frac{\partial(\widetilde{H})}{\partial\psi_2}=(p/2)\sin(\psi_1+\psi_2)\sin\theta_1\sin\theta_2\\
&\times\Big\{\ln\big[4^{-1}\big(1-p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2-p\cos\theta_1\cos\theta_2\big)\big]\\
&-\ln\big[4^{-1}\big(p\cos\theta_1\cos\theta_2+p\cos(\psi_1+\psi_2)\sin\theta_1\sin\theta_2+1\big)\big]\Big\}
\end{aligned}$$

and equating them to zero, we obtain the critical points $\Theta^0=(\theta_1^0,\theta_2^0,\psi_1^0,\psi_2^0)$ as follows:

- $\theta_1=\theta_2=\pi n\ (n=0,1,\dots)\ \ \forall\psi_1,\psi_2$;
- $\theta_1=(\pi/2)+\pi n,\ \theta_2=\pi n\ (n=0,1,\dots)\ \ \forall\psi_1,\psi_2$;
- $\theta_1=\pi n,\ \theta_2=(\pi/2)+\pi n\ (n=0,1,\dots)\ \ \forall\psi_1,\psi_2$;
- $\theta_1=\theta_2=(\pi/2)+\pi n\ (n=0,1,\dots),\ \ \psi_1+\psi_2=\pi m$ or $\psi_1+\psi_2=(\pi/2)+\pi m\ \ (m=0,1,\dots)$;
- $\theta_1=(\pi/2)+\pi n,\ \psi_1+\psi_2=(\pi/2)+\pi m\ \ \forall\theta_2\ \ (n,m=0,1,\dots)$;
- $\theta_2=(\pi/2)+\pi n,\ \psi_1+\psi_2=(\pi/2)+\pi m\ \ \forall\theta_1\ (n,m=0,1,\dots)$.

The second differential can be written in a quadratic form $d^2\widetilde{H}(\Theta)$ with the determinant

$$\begin{vmatrix}
\dfrac{\partial^2(\widetilde{H})}{\partial\theta_1^2} & \dfrac{\partial^2(\widetilde{H})}{\partial\theta_2\partial\theta_1} & \dfrac{\partial^2(\widetilde{H})}{\partial\psi_1\partial\theta_1}\\
\dfrac{\partial^2(\widetilde{H})}{\partial\theta_1\partial\theta_2} & \dfrac{\partial^2(\widetilde{H})}{\partial\theta_2^2} & \dfrac{\partial^2(\widetilde{H})}{\partial\psi_1\partial\theta_2}\\
2\,\dfrac{\partial^2(\widetilde{H})}{\partial\theta_1\partial\psi_1} & 2\,\dfrac{\partial^2(\widetilde{H})}{\partial\theta_2\partial\psi_1} & 2\,\dfrac{\partial^2(\widetilde{H})}{\partial\psi_1^2}
\end{vmatrix}, \tag{10}$$

where we see that $\dfrac{\partial^2(\widetilde{H})}{\partial\theta\,\partial\psi_1}=\dfrac{\partial^2(\widetilde{H})}{\partial\theta\,\partial\psi_2}$.

According to the sufficient condition for an extremum, we know that, if $d^2\widetilde{H}(\Theta^0)$ is a negatively defined quadratic form, Θ^0 is a strict maximum of the function $\widetilde{H}(\psi_1,\psi_2,\theta_1,\theta_2,p)$. In view of the Sylvester criterion, if all of the leading principal minors of (10) are negative, the quadratic form $d^2\widetilde{H}(\Theta^0)$ is negative.

For example, we can take $\theta_1 = \theta_2 = (\pi/2)+\pi n$ $(n = 0,1,\ldots)$; then determinant (10) for $\psi_1+\psi_2 = \pi m$ $(m = 0,1,\ldots)$ reads $\begin{vmatrix} (p/2)z & (p/2)z & 0 \\ (p/2)z & (p/2)z & 0 \\ 0 & 0 & (p/2)z \end{vmatrix}$, where $z = \ln\bigl[(1-p)/4\bigr] - \ln\bigl[(1+p)/4\bigr]$, and it is $\begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2p^2 \end{vmatrix}$ for $\psi_1+\psi_2 = (\pi/2)+\pi m$. It is easy to show that both determinants are equal to zero. Thus, $\Theta_1^0 = (\pi/2)+\pi n, (\pi/2)+\pi n\psi_1+\psi_2 = \pi m$ and $\psi_1+\psi_2 = (\pi/2)+\pi m$, $n,m = 0,1,\ldots$ are not the extremum points. Similarly, for all other stationary points, it can be proved that the second differential (10) becomes zero. Hence there is no global extremum of the function $\widetilde{H}(\psi_1,\psi_2,\theta_1,\theta_2,p)$.

Due to the form of stationary points, we can find θ_1^0 and θ_2^0 that maximize $\widetilde{H}(\psi_1,\psi_2,\theta_1,\theta_2,p)$ with fixed angles $\psi_1+\psi_2 = \pi m$ or $\psi_1+\psi_2 = (\pi/2)+\pi m$, $m = 0,1,\ldots$

The difference of quantum information I_q and maximum of the unitary tomographic information I_t is $I_q - I_t = \triangle I \geq 0$. For fixed angles ψ_1 and ψ_2, this difference is shown for $p = 0.9$ in Fig. 1 a and for $p = 0.999$ in Fig. 1 b. We see that with increase in the parameter p the minimal value of difference I_t changes. In Fig. 1 a, the minimum value of I_t is $\sim$0.65, and in Fig. 1 b it is $\sim$0.7. Let us find its limit at $p \to 1$.

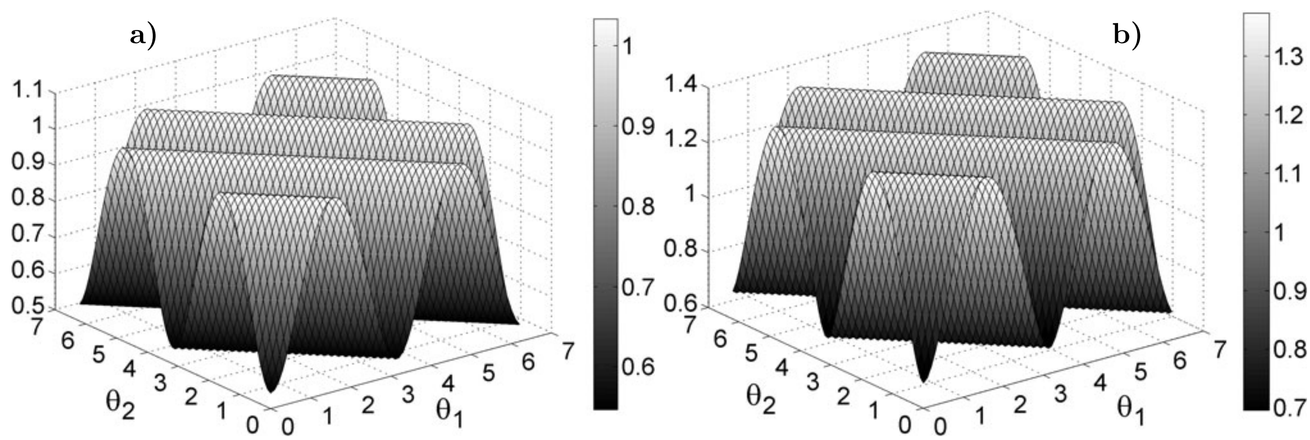

Fig. 1. Difference I_t for fixed angles ψ_1 and ψ_2 at $p = 0.9$ (a) and 0.999 (b).

To find the limit, we used the well-known relation $\lim\limits_{x\to 0} x\ln x = 0$. Now we obtain that S_{12} and I_q are

$$\lim_{p\to 1} S_{12} = 0, \quad \lim_{p\to -1/3} S_{12} = -\ln 3 \approx -1.098612, \quad \lim_{p\to 1} I_q = \ln 4, \quad \lim_{p\to -1/3} I_q = \ln 4 - \ln 3 \approx 0.287682.$$

For $\psi_1+\psi_2 = (\pi/2)+\pi m$ and $(\theta_1,\theta_2) = (\pi,\pi)$, the Shannon entropy (9) reads

$$\widetilde{H}(p) = \ln 4 - \ln[(1/4)-(p/4)][(p/2)-(1/2)] + \ln[(p/4)+(1/4)][(p/2)+(1/2)],$$

and its limits are $\lim\limits_{p\to 1} \widetilde{H}(p) = \ln 4 - \ln 2$ and $\lim\limits_{p\to -1/3} \widetilde{H}(p) = (5/3)\ln 2 + \ln 3$. We obtain the limit values of $I_q - I_t$ as follows: $\lim\limits_{p\to 1}(I_q - I_t) = \ln 2 \approx 0.693147$ and $\lim\limits_{p\to -1/3}(I_q - I_t) = (1/3)\ln 2 \approx 0.231049$.

For $\psi_1+\psi_2=(\pi/2)+\pi m$ and $(\theta_1,\theta_2)=(\pi,\pi/2)$, we have $\lim\limits_{p\to 1}\widetilde{H}(p)=0$ and $\lim\limits_{p\to -1/3}\widetilde{H}(p)=0$, and the limit values of I_q-I_t are $\lim\limits_{p\to 1}(I_q-I_t)=\ln 4\approx 1.386294$ and $\lim\limits_{p\to -1/3}(I_q-I_t)=\ln 4-\ln 3\approx 0.287682$. These limits are shown in all stationary points with varying p in Fig. 2 and with varying angle $\theta_1\in[0,2\pi]$ in Fig. 3. We see that the minimum value $\triangle I=I_q-I_t=\ln 2$ at $p\to 1$ and $(1/3)\ln 2$ at $p\to -1/3$.

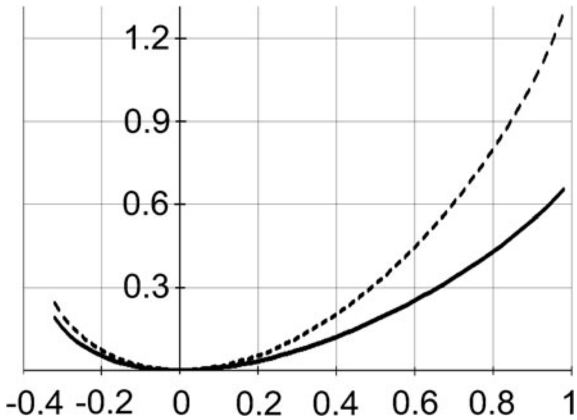

Fig. 2. $\Delta I=I_q-I_t$ at $(\theta_1=\pi,\ \theta_2=\pi,\ \psi_1+\psi_2=\pi m)$, $(\theta_1=\pi/2,\ \theta_2=\pi/2,\ \psi_1+\psi_2=\pi m)$, and $(\theta_1=\pi,\ \theta_2=\pi,\ \psi_1+\psi_2=(\pi/2)+\pi m)$ (solid curve) and $(\theta_1=\pi/2,\ \theta_2=\pi,\ \psi_1+\psi_2=\pi m)$, $(\theta_1=\pi,\ \theta_2=\pi/2,\ \psi_1+\psi_2=\pi m)$, $(\theta_1=\pi/2,\ \theta_2=\pi,\ \psi_1+\psi_2=\pi/2+\pi m)$, $(\theta_1=\pi,\ \theta_2=\pi/2,\ \psi_1+\psi_2=\pi/2+\pi m)$, and $(\theta_1=\pi/2,\ \theta_2=\pi/2,\ \psi_1+\psi_2=(\pi/2)+\pi m)$ (dashed curve).

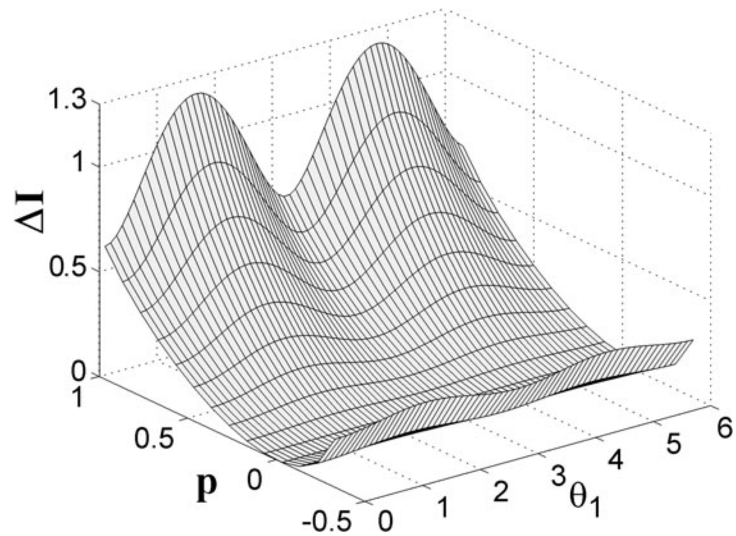

Fig. 3. $\Delta I=I_q-I_t$ for $\psi_1+\psi_2=\pi$, $-1/3<p<1$, and $\theta_1\in[0,\pi/2]$.

4. Summary

To conclude, we point out the main results of this study. We investigated correlations in the Werner state of two qubits. The difference of von Neumann information I_q and the maximum of tomographic information I_t associated with correlations in the system must be nonnegative. This is shown in Fig. 3 for a fixed $\overline{n}_2=(\theta_2,\psi_2)$ and varying $\overline{n}_1$. At $p\to 1$ (maximum entangled state), the difference goes to $\ln 2$. The studied difference characterizes the degree of quantum correlations in the two-qubit system.

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