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SUBADDITIVITY CONDITION FOR SPIN TOMOGRAMS AND DENSITY MATRICES OF ARBITRARY COMPOSITE AND NONCOMPOSITE QUDIT SYSTEMS

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Abstract

We obtain a new quantum entropic inequality for the states of a system of $n \geq 1$ qudits. The inequality has the form of the quantum subadditivity condition of a bipartite qudit system and coincides with the subadditivity condition for the system of two qudits. We formulate a general statement on the existence of the subadditivity condition for an arbitrary probability distribution and an arbitrary qudit-system tomogram. We discuss the nonlinear quantum channels creating the entangled states from separable states.

Keywords: entropy, information, tomographic probability, qubits, qudit, subadditivity condition, nonlinear quantum channels.

1. Introduction

Probability distributions are characterized by the Shannon entropy [1]. The states of quantum systems identified with the density matrices [2–5] are characterized by the von Neumann entropy. For the pure states identified with the wave functions, the von Neumann entropy is equal to zero. The entropies correspond to the order in systems [6]. For the complete order in the classical system, the Shannon entropy is equal to zero. For composite classical and quantum systems, there exist some inequalities related to the entropies of the system and its subsystems. The inequalities for the von Neumann entropies of a bipartite quantum system mean that the sum of the entropies of the subsystems is larger or equal to the entropy of the composite system. An analogous inequality holds for the Shannon entropy [1] of the bipartite system.

Recently, it was shown in [7–11] that the quantum states can be identified with tomographic probability distributions called quantum tomograms both for discrete spin (qudit) states and for the systems with continuous variables like the systems of interacting oscillators. In view of this, the inequalities known for classical probability distributions can be obtained also for quantum tomograms [12–17]. A recent review of the probability-vector properties both in classical and quantum domains is presented in [18]. Recently, it was clarified [19] that the inequalities like the subadditivity conditions known for

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bipartite systems can be found for noncomposite systems as well. The idea of this approach is based on the qubit-portrait method of qudit states suggested in [20]; it was applied to studying the entanglement properties of bipartite qudit systems in [21].

There exist [22] some inequalities for the von Neumann entropy of bipartite systems connecting "classical" and quantum entropies. The aim of our paper is to use the approach of extending the inequalities known for composite systems considered in [23] and obtain new inequalities for tomographic entropies for both the composite and noncomposite quantum systems. The model of quantum mechanics based on the classical Gaussian probability distributions was elaborated in [24–26].

This paper is organized as follows.

In Sec. 2, we discuss the probability vectors and entropic inequality for bipartite systems. In Sec. 3, we generalize the subadditivity condition for an arbitrary probability vector $\mathbf{P}$. In Sec. 4, we review the method of the portrait of the density matrices. In Sec. 5, we consider, as an example, the system states with density 6×6 matrices. In Sec. 6, we present nonlinear chains of maps of the probability vectors. We give the conclusions and prospectives in Sec. 7.

2. Probability Vectors and Entropic Inequalities for Bipartite Systems

We consider a set of N nonnegative numbers $p_1, p_2, \ldots, p_N$, such that $\sum_{k=1}^N p_k = 1$. The set of the numbers can be identified with a probability vector $\mathbf{P} = (p_1, p_2, \ldots, p_N)$, where the numbers p_k $(k = 1, 2, \ldots, N)$ are related to the results of measuring a system's random variable. The variable is assumed to give N different values. The numbers p_k provide the probability to get the k^{th} value of the random variable. For systems of qudits, the components of the probability vector $\mathbf{P}$ can be identified with n values of qudit-state tomograms $w(\mathbf{m}, u) = \langle \mathbf{m}|u\rho u^{+}|\mathbf{m}\rangle$, where ρ is the density matrix, u is the unitary matrix, and the vector $\mathbf{m} = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_n)$, with $\mathbf{m}_k = (-j_k, -j_k + 1, \ldots, j_k)$ being the spin j_k projection.

If one considers a system that contains two subsystems (bipartite system), measuring the values of two random variables gives a table of $n = NM$ nonnegative numbers p_{kj} $(k = 1, 2, \ldots, N,\ j = 1, 2, \ldots, M)$. The numbers provide the joint probability distribution associated with the results of measuring two random variables. The joint probability distribution is normalized, i.e.,

$$\sum_{k=1}^{N}\sum_{j=1}^{M} p_{kj} = 1. \tag{1}$$

If one measures only one of these two random variables, the joint probability distribution determines the marginal probability distribution

$$\mathcal{P}_k = \sum_{j=1}^{M} p_{kj}, \qquad \sum_{k=1}^{N} \mathcal{P}_k = 1. \tag{2}$$

The other marginal probability distribution describing the results of measuring the second random variable reads

$$\Pi_j = \sum_{k=1}^{N} p_{kj}, \qquad \sum_{j=1}^{M} \Pi_j = 1. \tag{3}$$

If the random variables are independent (there is no correlations between the subsystems of the bipartite system), the numbers p_{kj} have the factorized form

$$p_{kj} = \mathcal{P}_k \Pi_j. \tag{4}$$

The marginal distributions can be associated with two probability vectors $\vec{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_N)$ and $\mathbf{\Pi} = (\Pi_1, \Pi_2, \ldots, \Pi_M)$. The table of numbers p_{kj} can be described by the probability vector $\mathbf{P}$. In fact, any column vector can be considered as a rectangular matrix. Then the vector (rectangular matrix) $\mathbf{P}$ is expressed in terms of two rectangular matrices (vector $\vec{\mathcal{P}}$ and $\mathbf{\Pi}$) as their direct product

$$\mathbf{P} = \vec{\mathcal{P}} \otimes \mathbf{\Pi}, \qquad \mathbf{P} = (P_1, P_2, \quad \ldots, \quad P_{NM}). \tag{5}$$

This means that we use an invertible map of natural numbers onto pairs of integers $n \Longleftrightarrow (kj)$, which explicitly reads

$$1 \Longleftrightarrow 11, \quad 2 \Longleftrightarrow 21, \quad \ldots, \quad N \Longleftrightarrow N1, \quad N+1 \Longleftrightarrow 21, \quad \ldots, \quad n \Longleftrightarrow NM. \tag{6}$$

In fact, we code the natural numbers $1, 2, \ldots, n = NM$ by pairs of the natural numbers (kj) where $k = 1, 2, \ldots, N$, $j = 1, 2, \ldots, M$.

For simplicity, we assume that $N \leq M$.

Any probability distribution is characterized by the Shannon entropy [1]. For example, the joint probability distribution p_{kj} for a bipartite system has the Shannon entropy $H(1,2)$ determined as follows:

$$H(1,2) = -\sum_{k=1}^{N}\sum_{j=1}^{M} p_{kj} \ln\, p_{kj}. \tag{7}$$

The marginal probability distributions have the Shannon entropies $H(1)$ and $H(2)$ of the form

$$H(1) = -\sum_{k=1}^{N} \mathcal{P}_k \,\ln \mathcal{P}_k, \qquad H(2) = -\sum_{j=1}^{M} \Pi_j \,\ln \Pi_j. \tag{8}$$

It is worth noting that one can write the entropy $H(1,2)$ in the form

$$H \equiv H(1,2) = -\sum_{n=1}^{NM} P_n \ln\, P_n. \tag{9}$$

For all these entropies, we introduce the vector notation. The entropy

$$H = -\mathbf{P}\ln \mathbf{P}, \qquad H(1) = -\vec{\mathcal{P}}\,\ln\,\vec{\mathcal{P}}, \qquad H(2) = -\mathbf{\Pi}\,\ln \mathbf{\Pi}. \tag{10}$$

In formula (10) we used the definition $\mathbf{x}\ln \mathbf{x} \equiv \sum_{\alpha=1}^{L} x_\alpha \ln x_\alpha$, which means that $\mathbf{x} = (x_1, x_2, \ldots, x_L)$ and $\alpha = 1, 2, \ldots, L$.

Employment of the vector notation provides the possibility to describe the Shannon entropy of a bipartite system with two random variables associated with the joint probability distribution p_{kj} and the system with one random variable associated with the probability distribution p_k by identical formulas (10). The only difference in expressions for H, $H(1)$ and $H(2)$ consists in the fact that the “scalar product” in (10) is evaluated for the vectors with different numbers of components. In (10), the vector $\vec{\mathcal{P}}$ has N components, the vector $\mathbf{\Pi}$ has M components, and the vector $\mathbf{P}$ has $n = NM$ components. This difference can be removed. In fact, since $\lim_{x\to 0} x \ln x = 0$, we can consider vectors $\vec{\mathcal{P}}$ and $\mathbf{\Pi}$ as vectors with $n = NM$ components by adding the zero components to the initial vectors, i.e.,

$$\vec{\mathcal{P}} = (\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_N, 0, 0, \dots, \mathcal{P}_{MN} = 0), \tag{11}$$

$$\mathbf{\Pi} = (\Pi_1, \Pi_2, \dots, \Pi_M, 0, 0, \dots, \Pi_{NM} = 0). \tag{12}$$

In using these new vectors, we do not change the values of entropies, i.e., in formulas (10) we have the same expressions, but all the probability vectors $\mathbf{P}$, $\vec{\mathcal{P}}$ and $\mathbf{\Pi}$ are considered as vectors with $n = NM$ components.

It is known that the marginals $\mathcal{P}_k$ and Π_j of the joint probability distribution p_{kj} satisfy the entropic inequality called the subadditivity condition, which reads

$$H(1,2) \leq H(1) + H(2), \tag{13}$$

where the Shannon entropies are given by (6)–(9). In the explicit form, this inequality reads

$$-\sum_{k=1}^{N}\sum_{j=1}^{M} p_{kj} \ln p_{kj} \leq -\sum_{k=1}^{N} \mathcal{P}_k \ln \mathcal{P}_k - \sum_{j=1}^{M} \Pi_j \ln \Pi_j. \tag{14}$$

For the case of independent random variables $p_{kj} = \mathcal{P}_k \Pi_j$, one has equality

$$H(1,2) = H(1) + H(2). \tag{15}$$

The Shannon mutual information is defined as the difference of entropies

$$I = H(1) + H(2) - H(1,2). \tag{16}$$

This information satisfies the nonnegativity condition $I \geq 0$.

In view of the vector notation, we can rewrite the subadditivity condition (14) in the form

$$-\mathbf{P} \ln \mathbf{P} \leq -\vec{\mathcal{P}} \ln \vec{\mathcal{P}} - \mathbf{\Pi} \ln \mathbf{\Pi}, \tag{17}$$

where all the probability vectors have NM components.

The Shannon information is expressed in terms of the probability n-vectors as

$$I = -\vec{\mathcal{P}} \ln \vec{\mathcal{P}} - \mathbf{\Pi} \ln \mathbf{\Pi} + \mathbf{P} \ln \mathbf{P}, \tag{18}$$

where $n = NM$.

3. Generalization of the Subadditivity Condition for an Arbitrary Probability Vector P

The subadditivity condition (17) written as an inequality for three probability n-vectors $\mathbf{P}$, $\vec{\mathcal{P}}$, and $\mathbf{\Pi}$ provides the possibility to generalize the inequality and to prove that such inequality takes place for arbitrary probability n-vectors. To clarify this issue, we express the n-vectors $\vec{\mathcal{P}}$ and $\mathbf{\Pi}$ in terms of two stochastic $n \times n$ matrices M_{12} and M_{21} and the vector $\mathbf{P}$.

In fact, one can easily observe that the following equalities hold

$$\vec{\mathcal{P}} = M_{12}\mathbf{P}, \qquad \mathbf{\Pi} = M_{21}\mathbf{P}, \tag{19}$$

where the stochastic matrices M_{12} and M_{21} read

$$M_{12} = \begin{pmatrix} 1_M & 0_M & \dots & 0_M \\ 0_M & 1_M & \dots & 0_M \\ \dots & \dots & \dots & \dots \\ 0_M & 0_M & \dots & 1_M \\ & 0_S & & \end{pmatrix}, \qquad M_{21} = \begin{pmatrix} 1_N & 0_N & \dots & 0_N \\ 0_N & 1_N & \dots & 0_N \\ \dots & \dots & \dots & \dots \\ 0_N & 0_N & \dots & 1_N \\ & 0_Q & & \end{pmatrix}, \tag{20}$$

and the rectangular matrices 1_M and 0_M with one row and M columns are

$$1_M = (1, 1, \dots, 1), \qquad 0_M = (0, 0, \dots, 0). \tag{21}$$

The zero rectangular matrix 0_S has $NM - N$ rows and NM columns. The $N \times N$ blocks in the matrix M_{21} are the unity $N \times N$ matrix 1_N and zero $N \times N$ matrix 0_N. The zero matrix 0_Q contains $NM - M$ rows and NM columns. Using formula (19), we rewrite the subadditivity condition (17) known for the joint probability distribution of a bipartite system in the form

$$-\mathbf{P} \ln \mathbf{P} \leq -(M_{12}\mathbf{P}) \ln(M_{12}\mathbf{P}) - (M_{21}\mathbf{P}) \ln(M_{21}\mathbf{P}). \tag{22}$$

We obtain inequality (22) as a property of the joint probability distribution of the bipartite system. But it is obvious that this inequality is the inequality that is valid for an arbitrary set of $n = NM$ nonnegative numbers $(p_1, p_2, \dots, p_{NM})$. In view of this fact, one can formulate the general statement: Given arbitrary probability vector $\mathbf{P}$ with n components, where the integer n can be presented as the product of two integers $n = NM$, $N \leq M$, inequality (22) holds, where matrices (20) are two stochastic matrices containing only zeros and unities.

Inequality (22) is valid also for all $n!$ vectors $\mathbf{P}_{\rm per}$ obtained from the initial vector $\mathbf{P}$ by means of permutations of the indices $(1, 2, \dots, n)$ labeling the vector components; this means that

$$-\mathbf{P} \ln \mathbf{P} = -\mathbf{P}_{\rm per} \ln \mathbf{P}_{\rm per} \leq -(M_{12}\mathbf{P}_{\rm per}) \ln(M_{12}\mathbf{P}_{\rm per}) - (M_{21}\mathbf{P}_{\rm per}) \ln(M_{21}\mathbf{P}_{\rm per}). \tag{23}$$

It is worth noting that an integer n can have different product decompositions $n = \bar{N}\bar{M}$. Equalities (22) and (23) take place also for new matrices $\bar{M}_{12}, \bar{M}_{21}$ obtained from (20) by the substitutions $N \to \bar{N}$ and $M \to \bar{M}$. Inequality (22) holds for an arbitrary probability vector $\mathbf{P}$ corresponding to a point on the simplex, including vectors that have some zero components. We employ this remark to extend our inequality (22) to arbitrary probability n-vectors, including the case of prime number n.

To write the inequality for such probability n-vector $\mathbf{P}$, we construct a new vector

$$\mathbf{P'} = (p_1, p_2, \dots, p_n, 0, 0, \dots, p_{n'} = 0).$$

The n'-vector $\mathbf{P'}$ has n' components. We added the appropriate quantity of zero components to the initial n-vector $\mathbf{P}$, such that the new integer n' has the product form $n' = N'M'$. It is clear that there are many ways to construct such vectors with different integers $n' \geq n$. All these vectors will satisfy the subadditivity condition.

The other generalization of obtained inequality can be formulated for an arbitrary set of nonnegative numbers $x_1, x_2, \ldots, x_n$. These numbers correspond to a point on the cone.

Using the map $x_k \to p_k = \dfrac{x_k}{\sum_{j=1}^n x_j}$ and applying inequality (22) to the vector $\mathbf{P} = \dfrac{\mathbf{x}}{\sum_{j=1}^n x_j}$, we obtain the inequality for an arbitrary finite set of n nonnegative numbers x_k, i.e.,

$$-\mathbf{x}\ln\mathbf{x} \leq -(M_{12}\mathbf{x})\ln(M_{12}\mathbf{x}) - (M_{21}\mathbf{x})\ln(M_{21}\mathbf{x}) + \Big(\sum_{j=1}^n x_j\Big)\ln\Big(\sum_{j=1}^n x_j\Big). \tag{24}$$

Thus, we proved that the coordinates of a point on the cone satisfy an analog of the subadditivity condition with extra terms in the right-hand side of (24).

For arbitrary integers n, the stochastic matrices M_{12} and M_{21} can be written in a fixed canonical form. We can introduce information on the cone, which is the difference in the right-hand side and left-hand side of Eq. (24), namely,

$$I_{\mathbf{x}} = -(M_{12}\mathbf{x})\ln(M_{12}\mathbf{x}) - (M_{21}\mathbf{x})\ln(M_{21}\mathbf{x}) + \mathbf{x}\ln\mathbf{x} + \Big(\sum_{j=1}^n x_j\Big)\ln\Big(\sum_{j=1}^n x_j\Big). \tag{25}$$

If $\sum_{j=1}^n x_j = 1$, we have the point on the simplex and information $I_{\mathbf{x}}$ becomes an analog of the Shannon information, which we introduced for an arbitrary probability distribution described by the probability vector $\mathbf{P}$; it reads

$$I_{\mathbf{p}} = -(M_{12}\mathbf{P})\ln(M_{12}\mathbf{P}) - (M_{21}\mathbf{P})\ln(M_{21}\mathbf{P}) + \mathbf{P}\ln\mathbf{P} \geq 0. \tag{26}$$

There exist $n!$ informations $I_{\mathbf{p}}$ obtained from (26) by replacing the probability vector $\mathbf{P} \to \mathbf{P}_{\text{per}}$.

In the case of bipartite systems and $n = NM$, where N and M correspond to outcomes of two random variables, the information $I_{\mathbf{p}}$ coincides with the Shannon mutual information. The meaning of the introduced information $I_{\mathbf{p}_{\text{per}}}$ and information (26) introduced for an arbitrary probability vector $\mathbf{P}$ needs additional clarification.

4. Portrait of Density Matrices

We apply the analogous method to obtain the positive map of $n \times n$ density matrix $\rho(1,2)$ of a bipartite system with $n = NM$, $N \leq M$. In fact, if the state $\rho(1,2)$ is a simply separable state, i.e., $\rho(1,2) = \rho(1) \otimes \rho(2)$, $\rho(1)$ is the $N \times N$ matrix, and $\rho(2)$ is the $M \times M$ matrix, we see that the $N \times N$ matrix $\rho(1)$ is obtained by the following procedure.

The matrix elements $\rho_{kj}(1)$, $k, j, = 1, 2, \ldots, N$ are given as the first N vector components of NM vectors $\vec{\rho}_1(1), \vec{\rho}_2(1), \ldots, \vec{\rho}_N(1)$, where

$$\vec{\rho}_1(1) = M_{12}\mathbf{R}_1, \qquad \vec{\rho}_2(1) = M_{12}\mathbf{R}_2, \quad \ldots, \qquad \vec{\rho}_N(1) = M_{12}\mathbf{R}_N. \tag{27}$$

Here the NM matrix M_{12} is given by Eq. (20). The NM vectors $\mathbf{R}_j$, $j = 1, 2, \ldots N$, have the components

$$(\mathbf{R}_j)_{k\alpha} = \rho_{kj}(1)\rho_{\alpha\alpha}(2), \qquad k = 1, 2, \ldots, N,\ \alpha = 1, 2, \ldots, M. \tag{28}$$

Thus, we used the invertible map of integers $1, 2, \ldots, N$, $\alpha = 1, \ldots, M$ onto the pairs of integers k, α $1 \leftrightarrow 11$, $2 \leftrightarrow 21, \ldots$, $n \leftrightarrow NM$ to label the components of the vector $\mathbf{R}_j$.

If the matrix $\rho(1,2)$ has the generic form with matrix elements $\rho_{k\alpha j\beta}(1,2)$, we obtain the positive map $\rho(1,2) \to \rho(1)$ given by the same formula (27) with changed vectors $\mathbf{R}_j$. The vectors $\mathbf{R}_j$ have the components

$$(\mathbf{R}_j)_{k\alpha} = \rho_{k\alpha\, j\alpha}(1,2). \tag{29}$$

There is no summation over indices α. Thus, the described construction provides the map of the NM density matrix $\rho(1,2)$ onto the density matrix, which can also be considered as the NM matrix $\bar{\rho}(1)$ of the form

$$\bar{\rho}(1) = \begin{pmatrix} \rho(1) & 0_1 \\ 0_1^{\rm tr} & 0_{n-N} \end{pmatrix}, \tag{30}$$

where 0_1 is a zero rectangular matrix with N rows and $n-N$ columns, and the matrix 0_{n-N}, where $n = NM$ has the zero matrix elements.

An analogous construction can be applied to obtain the map $\rho(1,2) \to \rho(2) = {\rm Tr}_1 \rho(1,2)$. The explicit form of this map can be obtained from (30), in view of the known matrix of map of vectors $\mathbf{a}\otimes\mathbf{b} \longleftrightarrow \mathbf{b}\otimes\mathbf{a}$ given by the matrix S, such that

$$(\mathbf{a}\otimes\mathbf{b})_k = \sum_{m=1}^{n} S_{km} (\mathbf{b}\otimes\mathbf{a})_m. \tag{31}$$

Using the matrix S, we can reduce the problem of finding the expression for the matrix $\rho(2)$ to the problem discussed above with the replacement $1 \leftrightarrow 2$, $N \leftrightarrow M$.

The NM matrices $\bar{\rho}(1)$ and $\bar{\rho}(2)$ satisfy the subadditivity condition

$$-{\rm Tr}\bar{\rho}(1) \ln \bar{\rho}(1) - {\rm Tr}\bar{\rho}(2) \ln \bar{\rho}(2) \geq -{\rm Tr}\bar{\rho}(1,2) \ln \bar{\rho}(1,2). \tag{32}$$

Thus, for an arbitrary $n{\times}n$ matrix ρ, where $n = NM$, we can obtain two matrices $\rho(1)$ and $\rho(2)$ applying the map, which naturally can be applied to the bipartite matrix $\rho(1,2)$, to the initial matrix ρ. The matrix ρ can be considered as the density matrix of only one qudit. Nevertheless, the matrices $\bar{\rho}(1)$ and $\bar{\rho}(2)$ associated with it satisfy the subadditivity condition (32).

5. Example of System States with 6×6 Matrices

To demonstrate our approach, we consider the example of $n = 6$.

We can consider the density matrix ρ_{kj}, k, $j = 1, 2, \ldots, 6$ as the density matrix of one qudit with $j = 5/2$. The matrix ρ reads

$$\begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} & \rho_{15} & \rho_{16} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} & \rho_{25} & \rho_{26} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} & \rho_{35} & \rho_{36} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} & \rho_{45} & \rho_{46} \\ \rho_{51} & \rho_{52} & \rho_{53} & \rho_{54} & \rho_{55} & \rho_{56} \\ \rho_{61} & \rho_{62} & \rho_{63} & \rho_{64} & \rho_{65} & \rho_{66} \end{pmatrix} \equiv \begin{pmatrix} \rho^{(1)} & \rho^{(2)} \\ \rho^{(3)} & \rho^{(4)} \end{pmatrix}, \tag{33}$$

where the matrices $\rho^{(k)}$, $k = 1, 2, 3, 4$, are 3×3 matrices, which constitute ρ. We take integers $N = 2$ and $M = 3$. The 6×6 stochastic matrix M_{12} and 6-vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ read

$$M_{12} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{R}_1 = \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \rho_{33} \\ \rho_{41} \\ \rho_{52} \\ \rho_{63} \end{pmatrix}, \qquad \mathbf{R}_2 = \begin{pmatrix} \rho_{14} \\ \rho_{25} \\ \rho_{36} \\ \rho_{44} \\ \rho_{55} \\ \rho_{66} \end{pmatrix}. \tag{34}$$

Applying the matrix M_{12} to vectors $\mathbf{R}_1$ and $\mathbf{R}_2$, we obtain the 2×2 matrix $\rho(1)$ of the form

$$\rho(1) = \begin{pmatrix} \rho_{11} + \rho_{22} + \rho_{33} & \rho_{14} + \rho_{25} + \rho_{36} \\ \rho_{41} + \rho_{52} + \rho_{63} & \rho_{44} + \rho_{55} + \rho_{66} \end{pmatrix}. \tag{35}$$

The 6×6 matrix $\bar{\rho}(1)$ reads

$$\bar{\rho}(1) = \begin{pmatrix} \rho(1) & 0_{24} \\ 0_{24}^{tr} & 0_4 \end{pmatrix}, \qquad 0_{24} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{36}$$

and 0_4 is the 4×4 matrix with zero matrix elements. The matrix $\bar{\rho}(2)$ reads

$$\bar{\rho}(2) = \begin{pmatrix} \rho(2) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho(2) = \rho^{(1)} + \rho^{(4)}. \tag{37}$$

In the general case of $n \times n$ matrix ρ, one has an analogous map $\rho \to \bar{\rho}(1)$, $\rho \to \bar{\rho}(2)$.

The $M \times M$ matrix $\rho(2)$ is equal to the sum of N blocks of the matrix ρ, i.e.,

$$\rho(2) = \sum_{k=1}^{N} \rho^{(k)}. \tag{38}$$

Each block $\rho^{(k)}$ is the $M \times M$ matrix. The N blocks constitute the density matrix of the state, which is obtained by the "decoherence" map from the initial matrix ρ. Namely, from ρ we construct the block-diagonal matrix

$$\rho_d = \begin{pmatrix} \rho^{(1)} & 0 & 0 \\ 0 & \rho^{(2)} & 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & \rho^{(N)} \end{pmatrix},$$

keeping N of the $M \times M$ matrices, and the other matrix elements are assumed to be equal to zero. Then, we take the sum of all these blocks. As a result, we obtain the matrix $\rho(2)$. So, the matrix $\bar{\rho}(2)$ is a "portrait" of the initial matrix ρ. The other "portrait" is the matrix $\bar{\rho}(1)$, which for the initial 6×6 matrix ρ is given by Eq. (36). There is the possibility to construct another map of the matrix ρ onto two "portrait" matrices $\bar\rho(1)$ and $\bar\rho(2)$, namely, we take $N=3$ and $M=2$. Then the 3×3 matrix $\rho(1)$ and 2×2 matrix $\rho(2)$ read

$$\rho(1)=\begin{pmatrix}\rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} & \rho_{15}+\rho_{26}\\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} & \rho_{35}+\rho_{46}\\ \rho_{51}+\rho_{62} & \rho_{53}+\rho_{64} & \rho_{55}+\rho_{66}\end{pmatrix},\qquad \rho(2)=\begin{pmatrix}\rho_{11}+\rho_{33}+\rho_{55} & \rho_{12}+\rho_{34}+\rho_{56}\\ \rho_{21}+\rho_{43}+\rho_{65} & \rho_{22}+\rho_{44}+\rho_{66}\end{pmatrix}. \qquad (39)$$

The subadditivity inequality for all the pairs $\rho(1)$ and $\rho(2)$ obtained (i.e., $\bar\rho(1)$ and $\bar\rho(2)$) is given as

$$-\mathrm{Tr}\big(\rho(1)\ln\rho(1)\big)-\mathrm{Tr}\big(\rho(2)\ln\rho(2)\big)=-\mathrm{Tr}\big(\bar\rho(1)\ln\bar\rho(1)\big)-\mathrm{Tr}\big(\bar\rho(2)\ln\bar\rho(2)\big)\geq-\mathrm{Tr}(\rho\ln\rho). \qquad (40)$$

The von Neumann quantum mutual information is given by the difference

$$I_q\big(\bar\rho(1),\bar\rho(2)\big)=-\mathrm{Tr}\big(\bar\rho(1)\ln\bar\rho(1)\big)-\mathrm{Tr}\big(\bar\rho(2)\ln\bar\rho(2)\big)+\mathrm{Tr}(\rho\ln\rho). \qquad (41)$$

Inequalities for entropies (40) are valid for $(NM)!$ matrices $\bar\rho(1)$ and $\bar\rho(2)$ obtained by means of all permutations of integers $1,2,\ldots,n\to 1_p,2_p,\ldots,n_p$, determining matrix elements of the $n\times n$ matrix ρ. It is worth noting that if one has any $n\times n$ density matrix ρ, one can construct the matrix $\rho_{n'}=\begin{pmatrix}\rho & 0\\ 0 & 0\end{pmatrix}$, where $n'=n+p=NM$.

After this is done, one obtains by the described procedure all the maps $\rho_{n'}\to\bar\rho^{(n')}(1)$ and $\bar\rho^{(n')}(2)$. We write the new subadditivity conditions for these matrices

$$-\mathrm{Tr}\big(\bar\rho^{(n')}(1)\ln\bar\rho^{(n')}(1)\big)-\mathrm{Tr}\big(\bar\rho^{(n')}(2)\ln\bar\rho^{(n')}(2)\big)\geq-\mathrm{Tr}\big(\rho^{(n')}\ln\rho^{(n')}\big)=-\mathrm{Tr}(\rho\ln\rho). \qquad (42)$$

Analogously, the nonnegative mutual information is given by the difference

$$I'_\rho\big(\rho^{(n')}(1)),\rho^{(n')}(2)\big)=-\mathrm{Tr}\big(\bar\rho^{(n')}(1)\ln\bar\rho^{(n')}(1)\big)-\mathrm{Tr}\big(\bar\rho^{(n')}(2)\ln\bar\rho^{(n')}(2)\big)+\mathrm{Tr}(\rho\ln\rho). \qquad (43)$$

The information depends on the maps of the matrices $\rho\to\rho^{(n')}(1)$ and $\rho\to\rho^{(n')}(2)$.

For example, if one has the 5×5 density matrix ρ corresponding to qudit with $j=2$ (i.e., $n=5$, and one can take $n'=n+1=6$), the pairs of matrices $\rho(1)$ and $\rho(2)$ obtained by the described positive maps are

$$\rho(1)=\begin{pmatrix}\rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} & \rho_{15}\\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} & \rho_{35}\\ \rho_{51} & \rho_{53} & \rho_{55}\end{pmatrix},\qquad \rho(2)=\begin{pmatrix}\rho_{11}+\rho_{33}+\rho_{55} & \rho_{12}+\rho_{34}\\ \rho_{24}+\rho_{43} & \rho_{22}+\rho_{44}\end{pmatrix}. \qquad (44)$$

One has

$$-\mathrm{Tr}\big(\rho(1)\ln\rho(1)\big)-\mathrm{Tr}\big(\rho(2)\ln\rho(2)\big)\geq-\mathrm{Tr}(\rho\ln\rho). \qquad (45)$$

Other pairs are also obtained by means of coding the pairs for generic 6×6 density matrix ρ_{kj} and assuming that all the matrix elements ρ_{k6} and ρ_{6j} are equal to zero. Then all the subadditivity conditions for qudit $j=2$ states are obtained from the constructed one by permutations of the integers $1,2,3,4,5\mapsto 1_p,2_p,3_p,4_p,5_p$ and labeling matrix elements of the matrix ρ_{kj} $(k,j=1,2,3,4,5)$.

6. Nonlinear Maps of the Probability Vectors

In this section, we discuss the possibility to construct a general map of the probability vector $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ onto the probability vector $\vec{\Pi} = (\Pi_1(\mathbf{p}), \Pi_2(\mathbf{p}), \ldots, \Pi_m(\mathbf{p}))$ and the components of the vector $\vec{\Pi}$, i.e., $\Pi_k(\mathbf{p})$ are some functions of the vector $\mathbf{p}$. If $n = m$, for a particular case of linear functions, we have the map

$$\vec{\Pi}(\mathbf{p}) = M\mathbf{p}, \tag{46}$$

where the $n\times n$ matrix M has the matrix elements with the property $\sum_{k=1}^{n} M_{kj} = 1$. If Eq. (46) provides the linear map for all the vectors $\mathbf{p}$ belonging to a simplex, the matrices M are stochastic matrices with nonnegative matrix elements. If Eq. (46) provides the linear map for the vector belonging to some domain in the simplex, the matrices M can have negative matrix elements. In all the cases, the matrices M form a semigroup. (In particular, the stochastic matrices form the semigroup.) One can introduce nonlinear maps of the probability vectors by choosing specific functions $\Pi_k(\mathbf{p})$ that preserve the properties of the nonnegativity $\Pi_k(\mathbf{p}) \geq 0$ and the normalization $\sum_{k=1}^{m} \Pi_k(\mathbf{p}) = 1$.

A simple example of the nonlinear map is given by the rational function of the form

$$\Pi_k^{(s)}(\mathbf{p}) = \frac{p_k^s}{\sum_{k=n_1}^{n_2} p_k^s}, \quad 1 \leq k = n_1,\, n_1+1, \ldots, n_2 \leq n. \tag{47}$$

Such a map for $s = 1$ gives, for example, the conditional probability distribution. In fact, if a joint probability distribution $P(k, j)$ is written in the form of the probability vector

$$\mathbf{p} = \big(P(1,1), P(1,2), \ldots, P(1,n), P(2,1), \ldots, P(n,m)\big),$$

the Bayes formula for the conditional probability

$$\mathbf{p} \mapsto P(k|j) = \frac{P(k,j)}{\sum_{k=1}^{n} P(k,j)}$$

has the form (47), where the corresponding indices are chosen.

A particular case of this map takes place for $n_1 = 1$ and $n_2 = n$. For example, if $s = 2$, one has the map

$$\mathbf{p} \mapsto \vec{\Pi}^{(2)}(\mathbf{p}) = \left(\sum_{k=1}^{n} p_k^2\right)^{-1} \big(p_1^2, p_2^2, \ldots, p_n^2\big). \tag{48}$$

Such maps can be considered as examples of nonlinear classical channels.

In the quantum case, we define the nonlinear map of the density $n\times n$ matrix ρ onto the density $m\times m$ matrix R (i.e., $\rho_{kj} \mapsto R_{\alpha\beta}(\rho)$) preserving the properties of density matrices $R^\dagger = R$, $\mathrm{Tr}R = 1$, $R \geq 0$. The case of linear map of density matrices is a particular case of the map under discussion. For example, the positive linear map [27, 28] given in the form $R_{\alpha\beta}(\rho) = \sum_{k,j=1}^{n} B_{\alpha\beta,kj}\rho_{kj}$ and quantum channels corresponding to completely positive map of the density matrix play an important role in studying the quantum correlations in composite systems like the entanglement phenomenon. The properties of linear maps such as the positivity and complete positivity are coded by the properties of the matrix $B_{\alpha\beta,kj}$ [28]. The nonlinear maps of the density matrices, which we call the nonlinear quantum channels, are characterized by the functions $R_{\alpha\beta}(\rho_{kj})$. A simple example corresponds to the probability vector transform (47); it reads

$$R = \rho^s \big(\mathrm{Tr}\rho^s\big)^{-1}, \quad s = 2, 3, \ldots, \infty. \tag{49}$$

The map provides the new density matrix with larger purity, which in the generic case for $s \to \infty$ gives the pure state. The map can create the entanglement, e.g., for the two-qubit X-states. An analog of the classical Bayes formula for the conditional probability distribution given by nonlinear map (47) for the matrix ρ_{kj} $(k, j = 1, 2, \ldots, n)$ has the form

$$\rho_{kj} \mapsto R^{(m)}_{k'j'} = \frac{\rho_{k'j'}}{\sum_{k=1}^{m} \rho_{kk}}, \qquad k', j' = 1, 2, \ldots, m < n. \tag{50}$$

The nonlinear positive map can be given in the form of a map of the qudit tomogram. It is known (see, e.g., [29]) that the density matrix of an arbitrary qudit-system state with N subsystems and density matrix $\rho(1, 2, \ldots, N)$ is described by the tomographic probability distribution (qudit tomogram), which determines the density matrix. The probability vector $\vec{w}(u)$ corresponding to the density matrix has vector components depending on the unitary matrix u

$$\vec{w}(u) = |uu_o|^2 \vec{\rho}, \tag{51}$$

where $\vec{\rho}$ is the vector having components equal to the eigenvalues of the density matrix. The unitary matrix u_0 has the corresponding eigenvectors of the density matrix as the columns. Using expression (51) for the tomogram of any qudit-system state, we formulate the general statement.

We consider the first case where the tomographic probability vector $\vec{w}(u)$ has $n = NM$ components. Applying Eq. (22) to the vector, we obtain the inequality

$$-|uu_0|^2 \vec{\rho} \ln |uu_o|^2 \vec{\rho} \leq -M_{12} |uu_0|^2 \vec{\rho} \ln \left(M_{12} |uu_0|^2 \vec{\rho}\right) - M_{21} |uu_0|^2 \vec{\rho} \ln \left(M_{21} |uu_0|^2 \vec{\rho}\right). \tag{52}$$

This inequality is valid for the tomogram of a composite or noncomposite qudit system. The system is described by the density matrix with eigenvalues providing the NM-vector $\vec{\rho}$ and corresponding eigenvectors combined into the unitary matrix u_0. If $n \neq NM$, we introduce the vector with $n' = NM$ components, where $n' = n + s$, by adding s extra zero components to the vector $\vec{\rho}$. Also we extend the orthostochastic matrix $|uu_0|^2$ and replace it by the matrix $\begin{pmatrix} |uu_0|^2 & 0 \\ 0 & 1_s \end{pmatrix}$, where 1_s is the $s \times s$ matrix.

Any map of the density matrix, i.e., $\vec{\rho} \mapsto \vec{\rho}'$, $u_0 \mapsto u_0'$ provides the map of the tomographic vector $\vec{w}(u)$. The rational map of the density matrix (49) is equivalent to the map of the probability vector $\vec{\rho}$ given by (47) with $n_1 = 1$ and $n_2 = n$. Thus, any linear or nonlinear map of the probability vector $\vec{\rho}$, which has components equal to the density matrix eigenvalues, yields the nonlinear positive map of the density matrix. Other nonlinear positive maps can be associated with the change in the unitary matrix u_0, e.g., by means of the linear map $u_0 \mapsto u_0' = T u_0$, where T is the unitary transform of the eigenvectors of the density matrix $\rho(1, 2, \ldots, N)$.

One has the entropic inequalities for the probability vector (47) at $n_1 = 1$ and $n = n_2$

$$-\sum_{k=1}^{n} \Pi_k^{(s+1)}(\mathbf{p}) \ln \Pi_k^{(s+1)}(\mathbf{p}) \leq -\sum_{k=1}^{n} \Pi_k^{(s)}(\mathbf{p}) \Pi_k^{(s)}(\mathbf{p}) \leq -\mathbf{p} \ln \mathbf{p}, \quad s = 1, 2, 3, \ldots \tag{53}$$

The von Neumann entropy of the density matrix R (49) obeys the inequality

$$-\mathrm{Tr}\left[(\rho^s (\mathrm{Tr}\rho^s)^{-1}) \ln(\rho^s (\mathrm{Tr}\rho^s)^{-1})\right] \geq -\mathrm{Tr}\left[(\rho^{s+1} (\mathrm{Tr}\rho^{s+1})^{-1} \rho^{s+1}) \ln(\rho^{s+1} (\mathrm{Tr}\rho^{s+1})^{-1} \rho^{s+1})\right]. \tag{54}$$

One can introduce the positive map as the convex sum of the terms $\rho^s\big(\mathrm{Tr}\rho^s\big)^{-1}$, i.e.,

$$\rho \Rightarrow R = \sum_s p_s\Big[\rho^s\big(\mathrm{Tr}\rho^s\big)^{-1}\Big], \qquad 0 \le p_s \le 1, \qquad \sum_s p_s = 1.$$

The map provides an example of the nonlinear channel.

For composite bipartite systems, the unitary transform, which converts the entangled states to the separable states, is given by the matrix A, such that

$$A = u_{01} \otimes u_{02}, \tag{55}$$

where the matrices u_{ok} are unitary local transform matrices.

For example, if the bipartite system with the density matrix $\rho(1,2)$ has the eigenvectors providing the unitary matrix u_0, such that

$$\rho(1,2) = u_0\rho_d u_0^\dagger, \qquad \rho_d = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_2 & 0 & o \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \rho_n \end{pmatrix}, \tag{56}$$

any unitary matrix A of the form

$$A = (u_{01} \otimes u_{02})u_0^\dagger \tag{57}$$

gives the tomographic vector

$$\vec{w}_A(u) = |u(u_{01} \otimes u_{02})|^2\vec{\rho}, \tag{58}$$

which is the tomogram of the separable state. The described transform depends on the state.

7. Conclusions

To conclude, we formulate the main results of our work.

We obtained new inequalities for both probability vectors and density matrices. These inequalities are analogs of known subadditivity conditions, which are valid for composite systems, but we showed that the inequalities are valid for arbitrary probability vectors and arbitrary density matrices, including the case of systems without subsystems.

We discussed the positive nonlinear maps of the probability vectors and density matrices. The nonlinear maps can be used to create entangled states from the separable states. We considered explicitly the examples of the density matrix in six-dimensional Hilbert space, which can be identified either with a qubit–qutrit composite system state or with the state of a single qudit with $j = 5/2$.

It is worth noting that one can introduce Bell inequalities for noncomposite systems. Also one can study the violation of the inequalities. The Bell inequality for the qudit with $j = 3/2$ has the form of the inequality for a two-qubit system. The entangled states for the qudit with $j = 3/2$ are the states for which the equality for the density matrix of the state in the form of separability condition is not valid. The Bell inequality can be violated. We consider this problem in a future publication.

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