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ENTROPIC INEQUALITIES AND PROPERTIES OF SOME SPECIAL FUNCTIONS

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Abstract

Using known entropic and information inequalities, we obtain new inequalities for some classical polynomials. We consider examples of Jacobi and Legendre polynomials.

Keywords: Jacobi polynomials, Legendre polynomials, entropic inequalities, information inequalities.

1. Introduction

It is known [1] that there exists a possibility to derive some relations for special functions, which turn out to be the matrix elements of irreducible unitary representation of compact and noncompact groups. On the other hand, in the classical probability theory and the quantum tomographic approach [2, 3] for the description of quantum states, the specific probability distributions expressed in terms of some special functions appear in a natural way. Quantum entropic inequalities characterize the correlation properties of system states. Analogous properties are connected with different kinds of uncertainties [4]. Since entropies are determined by the probability distributions and there exist relations, in particular, in the form of entropic and information inequalities, one can apply these inequalities to derive some new inequalities for the special functions.

For a random variable, the probability distribution, which appears as a result of experiments with a finite number of outcomes, is characterized by the Shannon entropy [5]. The results of experiments, where two random variables are measured, can be associated with a joint probability distribution. The distribution is connected with $N = N_1 N_2$ outcomes, where for the first random variable we have N_1 results and for second random variable, N_2 results. The joint probability distribution and dependence between two random variables determine two marginal probability distributions by the Sklar theorem [6]. For these three probability distributions, one can calculate Shannon entropies [5].

These entropies satisfy the inequality called the subadditivity condition [7, 8]. Different entropic and information inequalities were studied in [9–12]. The entropic inequalities for the bipartite systems were used in [13, 14], within the framework of the tomographic probability representation of quantum

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mechanics, to characterize two degrees of quantum correlations in the systems. On the other hand, the mathematical structure of the subadditivity condition permits one to apply this inequality in all cases where the nonnegative numbers or functions appears and the sum of the numbers or functions equals unity.

The aim of our work is to consider the unitary matrices connected with the irreducible representation of the rotation group and other groups and construct probability distributions, creating from the entropic inequalities the inequalities for such special functions as Jacobi and Legendre polynomials.

2. Inequalities for the $SU(2)$ Representation Matrix Elements

It is known that unitary irreducible representations of the rotation group with spins (or $SU(2)$ group) are expressed in terms of Jacobi polynomials [1,15]. The squared moduli of the matrix elements read

$$|d^{(j)}_{m',m}(\beta)|^2=\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\left[\cos(\beta/2)^{m'+m}\sin(\beta/2)^{m'-m}P^{(m'-m,m'+m)}_{j-m'}(\cos\beta)\right]^2, \tag{1}$$

where $P^{(m'-m,m'+m)}_{j-m'}(\cos\beta)$ are the Jacobi polynomials [15]

$$P^{(a,b)}_n(z) \;=\; \frac{(-1)^n}{2^n n!}(1-z)^{-a}(1+z)^{-b}\frac{d^n}{dz^n}(1-z)^{a+n}(1+z)^{b+n}.$$

The relation

$$d^{(j)}_{m',m}(\beta)=(-1)^{m'-m}d^{(j)}_{m,m'}(\beta)=d^{(j)}_{-m,-m'}(\beta) \tag{2}$$

holds. We apply the generic inequalities for probabilities expressed in terms of Shannon entropies to these matrix elements.

The point is that one has $|d^{(j)}_{m'm}(\beta)|^2\geq 0$, $\sum\limits_{m'=-j}^{j}|d^{(j)}_{m',m}(\beta)|^2=1$, and $\sum\limits_{m=-j}^{j}|d^{(j)}_{m',m}(\beta)|^2=1$. Thus, the values $|d^{(j)}_{m'm}(\beta)|^2$ can be considered as probabilities. We denote these probabilities as

$$P^{(j)}_{m'}(\beta) \;=\; |d^{(j)}_{m',m}(\beta)|^2, \qquad P^{(j)}_{m}(\beta)=|d^{(j)}_{m',m}(\beta)|^2. \tag{3}$$

We use the map of the numbers m' and m onto the numbers $1,2,\ldots,N=2j+1$ following the rule: $-j\Rightarrow 1,\ -j+1\Rightarrow 2,\ \ldots,\ j\Rightarrow N$. Now we study the relation obtained, in view of the probability vector $\overrightarrow{p}=(p_1,p_2,\ldots,p_N)$, where $\sum\limits_{k=1}^{N}p_k=1$ and $p_k\geq 0$.

The Shannon entropy associated with the probability vector $\overrightarrow{p}$ is determined as

$$H_p=-\sum_k p_k\ln p_k. \tag{4}$$

3. Example of Spin $j=3/2$

We discuss an arbitrary probability distribution identified with the four-vector $\overrightarrow{p}=(p_1,p_2,p_3,p_4)$. For the probability vector $\overrightarrow{p}$, we introduce the invertible map of the indices:

$$1\Leftrightarrow 11,\ 2\Leftrightarrow 12,\ 3\Leftrightarrow 21,\ 4\Leftrightarrow 22. \tag{5}$$

Then the probabilities are given in the form of a matrix (p_{il}), which can be interpreted as a joint probability of the bipartite system (say, two coins), $p_1 \Leftrightarrow p_{11},\ p_2 \Leftrightarrow p_{12},\ p_3 \Leftrightarrow p_{21},\ p_4 \Leftrightarrow p_{22}$. This means that we map the probabilities for the system without subsystems into the joint probability distribution associated with the bipartite system. Marginal probability distributions determined by the joint probability distribution read

$$\pi_i = \sum_{l=1}^{2} p_{il} = p_{i1} + p_{i2}, \qquad \Pi_l = \sum_{i=1}^{2} p_{il} = p_{1l} + p_{2l}. \tag{6}$$

According to the definition of Shannon entropy (4), the entropies associated with the initial probability distribution and two marginals are

$$\begin{aligned} H_p &= -p_1 \ln p_1 - p_2 \ln p_2 - p_3 \ln p_3 - p_4 \ln p_4, \\ H_\pi &= -(p_1 + p_2)\ln(p_1 + p_2) - (p_3 + p_4)\ln(p_3 + p_4), \\ H_\Pi &= -(p_1 + p_3)\ln(p_1 + p_3) - (p_2 + p_4)\ln(p_2 + p_4). \end{aligned} \tag{7}$$

It is known that the Shannon entropies associated with the bipartite system satisfy some inequalities. The following inequality, called the subadditivity condition, reads

$$H_\pi + H_\Pi \geq H_p.$$

The Shannon information is defined as the difference of the sum of the entropies of the subsystems and entropy of the bipartite system, i.e.,

$$I(\beta) = H_\pi(\beta) + H_\Pi(\beta) - H_p(\beta). \tag{8}$$

Obviously, the Shannon information satisfies the inequality $I(\beta) \geq 0$ for all angles β.

Now we concentrate on a particular probability distribution determined by matrix elements of the four-dimensional irreducible representation of the group $SU(2)$, which corresponds to spin $j = 3/2$.

For $j = 3/2$, the numbers m and m', being the spin projections on the z axis, take the values $-3/2$, $-1/2$, $1/2$, and $3/2$. All terms of (1) are presented in Table 1, where the notation $|d^{(j)}_{m',m}(\beta)|^2 \equiv \widetilde{d}^{(j)}_{m',m}(\beta)$ is used.

Table 1. Probability Distributions $\widetilde{d}^{(j)}_{m',m}(\beta)$.

	m			
m'	3/2	1/2	−1/2	−3/2
3/2	$\widetilde{d}^{(j)}_{3/2,3/2}(\beta)$	$\widetilde{d}^{(j)}_{3/2,1/2}(\beta)$	$\widetilde{d}^{(j)}_{3/2,-1/2}(\beta)$	$\widetilde{d}^{(j)}_{3/2,-3/2}(\beta)$
1/2	$\widetilde{d}^{(j)}_{1/2,3/2}(\beta)$	$\widetilde{d}^{(j)}_{1/2,1/2}(\beta)$	$\widetilde{d}^{(j)}_{1/2,-1/2}(\beta)$	$\widetilde{d}^{(j)}_{1/2,-3/2}(\beta)$
−1/2	$\widetilde{d}^{(j)}_{-1/2,3/2}(\beta)$	$\widetilde{d}^{(j)}_{-1/2,1/2}(\beta)$	$\widetilde{d}^{(j)}_{-1/2,-1/2}(\beta)$	$\widetilde{d}^{(j)}_{-1/2,-3/2}(\beta)$
−3/2	$\widetilde{d}^{(j)}_{-3/2,3/2}(\beta)$	$\widetilde{d}^{(j)}_{-3/2,1/2}(\beta)$	$\widetilde{d}^{(j)}_{-3/2,-1/2}(\beta)$	$\widetilde{d}^{(j)}_{-3/2,-3/2}(\beta)$

In view of (2), we can write the following relation for $\widetilde{d}^{(j)}_{m',m}(\beta)$:

$$\widetilde{d}^{(j)}_{m',m}(\beta) \;=\; \widetilde{d}^{(j)}_{m,m'}(\beta) = \widetilde{d}^{(j)}_{-m,-m'}(\beta).$$

The following equations for the matrix elements are straightforward:

$$\begin{aligned}
\widetilde{d}^{(j)}_{3/2,\,3/2}(\beta) &= \widetilde{d}^{(j)}_{-3/2,-3/2}(\beta), \qquad \widetilde{d}^{(j)}_{-3/2,\,3/2}(\beta) = \widetilde{d}^{(j)}_{3/2,-3/2}(\beta),\\
\widetilde{d}^{(j)}_{1/2,\,1/2}(\beta) &= \widetilde{d}^{(j)}_{-1/2,-1/2}(\beta), \qquad \widetilde{d}^{(j)}_{-1/2,\,1/2}(\beta) = \widetilde{d}^{(j)}_{1/2,-1/2}(\beta),\\
\widetilde{d}^{(j)}_{3/2,\,1/2}(\beta) &= \widetilde{d}^{(j)}_{1/2,\,3/2}(\beta) = \widetilde{d}^{(j)}_{-3/2,-1/2}(\beta) = \widetilde{d}^{(j)}_{-1/2,-3/2}(\beta),\\
\widetilde{d}^{(j)}_{3/2,-1/2}(\beta) &= \widetilde{d}^{(j)}_{-1/2,\,3/2}(\beta) = \widetilde{d}^{(j)}_{3/2,-1/2}(\beta) = \widetilde{d}^{(j)}_{1/2,-3/2}(\beta).
\end{aligned} \tag{9}$$

The probabilities associated with these matrix elements are also equal.

The probability vector $\overrightarrow{p}$ can be chosen as in (3). For example, if we fix $m = 3/2$ and consider all elements with different m', i.e., the elements of the first column in Table 1, we obtain

$$\begin{aligned}
p_1(\beta) &= \widetilde{d}^{(j)}_{3/2,\,3/2}(\beta) = \cos(\beta/2)^6\, P_0^{(0,\,3)}(\cos\beta)^2 = (\cos\beta+1)^3/8,\\
p_2(\beta) &= \widetilde{d}^{(j)}_{1/2,\,3/2}(\beta) = \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2} P_1^{(-1,\,2)}(\cos\beta)^2 = 3\sin(\beta/2)^2\left(\sin(\beta/2)^2-1\right)^2,\\
p_3(\beta) &= \widetilde{d}^{(j)}_{-1/2,\,3/2}(\beta) = \frac{\cos(\beta/2)^2}{3\sin(\beta/2)^4} P_2^{(-2,\,1)}(\cos\beta)^2 = \frac{3}{8}(\cos\beta-1)^2(\cos\beta+1),\\
p_4(\beta) &= \widetilde{d}^{(j)}_{-3/2,\,3/2}(\beta) = \sin(\beta/2)^{-6}\, P_3^{(-3,\,0)}(\cos\beta)^2 = -(\cos\beta-1)^3/8.
\end{aligned} \tag{10}$$

The sum of the probabilities $\sum\limits_k p_k(\beta) = 1$ for any angle β. For example, we calculated the probabilities for $\beta = 1$; they are $p_1(1) = 0.4568019$, $p_2(1) = 0.40899267$, $p_3(1) = 0.1220624$, and $p_4(1) = 0.012143$. Of course, their sum is equal to 1.

One can fix m' and consider all matrix elements in the m'th row of Table 1. It is possible to construct a vector $\overrightarrow{p}$ using other combinations of the elements in Table 1. This conclusion is a direct consequence of equality (9). It is only necessary to have in mind that the sum of the elements of such a vector must always be equal to unity. This fact provides one with an opportunity to construct a large number of inequalities based on inequality (8) for the Shannon information.

We introduce the notation $\left(P_{j-m'}^{(m'-m,m'+m)}(\cos\beta)\right)^2 \equiv \widetilde{P}_{j-m'}^{(m'-m,m'+m)}$. Substituting (10) in (7) and (8), we arrive at

$$\begin{aligned}
&-\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big)\ln\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big)\\
&-\Big(\cos(\beta/2)^6\widetilde{P}_0^{(0,3)} + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\ln\Big(\cos(\beta/2)^6\widetilde{P}_0^{(0,3)} + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\\
&-\Big(\cos(\beta/2)^6\widetilde{P}_0^{(0,3)} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big)\ln\Big(\cos(\beta/2)^6\widetilde{P}_0^{(0,3)} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big)\\
&-\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\ln\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\\
&+\cos(\beta/2)^6\widetilde{P}_0^{(0,3)}\ln[\cos(\beta/2)^6\widetilde{P}_0^{(0,3)}] + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\ln\Big(\frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\\
&+\frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\ln\Big(\frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big) + \frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6}\ln\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6}\Big) \geq 0.
\end{aligned} \tag{11}$$

Entropies $H_\Pi(\beta)$, $H_\pi(\beta)$, and $H_p(\beta)$, along with the Shannon information $I(\beta)$ for probabilities (10) are shown in Figs. 1 and 2. Obviously, the information reaches its maximum value at the point $\beta = \pi/2$.

Now we consider how the permutation of probabilities (10) affects the entropies H_π and H_Π (8). We introduce a new vector $\widetilde{p} = (\widetilde{p}_1, \widetilde{p}_2, \widetilde{p}_3, \widetilde{p}_4)$ determined by

$$\widetilde{p}_1 \equiv p_4, \quad \widetilde{p}_2 \equiv p_1, \quad \widetilde{p}_3 \equiv p_2, \quad \widetilde{p}_4 \equiv p_3. \tag{12}$$

Entropies $H_\Pi(\beta)$, $H_\pi(\beta)$, and $H_p(\beta)$ and information $I(\beta)$ for the probability vector $\widetilde{p}$ are shown in Figs. 3 and 4.

One can easily see that permutations impact only the entropy H_π. The information turns to zero at the point $\beta = \pi/2$. After substituting (12) in (7) and (8), we arrive at the following inequality:

$$\begin{aligned}
&-\Big(\cos(\beta/2)^6 \widetilde{P}_0^{(0,3)} + \frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6}\Big) \ln\Big(\cos(\beta/2)^6 \widetilde{P}_0^{(0,3)} + \frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6}\Big)\\
&-\Big(\frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big) \ln\Big(\frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big)\\
&-\Big(\cos(\beta/2)^6 \widetilde{P}_0^{(0,3)} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big) \ln\Big(\cos(\beta/2)^6 \widetilde{P}_0^{(0,3)} + \frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2^4}\Big)\\
&-\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big) \ln\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\\
&+\cos(\beta/2)^6 \widetilde{P}_0^{(0,3)} \ln\big(\cos(\beta/2)^6 \widetilde{P}_0^{(0,3)}\big) + \frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)} \ln\Big(\frac{\cos(\beta/2)^4}{3\sin(\beta/2)^2}\widetilde{P}_1^{(-1,2)}\Big)\\
&+\frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4} \ln\Big(\frac{\widetilde{P}_2^{(-2,1)}\cos(\beta/2)^2}{3\sin(\beta/2)^4}\Big) + \frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6} \ln\Big(\frac{\widetilde{P}_3^{(-3,0)}}{\sin(\beta/2)^6}\Big) \geq 0.
\end{aligned} \tag{13}$$

Now we select the probability vector $\overrightarrow{p}$ using the other combination of $\widetilde{d}^{(j)}_{m',m}(\beta)$. We fix $m = 1/2$ and take elements in the second column of Table 1. The probability vector has the following components:

$$\begin{aligned}
p_1(\beta) &= \widetilde{d}^{(j)}_{3/2,\,1/2}(\beta) = 3\cos(\beta/2)^4 \sin(\beta/2)^2\, P_0^{(1,2)}(\cos\beta)^2 = 3\cos(\beta/2)^4 \sin(\beta/2)^2,\\
p_2(\beta) &= \widetilde{d}^{(j)}_{1/2,\,1/2}(\beta) = \frac{\cos\beta + 1}{2} P_1^{(0,1)}(\cos\beta)^2 = \frac{(3\cos\beta - 1)^2}{8}(\cos\beta + 1),\\
p_3(\beta) &= \widetilde{d}^{(j)}_{-1/2,1/2}(\beta) = \frac{P_2^{(-1,0)}(\cos\beta)^2}{\sin(\beta/2)^2} = -\frac{(3\cos\beta + 1)^2}{8}(\cos\beta - 1),\\
p_4(\beta) &= \widetilde{d}^{(j)}_{-3/2,\,1/2}(\beta) = \frac{3P_3^{(-2,-1)}(\cos\beta)^2}{\cos(\beta/2)^2(\cos(\beta/2)^2 - 1)^2} = \frac{3(\cos\beta - 1)^2}{8}(\cos\beta + 1).
\end{aligned} \tag{14}$$

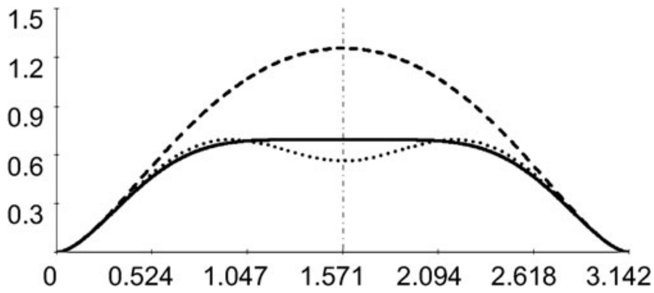

Fig. 1. Entropies $H_\Pi(\beta)$ (solid curve), $H_\pi(\beta)$ (dotted curve), and $H_p(\beta)$ (dashed curve) for probabilities (10). The dash-dotted line corresponds to $\pi/2$.

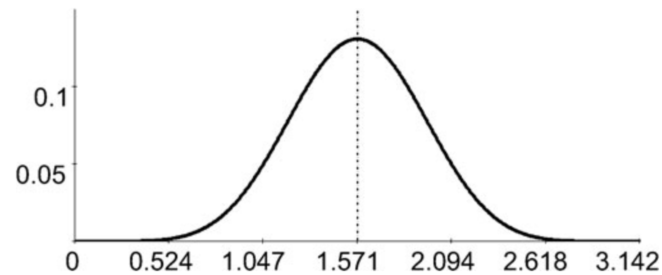

Fig. 2. Information $I(\beta)$ for probabilities (10). The dash-dotted line corresponds to $\pi/2$.

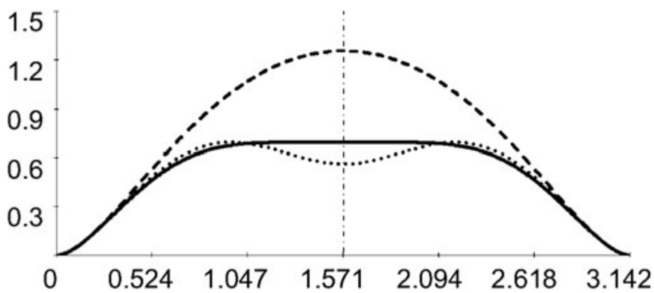

Fig. 3. Entropies $H_\Pi(\beta)$ (solid curve), $H_\pi(\beta)$ (dotted curve), and $H_p(\beta)$ (dashed curve) for probabilities (12). The dash-dotted line corresponds to $\pi/2$.

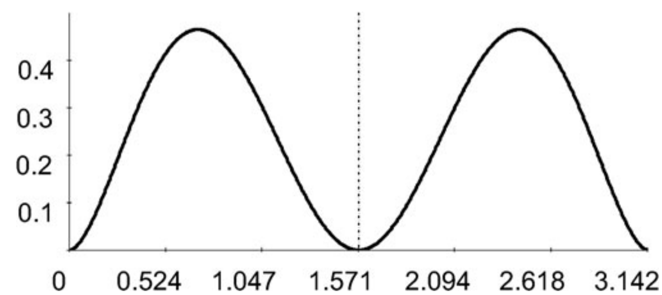

Fig. 4. Information $I(\beta)$ for probabilities (12). The dash-dotted line corresponds to $\pi/2$.

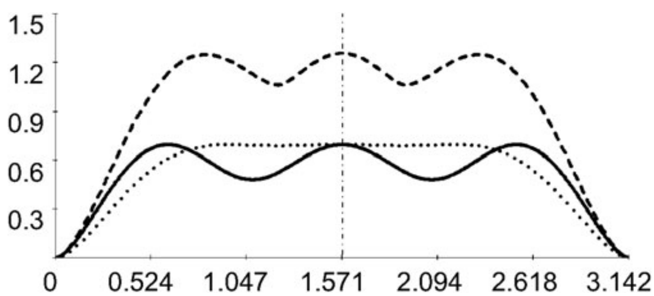

Fig. 5. Entropies $H_\Pi(\beta)$ (solid curve), $H_\pi(\beta)$ (dotted curve), and $H_p(\beta)$ (dashed curve) for probabilities (14). The dash-dotted line corresponds to $\pi/2$.

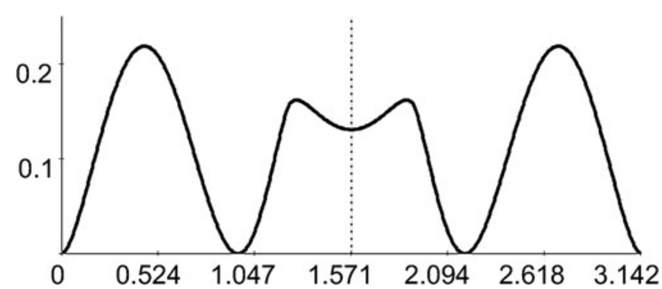

Fig. 6. Information $I(\beta)$ for probabilities (14). The dash-dotted line corresponds to $\pi/2$.

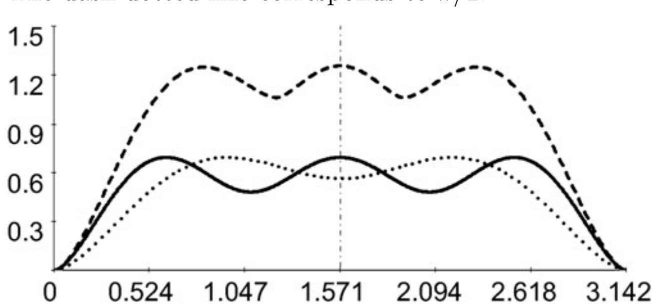

Fig. 7. Entropies $H_\Pi(\beta)$ (solid curve), $H_\pi(\beta)$ (dotted curve), and $H_p(\beta)$ (dashed curve) for probabilities (16). The dash-dotted line corresponds to $\pi/2$.

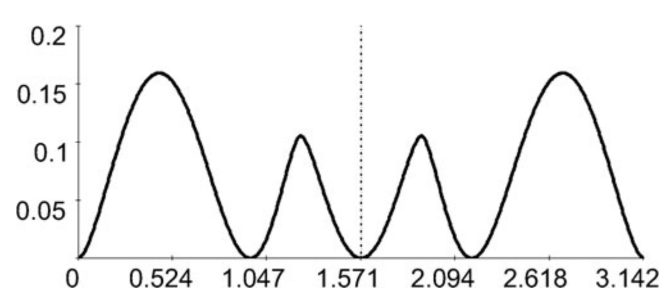

Fig. 8. Information $I(\beta)$ for probabilities (16). The vertical line corresponds to $\pi/2$.

Substituting (14) in (8), we arrive at the following inequality:

$$
\begin{aligned}
&-\Big(\frac{\widetilde{P}_2^{(-1,0)}}{\sin(\beta/2)^2}+\frac{3\widetilde{P}_3^{(-2,-1)}}{\cos(\beta/2)^2(\cos(\beta/2)^2-1)^2}\Big)\ln\Big(\frac{\widetilde{P}_2^{(-1,0)}(\cos\beta)^2}{\sin(\beta/2)^2}+\frac{3\widetilde{P}_3^{(-2,-1)}}{\cos(\beta/2)^2(\cos(\beta/2)^2-1)^2}\Big)\\
&-\Big(\frac{\cos\beta+1}{2}\widetilde{P}_1^{(0,1)}+3\cos\Big(\frac{\beta}{2}\Big)^4\sin\Big(\frac{\beta}{2}\Big)^2\widetilde{P}_0^{(1,2)}\Big)\ln\Big(\frac{\cos\beta+1}{2}\widetilde{P}_1^{(0,1)}+3\cos\Big(\frac{\beta}{2}\Big)^4\sin\Big(\frac{\beta}{2}\Big)^2\widetilde{P}_0^{(1,2)}\Big)\\
&-\Big(\frac{\cos\beta+1}{2}\widetilde{P}_1^{(0,1)}+\frac{3\widetilde{P}_3^{(-2,-1)}}{\cos(\beta/2)^2(\cos(\beta/2)^2-1)^2}\Big)\ln\Big(\frac{\cos\beta+1}{2}\widetilde{P}_1^{(0,1)}+\frac{3\widetilde{P}_3^{(-2,-1)}}{\cos(\beta/2)^2(\cos(\beta/2)^2-1)^2}\Big)\\
&-\Big(\frac{\widetilde{P}_2^{(-1,0)}}{\sin(\beta/2)^2}+3\cos(\beta/2)^4\sin(\beta/2)^2\widetilde{P}_0^{(1,2)}\Big)\ln\Big(\frac{\widetilde{P}_2^{(-1,0)}}{\sin(\beta/2)^2}+3\cos(\beta/2)^4\sin(\beta/2)^2\widetilde{P}_0^{(1,2)}\Big)\\
&+\Big(3\cos(\beta/2)^4\sin(\beta/2)^2\widetilde{P}_0^{(1,2)}\Big)\ln\Big(3\cos(\beta/2)^4\sin(\beta/2)^2\widetilde{P}_0^{(1,2)}(\cos\beta)^2\Big)\\
&+\Big(\frac{\widetilde{P}_2^{(-1,0)}(\cos\beta)^2}{\sin(\beta/2)^2}\Big)\ln\Big(\frac{\widetilde{P}_2^{(-1,0)}(\cos\beta)^2}{\sin(\beta/2)^2}\Big)+\Big(\frac{\cos\beta+1}{2}\widetilde{P}_1^{(0,1)}\Big)\ln\Big(\frac{\cos\beta+1}{2}\widetilde{P}_1^{(0,1)}(\cos\beta)^2\Big)\\
&+\Big(\frac{3\widetilde{P}_3^{(-2,-1)}}{\cos(\beta/2)^2(\cos(\beta/2)^2-1)^2}\Big)\ln\Big(\frac{3\widetilde{P}_3^{(-2,-1)}}{\cos(\beta/2)^2(\cos(\beta/2)^2-1)^2}\Big)\geq 0.
\end{aligned}
\tag{15}
$$

Entropies $H_\Pi(\beta)$, $H_\pi(\beta)$, and $H_p(\beta)$ along with information $I(\beta)$ for the probability vector (14) are shown in Figs. 5 and 6. One can easily see that they differ from the entropies and information constructed by the polynomials based on the first column of Table 1.

Also it is interesting to see how the permutation of the components of the probability vector $\overrightarrow{p}$ changes information. To this end, we perform the same procedure as in (12).

The new vector $\widehat{p}=(\widehat{p}_1,\widehat{p}_2,\widehat{p}_3,\widehat{p}_4)$ can be constructed as

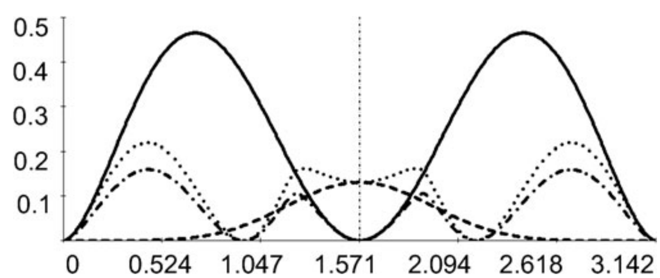

Fig. 9. Information for four probability vectors: Eq. (10) (dashed curve), Eq. (12) (solid curve), Eq. (14) (dotted curve), and Eq. (16) (dash-dotted curve). The vertical line corresponds to $\pi/2$.

$$\widehat{p}_1\equiv p_4,\ \widehat{p}_2\equiv p_1,\ \widehat{p}_3\equiv p_2,\ \widehat{p}_4\equiv p_3. \tag{16}$$

Substituting (14) in (7) and (8), we obtain the new entropies and information shown in Figs. 7 and 8. In contrast to Fig. 6, the information for permuted vector $\widehat{p}$ is equal to zero at the point $\beta=\pi/2$.

Finally, we summarize the results shown in Figs. 2, 4, 6, and 8 and present the final results in Fig. 9. We see that for systems (10) and (14) information is not equal to zero at $\beta=\pi/2$, and for permuted vectors information is equal to zero.

From (9), we can conclude that the two examples given by (10) and (12) cover all possible probabilities. The other combinations taken from Table 1 determine their permutations, but the Jacobi polynomials corresponding to them are, of course, not the same. This fact provide us with the possibility to obtain many different inequalities of the form (11).

4. Case of the N-Component Probability Vector

In this section, we extend our consideration to the case where the probability vector has N components, i.e., $\overrightarrow{p} = (p_1, p_2, \ldots, p_N)$. If N is an even number, we can introduce the map of indices similar to (5), namely,

$$\begin{aligned} 1 &\Leftrightarrow 11, \quad 2 \Leftrightarrow 12, \quad 3 \Leftrightarrow 13, \quad \ldots, \quad N/2 \Leftrightarrow 1\,N/2, \\ N/2+1 &\Leftrightarrow 21, \quad N/2+2 \Leftrightarrow 22, \quad N/2+3 \Leftrightarrow 23, \quad \ldots, \quad N \Leftrightarrow 2\,N/2. \end{aligned}$$

Hence the probabilities are given in the form of a matrix (p_{il}), $i = 1, 2$, $l = 1, 2, \ldots, N/2$ with matrix elements

$$\begin{aligned} p_1 &\Leftrightarrow p_{11}, \quad p_2 \Leftrightarrow p_{12}, \quad p_3 \Leftrightarrow p_{13}, \quad \ldots, \quad p_{N/2} \Leftrightarrow p_{1\,N/2}, \\ p_{(N/2)+1} &\Leftrightarrow p_{21}, \quad p_{(N/2)+2} \Leftrightarrow p_{22}, \quad p_{(N/2)+3} \Leftrightarrow p_{23}, \quad \ldots, \quad p_N \Leftrightarrow p_{2\,N/2}. \end{aligned} \tag{17}$$

If N is an odd number, we add a zero vector $p_{N+1} = 0$ to the N-component vector $\overrightarrow{p}$, and obtain the $(N+1)$-component vector $\overrightarrow{p} = (p_1, p_2, \ldots, p_N, p_{N+1})$; the invertible map of the indices is

$$\begin{aligned} 1 &\Leftrightarrow 11, \quad 2 \Leftrightarrow 12, \quad 3 \Leftrightarrow 13, \quad \ldots, \quad (N+1)/2 \Leftrightarrow 1\,(N+1)/2, \\ (N+1)/2+1 &\Leftrightarrow 21, \quad (N+1)/2+2 \Leftrightarrow 22, \quad (N+1)/2+3 \Leftrightarrow 23, \quad \ldots, \quad (N+1) \Leftrightarrow 2\,(N+1)/2. \end{aligned}$$

The probabilities are given in the form of a matrix (p_{il}), $i = 1, 2$, $l = 1, 2, \ldots, (N+1)/2$ with matrix elements

$$\begin{aligned} p_1 &\Leftrightarrow p_{11}, \quad p_2 \Leftrightarrow p_{12}, \quad p_3 \Leftrightarrow p_{13}, \quad \ldots, \quad p_{(N+1)/2} \Leftrightarrow p_{1(N+1)/2}, \\ p_{[(N+1)/2]+1} &\Leftrightarrow p_{21}, \quad p_{[(N+1)/2]+2} \Leftrightarrow p_{22}, \quad p_{[(N+1)/2]+3} \Leftrightarrow p_{23}, \quad \ldots, \quad p_{N+1} \Leftrightarrow p_{2\,(N+1)/2}. \end{aligned} \tag{18}$$

Similarly to (6), we can write marginal probability distributions determined by the joint probability distribution. For an odd N, they are as follows:

$$\begin{aligned} \pi_i &= \sum_{l=1}^{(N+1)/2} p_{il} = p_{i1} + p_{i2} + p_{i3} + \cdots + p_{i\,(N+1)/2}, \quad i = 1, 2 \\ \Pi_l &= \sum_{i=1}^{2} p_{il} = p_{1l} + p_{2l}, \quad l = 1, 2, \ldots, (N+1)/2. \end{aligned} \tag{19}$$

For (19), we obtain inequalities similar to (11).

To this end, we define the Shannon entropies as

$$\begin{aligned} &H_p = -\sum_{t=1}^{N+1} p_t \ln p_t, \quad H_{\pi_1} = -\sum_{t=1}^{(N+1)/2} p_t \ln \sum_{t=1}^{(N+1)/2} p_t, \quad H_{\pi_2} = -\sum_{t=[(N+1)/2]+1}^{N+1} p_t \ln \sum_{t=[(N+1)/2]+1}^{N+1} p_t, \\ &H_{\Pi_l} = -\left(p_l + p_{[(N+1)/2]+l}\right) \ln p_l + p_{[(N+1)/2]+l}, \qquad l = 1, 2, \ldots, (N+1)/2. \end{aligned} \tag{20}$$

Then the Shannon information (8) based on entropies (20) reads

$$I(\beta)_{tl} = H_{\pi_t}(\beta) + H_{\Pi_l}(\beta) - H_p(\beta), \quad t = 1, 2 \quad l = 1, 2, \ldots, (N+1)/2. \tag{21}$$

Obviously, the inequality $I(\beta) \geq 0$ is valid for all the angles β.

For spin j, the spin projections m and m' can take $2j+1 = N$ values. If j is a fractional number (N is even), the spin projections can be $-j, -j+1, \ldots, j-1, j$. Then the components of the probability $\overrightarrow{p}$ are given by (17). On the other hand, if j is an integer number (N is odd), than the spin projections can be $-j, -j+1, \ldots, 0, \ldots, j-1, j$. The components of the probability vector are given by (18).

One can take the polynomials $|d^{(j)}_{m',m}(\beta)|^2$ as the probabilities for a fixed m' and all m. One of the possible choices for an odd N is as follows:

$$p_1 = |d^{(j)}_{m',j}(\beta)|^2, p_2 = |d^{(j)}_{m',j-1}(\beta)|^2, \ldots, p_{(N+1)/2} = |d^{(j)}_{m',0}(\beta)|^2, \ldots, p_N = |d^{(j)}_{m',-j}(\beta)|^2, p_{N+1} = 0. \tag{22}$$

In such notation, inequality (21) for $t = 1$ and $l = (N+1)/2$ reads

$$-\sum_{m=0}^{j} \widetilde{d}^{(j)}_{m',m}(\beta) \ln \sum_{m=0}^{j} \widetilde{d}^{(j)}_{m',m}(\beta) - \widetilde{d}^{(j)}_{m',0}(\beta) \ln \widetilde{d}^{(j)}_{m',0}(\beta) + \sum_{m=-j}^{j} \widetilde{d}^{(j)}_{m',m}(\beta) \ln \sum_{m=-j}^{j} \widetilde{d}^{(j)}_{m',m}(\beta) \geq 0, \tag{23}$$

and for $t = 2$ and $l = 1$ it is

$$\begin{aligned} &- \sum_{m=-j}^{-1} \widetilde{d}^{(j)}_{m',m}(\beta) \ln \sum_{m=-j}^{-1} \widetilde{d}^{(j)}_{m',m}(\beta) - \big(\widetilde{d}^{(j)}_{m',j}(\beta) + \widetilde{d}^{(j)}_{m',-1}(\beta)\big) \ln \big(\widetilde{d}^{(j)}_{m',j}(\beta) + \widetilde{d}^{(j)}_{m',-1}(\beta)\big) \\ &+ \sum_{m=-j}^{j} \widetilde{d}^{(j)}_{m',m}(\beta) \ln \sum_{m=-j}^{j} \widetilde{d}^{(j)}_{m',m}(\beta) \geq 0. \end{aligned} \tag{24}$$

Our aim now is to represent (23) and (24) by Jacobi polynomials. To this end, we introduce the new notation in (1)

$$\widetilde{d}^{(j)}_{m',m}(\beta) = |d^{(j)}_{m',m}(\beta)|^2 = G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'},$$

where

$$G^{j}_{m',m}(\beta) = \frac{(j+m')!(j-m')!}{(j+m)!(j-m)!} \big(\cos(\beta/2)^{m'+m} \sin(\beta/2)^{m'-m}\big)^2.$$

We rewrite inequalities (23) and (24) in new terms as follows:

$$\begin{aligned} &-\sum_{m=0}^{j} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \ln \sum_{m=0}^{j} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} - G^{j}_{m',0}(\beta) \widetilde{P}^{(m',m')}_{j-m'} \ln \big(G^{j}_{m',0}(\beta) \widetilde{P}^{(m',m')}_{j-m'}\big) \\ &+ \sum_{m=-j}^{j} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \ln \sum_{m=-j}^{j} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \geq 0, \end{aligned}$$

$$\begin{aligned} &- \sum_{m=-j}^{-1} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \ln \sum_{m=-j}^{-1} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \\ &-\big(G^{j}_{m',j}(\beta) \widetilde{P}^{(m'-j,m'+j)}_{j-m'} + G^{j}_{m',-1}(\beta) \widetilde{P}^{(m'+1,m'-1)}_{j-m'}\big) \ln \big(G^{j}_{m',j}(\beta \widetilde{P}^{(m'-j,m'+j)}_{j-m'} + G^{j}_{m',-1}(\beta) \widetilde{P}^{(m'+1,m'-1)}_{j-m'}\big) \\ &+ \sum_{m=-j}^{j} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \ln \sum_{m=-j}^{j} G^{j}_{m',m}(\beta) \widetilde{P}^{(m'-m,m'+m)}_{j-m'} \geq 0. \end{aligned}$$

Now we consider a special case where $j = c$, with c being an integer number and $m' = 0$. Then we can rewrite (1) as

$$|d^{(c)}_{0,m}(\beta)|^2 = |d^{(c)}_{m,0}(\beta)|^2 = \frac{(c-m)!}{(c+m)!}\big(P^m_c(\cos\beta)\big)^2, \tag{25}$$

where $P^m_c(\cos\beta)$ are the Legendre polynomials [15]. In this case, (23) reads

$$-\sum_{m=0}^{j}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c \ln\Big(\sum_{m=0}^{j}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c\Big) - \widetilde{P}^0_c\ln\big(\widetilde{P}^0_c\big) + \sum_{m=-j}^{j}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c\ln\sum_{m=-j}^{j}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c \geq 0,$$

and (24) is

$$-\sum_{m=-j}^{-1}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c\ln\Big(\sum_{m=-j}^{-1}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c\Big) - \frac{(c+1)!}{(c-1)!}\widetilde{P}^{-1}_c\ln\frac{(c+1)!}{(c-1)!}\widetilde{P}^{-1}_c$$
$$+\sum_{m=-j}^{j}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c\ln\sum_{m=-j}^{j}\frac{(c-m)!}{(c+m)!}\widetilde{P}^m_c \geq 0,$$

where we use the notation $\big(P^m_c(\cos\beta)\big)^2 \equiv \widetilde{P}^m_c$ for Legendre polynomials.

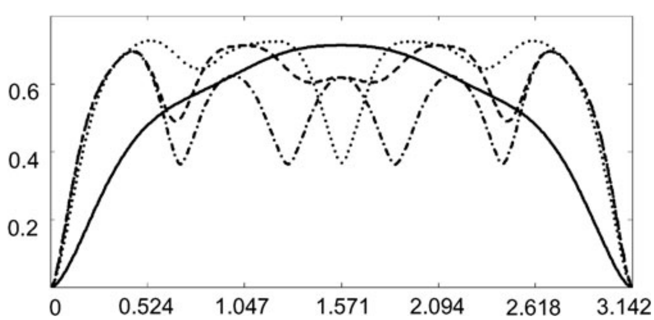

Fig. 10. Shannon information (21) for probabilities (22) at $t = 1$; $l = 1$ and $j = 1$ (solid curve), $l = 3$ and $j = 4$ (dotted curve), $l = 4$ and $j = 5$ (dashed curve), and $l = 6$ and $j = 5$ (dash-dotted curve).

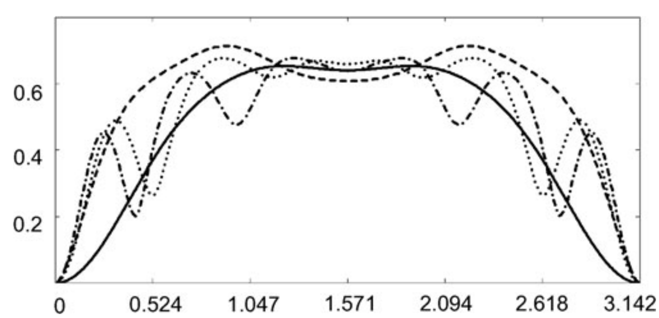

Fig. 11. Shannon information (21) for probabilities (22) at $t = 2$; $l = 1$ and $j = 1$ (solid curve), $l = 4$ and $j = 4$ (dotted curve), $l = 2$ and $j = 3$ (dashed curve), and $l = 5$ and $j = 5$ (dash-dotted curve).

Informations (21) are shown in Figs. 10 and 11 for entropies (20) and various parameters t and l, and spins j.

5. Summary

To conclude, we point out the main results of the study.

We considered the matrix elements of the unitary irreducible representations of the $SU(2)$ group and applied the known subadditivity condition for joint probability distributions, constructed from these matrix elements, to obtain new inequalities for the Jacobi and Legendre polynomials. The inequalities correspond to entropic inequalities for Shannon entropies of bipartite classical systems. We demonstrated the results on the example of spin $j = 3/2$, where the Shannon information of the bipartite system is expressed in terms of the polynomials. We formulated a general approach to derive analogous information and entropic inequalities for arbitrary spins j.

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