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ATOMIC ENTROPY SQUEEZING IN THREE-LEVEL SYSTEMS

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Abstract

We consider the problem of an atomic three-level system (in a ladder configuration) interacting with a radiation field. Assuming a coherent state as the initial state, we solve exactly the time evolution of the system. We discuss the appearance of atomic squeezing and calculate the atomic spin squeezing and the atomic entropy squeezing. We show that both parameters predict similar angular and time dependences.

Keywords: atomic squeezing, atomic entropy squeezing, three-level atoms.

1. Introduction

The study of entanglement has received a great deal of attention, particularly, in connection with recent developments in the field of quantum information [1–5]. Several quantum protocols, such as teleportation [6–10], are based on entangled states. One of the relevant physical quantities that can be used to measure entanglement is entropy [11]. Araki and Lieb have established a well-known relationship among the total entropy of the system and the reduced entropies of the components [12, 13]. From this relation, Lindblad [14], Barnett and Phoenix [15], and Knight and Phoenix [16] have introduced the index of correlations to analyze the presence of entanglement. Associated with the entropy of a system, in early 1957 Hirschman derived the entropic uncertainty relations (EUR) [17]. Since then, a lot of work has been devoted to applications of the EUR to different physical systems. In particular, Kraus [18] and Maasen and Uffink [19] have discussed the optimum entropy uncertainty relation for a pair of complementary observables in a finite Hilbert space. This concept was extended by Sánchez Ruiz [20] for a set of N complementary observables. Margarita Man'ko and coworkers have reviewed the EUR and presented new inequalities for symplectic tomographic entropies [21–23].

Among many-body quantum entangled states, squeezed atomic states play an essential role in precision measurements, quantum information, and fundamental tests of quantum mechanics [24]. Squeezed states are quantum correlated states for which the fluctuations of a given observable are reduced under the quantum limit at the expense of the increase in fluctuations of its conjugate. Recently, Obada and coworkers [25–30] have introduced the so-called entropy-squeezing parameter. They have analyzed the

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occurrence of entanglement in different physical systems using the entropy-squeezing parameter as an indicator. The variance- and entropy-squeezing for two-level atoms in interaction with a nondegenerate parametric amplifier was considered in [28]. The characterization of entropy squeezing, as an indicator of entanglement in a three-level system interacting with a cavity field, was presented in [29, 30]. It was found that the setting of the initial state and the activation of the atom-field coupling affect the field entropy squeezing rather drastically. Also, the link between the entanglement and entropy has been used to study coherence and entanglement in the ground state of a bosonic Josephson junction [31].

In recent works, the use of collisions in ultra-cold gases to induce quadrature spin squeezing in two-component Bose condensates has been reported [32, 33]. A generalization to a higher-dimensional spin space by measuring squeezing in a spin-1 Bose condensate was proposed in [34]. In this work, we study the appearance of atomic squeezing for a system of N three-level atoms interacting among themselves and with a radiation field. The atomic excitations are modeled by the algebra associated to the SU(3) group [35, 36]. We analyze the time and angular dependences of the entropy squeezing parameter and the spin squeezing parameter.

This paper is organized as follows.

We present the formalism of our model in Sec. 2. In Sec. 3 we discuss the results of our calculations performed for the model proposed. We draw our conclusions in Sec. 4.

2. Formalism

We consider a system of N identical three-level atoms of ^{87}Rb interacting with a radiation field in a cavity [37–42]. The dipole–dipole interaction of the atomic sector is modeled in view of [37, 43–45]. This is a suitable representation for the ladder atomic configuration considered, and it is a generalization of the two-level case [46].

Following [34–36, 47], we write the Hamiltonian of a system of three-level atoms in the ladder-configuration as

$$H = \frac{1}{3}\Omega N + \omega_a \left(a^\dagger a + \frac{1}{2}\right) + \omega\, S_z + \frac{1}{2}\Delta\, Q_{zz} + \zeta(a^\dagger S_- + S_+ a) + \lambda\, S_+ S_-, \tag{1}$$

where $a^\dagger(a)$ is the photon creation (annihilation) operator of the photon mode of energy ω_a, and $S_\pm$ and $Q_{\alpha\beta}$ are the spin and quadrupole operators of the Cartesian dipole–quadrupole decomposition of the su(3) Lie algebra [34–36]. The energies Ω, ω, and Δ are related to the level energies ω_i $(i = 0, 1, 2)$ by $\Omega = \omega_0 + \omega_1 + \omega_2$, $\omega = (\omega_2 - \omega_0)/2$, and $\Delta = [(\omega_2 + \omega_0)/2] - \omega_1$ $(\hbar = 1$ everywhere). In this scheme, the transitions take place between the atomic levels ordered in the sequence $\omega_2 > \omega_1 > \omega_0$. The operators of (1) are defined as [34]

$$S_\alpha = -i \sum_{\alpha,\beta,\gamma} \epsilon_{\alpha\beta\gamma} c^\dagger_\gamma c_\beta, \qquad Q_{\alpha\beta} = -c^\dagger_\beta c_\alpha - c^\dagger_\alpha c_\beta + \frac{2}{3}\delta_{\alpha\beta}\sum_\gamma c^\dagger_\gamma c_\gamma, \tag{2}$$

where the operators $c^\dagger_\alpha$ are expressed as

$$c^\dagger_x = \frac{1}{\sqrt{2}}\left(-b^\dagger_2 + b^\dagger_0\right), \qquad c^\dagger_y = \frac{i}{\sqrt{2}}\left(b^\dagger_2 + b^\dagger_0\right), \qquad c^\dagger_z = b^\dagger_1 \tag{3}$$

in terms of the boson creation (annihilation) operators $b_i^\dagger$ (b_i) associated to the excitation of the ith atomic level ($i = 0, 1, 2$). Defining $S^{ij} = b_j^\dagger b_i$, we write the Cartesian components of the operators (2) as follows:

$$S_x = \frac{1}{\sqrt{2}}(S^{01} + S^{12} + S^{21} + S^{10}), \qquad S_y = -\frac{\mathrm{i}}{\sqrt{2}}(S^{01} + S^{12} - S^{21} - S^{10}), \qquad S_z = (S^{22} - S^{00}),$$

$$Q_{xz} = \frac{1}{\sqrt{2}}(-S^{01} + S^{12} + S^{21} - S^{10}), \qquad Q_{xy} = \mathrm{i}(S^{20} - S^{02}),$$

$$Q_{yz} = -\frac{\mathrm{i}}{\sqrt{2}}(-S^{01} + S^{12} - S^{21} + S^{10}), \qquad Q_{xx} = \frac{2}{3}N - (S^{00} + S^{22} - S^{02} - S^{20}),$$

$$Q_{yy} = \frac{2}{3}N - (S^{00} + S^{22} + S^{02} + S^{20}), \qquad Q_{zz} = -\frac{4}{3}N + 2(S^{00} + S^{22}) = -(Q_{xx} + Q_{yy}),$$

where $S_\pm = S_x \pm \mathrm{i}S_y$.

Notice that the term $\Delta\, Q_{zz}$ produces the effect of a quadratic Zeeman operator [34].

The basis for the proposed system can be constructed as a direct product of the photon basis and the collective basis of the atoms

$$\mid n_a,\ k\rangle = \mid n_a\rangle \otimes \mid N, k\rangle. \tag{4}$$

The state with n_a photons $\mid n_a\rangle$ is written

$$\mid n_a\rangle = \frac{1}{\sqrt{n_a!}}\, a^{\dagger n_a} \mid 0\rangle, \tag{5}$$

and $\mid \mathrm{N}, k\rangle$ is the collective atomic state

$$\mid \mathrm{N}, \mathrm{k}\rangle = \sqrt{\frac{(2\mathrm{N} - \mathrm{k})!}{\mathrm{k}!(2\mathrm{N})!}}\mathrm{S}_+^{\mathrm{k}} \mid \mathrm{n}_0 = \mathrm{N}, \mathrm{n}_2 = 0\rangle. \tag{6}$$

In the previous expression, the states $\mid n_0, n_2\rangle$ span the bare atomic basis, with n_0 particles at level 0 and n_2 particles at level 2. The number of particles at level 1 is fixed by the constrain $N = n_0 + n_1 + n_2$. In the explicit form,

$$\mid \mathrm{N}, \mathrm{k}\rangle = \begin{pmatrix} 2N \\ k \end{pmatrix}^{-1/2} \sum_{\substack{p=0 \\ [(k-p)/2 \in \mathbf{N}]}}^{\mathrm{k}} \left[2^{\mathrm{p}} \begin{pmatrix} N \\ p \end{pmatrix} \begin{pmatrix} N-p \\ (k-p)/2 \end{pmatrix}\right]^{1/2} \left|\mathrm{N} - \mathrm{p} - (\mathrm{k} - \mathrm{p})/2,\ (\mathrm{k} - \mathrm{p})/2\right\rangle,$$

The Hamiltonian (1) commutes with the operator

$$P = a^\dagger a + S_z + N. \tag{7}$$

Thus, the vectors of the basis can be written in terms of the eigenvalues of P, $L = n_a + k$. Because of the symmetry (7), we can write the basis of product states as

$$\mid \mathrm{N\ L\ k}\rangle = \mid \mathrm{L} - \mathrm{k}\rangle \otimes \mid\ \mathrm{N}, \mathrm{k}\rangle. \tag{8}$$

In the basis of states with fixed values of N and L, the exact solution reads [41, 42]

$$\mid \Psi_{\mathrm{N,\,L},\,\alpha}\rangle = \sum_k c_{\mathrm{N,\,L},\,\alpha}(k) \ \mid L-k\rangle \otimes \mid \mathrm{N,\,k}\rangle. \tag{9}$$

The adopted su(3) representation in terms of the spin and quadrupole operators of [34–36, 47] allows us to simplify the construction of the exact solution. In particular, in the case $\Delta = 0$ the interaction described by the Hamiltonian of Eq. (1) reduces to that of a two atomic-level system in a $(2\,N+1)$-dimensional space

$$H = \frac{1}{3}\Omega N + \omega_a \left(a^\dagger a + 1/2\right) + \omega S_z + \zeta(a^\dagger S_- + S_+ a) + \lambda S_+ S_-. \tag{10}$$

2.1. The Initial Condition

To study the time evolution of the states and observables described by the previous models, we follow the formalism presented in [41, 42, 47, 48]. We assume that the initial state is the direct product of the coherent photon state and the coherent spin state (CSS)

$$\mid I\rangle = \mid z_{\mathrm{ph}}\rangle \otimes \mid z_{\mathrm{at}}\rangle, \tag{11}$$

with $\mid z_{\mathrm{ph}}\rangle = \mathcal{N}_{\mathrm{ph}}\ \mathrm{e}^{z_{\mathrm{ph}} a^\dagger} \mid 0\rangle$ and $\mid z_{\mathrm{at}}\rangle = \mathcal{N}_{\mathrm{at}}\ \mathrm{e}^{z_{\mathrm{at}} S_+} \mid 0\rangle$. The parameter z_{ph} is related to the mean value of photons in the system through $\mid z_{\mathrm{ph}} \mid^2 = \langle n_a\rangle$, while $z_{\mathrm{at}} = -\mathrm{e}^{-i\phi_0}\tan(\theta_0/2)$. The angles (θ_0, ϕ_0) define the direction $\vec{n_0} = (\sin\theta_0\cos\phi_0,\, \sin\theta_0\sin\phi_0,\, \cos\theta_0)$, such that $\vec{S}\cdot\vec{n_0} \mid z_{\mathrm{at}}\rangle = -S \mid z_{\mathrm{at}}\rangle$, with $S = N$ [49].

2.2. Spin Squeezing Parameter

Atomic spin squeezed states are quantum correlated systems with reduced fluctuations in one of the collective spin components. Following Ueda and Kitagawa [50], we define a set of orthogonal axes $\{\mathbf{n_{x'}},\, \mathbf{n_{y'}},\, \mathbf{n_{z'}}\}$, such that $\mathbf{n_{z'}}$ is along the direction of $\langle \mathbf{S}\rangle$. We define the squeezing factor in the σ' direction as

$$\zeta^2(\sigma') = \frac{2(\Delta S_{\sigma'})^2}{|\langle \mathbf{S}\rangle|}. \tag{12}$$

Then the system is squeezed in the σ' direction if $\zeta^2_{\sigma'} < 1$. So defined, the parameter (12) is su(2) invariant [51]. In what follows, we fix the direction x' as the direction for which the squeezing parameter reaches its minimum value. Clearly, $\zeta^2(x')\zeta^2(y') \geq 1$.

2.3. Entropy Squeezing

The importance of squeezed states of light for optical devices employed in quantum measurements has been demonstrated during the last decade; in particular, in connection with realizations of quantum communications, teleportation, and cryptography. Generally speaking, squeezed light is a natural tool in quantum information theory (see [25, 27] and references therein). In this section, we review briefly the concept of entropy squeezing in order to relate it to the spin-squeezing mechanism of the previous section.

In both cases, we start with the Heisenberg uncertainty relations and include fluctuations. In this section, we generalize the definitions obtained by the other authors for the case of two-level atoms [25,27] to the present case of three-level atoms.

The information entropy $H(S_\sigma)$ [25,27] of the operators S_σ ($\sigma = x', y', z'$) is given by

$$H(S_\sigma) = -\sum_{j=0}^{2N} P_j(\sigma)\ \log(P_j(\sigma)), \tag{13}$$

where

$$P_j(\sigma) = \langle \sigma, j | \rho_{\rm at}(t) \mid \sigma, j \rangle \tag{14}$$

is the expectation value of the reduced atomic entropy $\rho_{\rm at} = {\rm Tr}_{\rm ph}\rho(t)$ on the jth eigenstate of the operator S_σ.

For the present case of three-level atoms, we generalize the expressions valid for the SU(2) case (two-level atoms) [25, 27] to the SU(3) representation. The quantities $H(S_{x'})$, $H(S_{y'})$, and $H(S_{z'})$, when $\mathbf{n_{z'}}$ is along the direction of $\langle \mathbf{S} \rangle$, satisfy the condition [47]

$$H(S_{x'}) + H(S_{y'}) + H(S_{z'}) \geq f(N), \tag{15}$$

with

$$f(N) = 2\log(2^{2N}) - \frac{2}{2^{2N}} \sum_k \left(\begin{array}{c} 2N \\ k \end{array} \right) \log \left(\begin{array}{c} 2N \\ k \end{array} \right).$$

This condition can also be written as

$$\delta H(S_{x'}) \delta H(S_{y'}) \geq \frac{\delta f(N)}{\delta H(S_{z'})}, \tag{16}$$

with $\delta H(S_\sigma) = e^{H(S_\sigma)}$ and $\delta f(N) = e^{f(N)}$. For a detailed derivation, the reader is referred to [47].

The atomic squeezing of the system is determined in view of the entropy uncertainty relation (16). The fluctuations in component S_σ of the spin of the atomic system are said to be squeezed if the information entropy $H(S_{\sigma)}$ satisfies the constraint

$$\zeta_E^2(\sigma') = \delta H(S'_\sigma) / \sqrt{\delta f(N)/\delta H(S_{z'})}\ < 1. \tag{17}$$

In deriving the previous equations, we used the same arguments introduced previously for spin squeezing. In the following section, we show with the help of numerical results that the time and angular dependences of both observables, spin squeezing and entropy squeezing, are rather similar and both of them can be used to characterize the degree of squeezing of the spin of atomic systems.

3. Results and Discussion

In this section, we present the results of calculations that we performed by applying the formalism given by the Hamiltonian (1). We calculated entropy $\zeta_E^2(\sigma')$ and spin $\zeta^2(\sigma')$ squeezing parameters for a

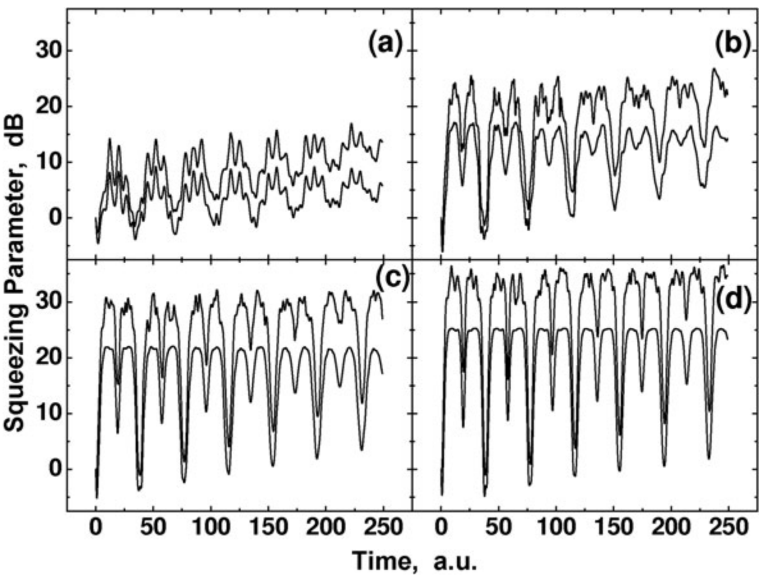

Fig. 1. Time dependence of the entropy squeezing parameter $\zeta_E^2(x')$ (upper curve) and the spin squeezing parameter $\zeta^2(x')$ (lower curve), both in the x' direction. Both parameters are given in [dB], while the time scale is given in arbitrary units. The atoms interact with couplings $\lambda = 0.08$ and $\zeta = 0.01$ for the Hamiltonian (1). The initial state corresponds to the initial coherent spin state with $\theta_0 = \pi/4$ and $\phi_0 = 0$, and the initial coherent photon state with $n_a = 60$. Here, $N = 5$ (a), 10 (b), 15 (c), and 20 (d) atoms.

system consisting of atoms of ^{87}Rb. The effective level scheme includes for the ladder configuration the state $5^2S_{1/2}$ as the lower state $|\,0\rangle$, the state $5^2P_{3/2}$ as the intermediate state $|\,1\rangle$, and the state $5^2D_{5/2}$ as the upper state $|\,2\rangle$. The energy of the photon sector of the Hamiltonian corresponds to the resonant case [48,52,53].

In the analysis, we fixed the interaction constants of the Hamiltonian (1) to realistic values extracted from our previous work on ^{87}Rb [46,48]. Since we are using natural units with $\hbar = 1$, the couplings and frequencies are given in units of energy (the scale is arbitrary; for comparison with the energy scale of the atomic case, see [46,48]), and the time variable is measured in units of the inverse energy. Both squeezing parameters are given in dB units. We discuss the results corresponding to the initial coherent state in the atomic sector with $\theta_0 = \pi/4$ and $\phi_0 = 0$, and the coherent state in the photonic sector with $\langle n_a \rangle = 60$.

Figure 1 shows the time dependence of the spin squeezing parameter $\zeta^2(x')$ and the entropy squeezing parameter $\zeta_E^2(x')$ for different numbers of atoms $N =$ 5 (a), 10 (b), 15 (c), and 20 (d). In all cases, the upper curve corresponds to the entropy squeezing parameter, and the lower curve corresponds to the spin squeezing parameter. Overall, both parameters predict similar behavior as a function of time, though the entropy squeezing parameter in the low time regime is more restrictive than the spin squeezing parameter. As the number of atoms increases, the value of the squeezing parameter also increases.

In Fig. 2, we show the dependence of the atomic squeezing in the plane $(\theta - \phi)$ perpendicular to the direction of $\langle \mathbf{S} \rangle$ at a fixed value of time for the system of $N = 20$ atoms. The results for the spin squeezing parameter are shown on panel a and for the entropy squeezing parameter on panel b, respectively. Light gray areas show the regions that exhibit squeezing ($\zeta^2(x') < 0$ and $\zeta_E^2(x') < 0$) in [dB]. From Figs. 1 and 2, we see that the regions of optimum atomic squeezing predicted by both parameters in the $(\theta - \phi)$ plane coincide. From the results presented, the range of values of θ and ϕ, for which the appearance of atomic squeezing is predicted, is larger for the spin squeezing parameter than for the entropy squeezing parameter. We performed the analysis of the angular dependence in the $(\theta - \phi)$ plane for systems of $N = 5$, 10, and 15 atoms, and obtained results are similar to the case of $N = 20$ atoms.

The behavior of the system for different initial states and different ranges of possible coupling con-

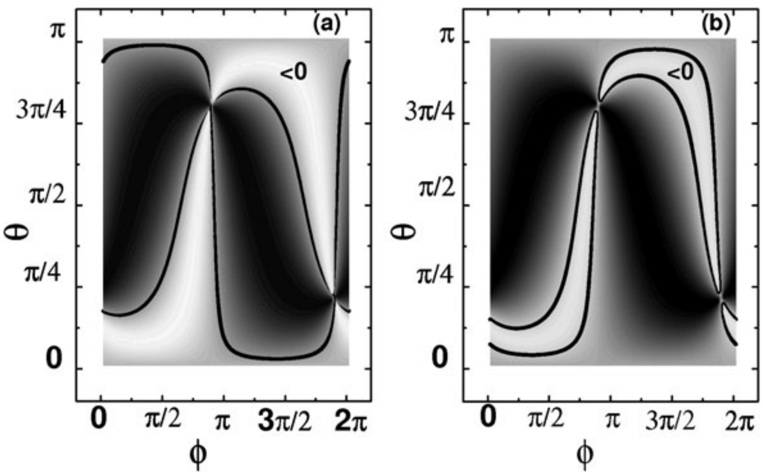

Fig. 2. Angular dependence of the atomic squeezing for a system with $N = 20$ atoms at $t = 38$ in the $(\theta - \phi)$ plane. The results for the spin squeezing parameter (a) and the corresponding results for the entropy squeezing parameter (b). Light gray areas show the regions that exhibit squeezing ($\zeta^2 < 0$ and $\zeta_E^2 < 0$) in [dB]. The other parameters are the same as in Fig. 1.

stants has been analyzed in [47]. The results seemingly indicate that both parameters provide the same information on the persistence of the orientation of the spin of the system.

4. Conclusions

We studied the appearance of atomic squeezing for three-level ^{87}Rb atoms in the ladder configuration interacting with a radiation field. We found that the use of the su(3) Cartesian dipole–quadrupole representation of the algebra simplifies the structure of the exact solution. Also, in the case of symmetry spacing levels, the system can be solved exactly in the subspace generated by the spin su(2) subalgebra. From the numerical analysis of the results, we observe that both the spin-squeezing and the entropy-squeezing parameters predict overall similar behavior concerning the time and angular dependences. However, the entropy squeezing parameter is more restrictive in the angular dependence and lower time regime than the spin squeezing parameter.

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References

1. M. A. Nielsen and I. Chuang, *Quantum Computation and Quantum Communication*, Cambridge University Press, Cambridge, UK (2000).
2. C. Gabriel, A. Aiello, W. Zhong, et al., *Phys. Rev. Lett.*, **106**, 060502 (2011).
3. L. Amico, R. Fazio, A. Osterloh, and B. Vedral, *Rev. Mod. Phys.*, **80**, 517 (2008).
4. J. Eisert, M. Cramer, and M. B. Plenio, *Rev. Mod. Phys.*, **82**, 277 (2010).
5. M. D. Reid and P. D. Drummond, *Rev. Mod. Phys.*, **81**, 1727 (2009).
6. C. Bennett, G. Brassard, C. Crepeau, et al., *Phys. Rev. Lett.*, **70**, 1895 (1993).
7. I. Khan, C. Wittmann, N. Jain, et al., *Phys. Rev. A*, **88**, 010302(R) (2013).

8. Xiao-song Ma, S. Kropatschek, W. Naylor, et al., *Opt. Exp.*, **20**, 23126 (2012).
9. Xiao-song Ma, T. Herbst, T. Scheidl, et al., *Nature*, **489**, 269 (2012).
10. U. L. Andersen and C. Timothy, *Phys. Rev. Lett.*, **111**, 050504 (2013).
11. C. E. Shanon, *Bell Syst. Techn. J.*, **27**, 379 (1948).
12. H. Araki and E. Lieb, *Commun. Math. Phys.*, **18**, 160 (1970).
13. A. Wehrl, *Rev. Mod. Phys.*, **50**, 221, (1978).
14. G. Lindblad, *Commun. Math. Phys.*, **33**, 305 (1973).
15. S. M. Barnett and S. J. D. Phoenix, *Phys. Rev. A*, **40**, 2404 (1989).
16. S. J. D. Phoenix and P. L. Knight, *Phys. Rev. A*, **44**, 6023 (1991).
17. I. I. Hirschman, *Am. J. Math.*, **79**, 152 (1957).
18. H. Kraus, *Phys. Rev. D*, **35**, 3070 (1987).
19. H. Maassen and J. B. M. Uffink, *Phys. Rev. Lett.*, **60**, 1103 (1988).
20. J. Sanchez-Ruiz, *Phys. Lett. A*, **201**, 125 (1995).
21. M. A. Man'ko, V. I. Man'ko and R. V. Mendes, *Russ. Laser Res.*, **27**, 507 (2006).
22. M. A. Man'ko, *Phys. Scr.*, **82**, 038109 (2010).
23. M. A. Man'ko and V. I. Man'ko, *Phys. Scr. T*, **147**, 014020 (2012).
24. J. Ma, X. Wang, C. P. Sun, and F. Nori, *Phys. Rep.*, **509**, 89 (2011).
25. Mahmoud Abdel-Aty, M. Sebawe Abdalla, and A. S. F. Obada, *J. Opt. B: Quantum Semiclass. Opt.*, **4**, 134 (2002).
26. M. Sebawe Abdalla and M. M. A. Ahmed, *Opt. Commun.*, **284**, 1933 (2011).
27. Qinghong Liao, Guangyu Fang, Yueyuan Wangc Muhammad Ashfaq Ahmad, and Shutian Liu, *Opt.*, **122**, 1392 (2011).
28. E. M. Khalil, M. Sebawe Abdalla, and A. S. F. Obada, *J. Phys. B: At. Mol. Opt. Phys.*, **43**, 095507 (2010).
29. M. M. Nassar, H. Eleuch, M. Abdel-Aty, and A. S. F. Obada, *J. At. Mol. Sci.*, **1**, 54 (2010).
30. A. S. F. Obada, S. A. Hanoura, and A. A. Eied, *Laser Phys.*, **23**, 055201 (2013).
31. G. Mazzarella, L. Salasnich, A. Parola, and F. Toigo, *Phys. Rev. A*, **83**, 053607 (2011).
32. C. Gross, T. Zibold, E. Nicklas, et al., *Nature*, **464**, 1165 (2010).
33. M. F. Riedel, P. Böhi, Yun Li, et al., *Nature*, **464**, 1170 (2010).
34. C. D. Hamley, C. S. Gerving, T. M. Hoang, et al., *Nature Phys.*, **8**, 305 (2012).
35. J. P. Elliot, *Proc. R. Soc. London A*, **245**, 125 (1958).
36. J. P. Elliot, *Proc. R. Soc. London A*, **245**, 562 (1958).
37. W. Vogel and R. Blatt, *Phys. Rev. A*, **45**, 3319 (1992).
38. B. J. Dalton, Z. Ficek, and P. Knight, *Phys. Rev. A*, **50**, 2646 (1954).
39. Z. Ficek, B. J. Dalton, and P. L. Knight, *Phys. Rev. A*, **51**, 4062 (1995).
40. A. B. Klimov, J. Negro, R. Farias, and S. M. Chumakov, *J. Opt. B: Quantum Semiclass. Opt.*, **1**, 562 (1999).
41. O. Civitarese and M. Reboiro, *Phys. Lett. A*, **357**, 224 (2006).
42. M. Reboiro, *Phys. Scr.*, **78**, 045006 (2008).
43. A. Dantan, M. Pinard, V. Josse, et al., *Phys. Rev. A*, **67**, 045801 (2003).
44. G. V. Varada and G. S. Agarwal, *Phys. Rev. A*, **45**, 6721 (1991).
45. C. S. Agarwal and A. K. Patnaik, *Phys. Rev. A*, **63**, 043805 (2001).
46. M. Reboiro, O. Civitarese, L. and Rebón, *Phys. Lett. A*, **366**, 241 (2007).
47. M. Reboiro, O. Civitarese, and D. Tielas, *Int. J. Mod. Phys. B*, **27**, 1350117 (2013).

48. O. Civitarese, M. Reboiro, L. Rebón, and D. Tielas, *Phys. Lett. A*, **374** 2117 (2010).
49. K. T. Hecht, *Lecture Notes in Physics*, Springer-Verlag, Berlin (1987), Vol. 290, p. 39.
50. M. Kitagawa and M. Ueda, *Phys. Rev. A*, **47**, 5138 (1993).
51. A. Luis and N. Korolkova, *Phys. Rev. A*, **74**, 043817 (2006).
52. S. Brattke, U. Kallmann and W. D. Hartman, *Eur. J. Phys. D*, **3**, 159 (1998).
53. M. J. Snadden, A. S. Bell, E. Riis, and A. I. Ferguson, *Opt. Commun.*, **125**, 70 (1996).