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TOMOGRAPHIC AND IMPROVED SUBADDITIVITY CONDITIONS FOR TWO QUBITS AND A QUDIT WITH $j = 3/2$

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Abstract

We obtain a new entropic inequality for quantum and tomographic Shannon information for systems of two qubits. We derive the inequality relating quantum information and spin-tomographic information for particles with spin $j = 3/2$. We recommend the method for obtaining new entropic and information inequalities for composite systems of qudits, as well as for one qudit.

Keywords: entropy, information, tomographic probability, qubits, qudit, subadditivity condition.

1. Introduction

The states of classical systems with fluctuating observables due to the interaction with the environment are described by the probability distributions. The probability distributions associated with one random variable q in the case of finite number of outcomes equal to N are identified with probability vectors $\mathbf{p} = (p_1, p_2, \ldots, p_N)$, where $1 \geq p_k \geq 0$, $k = 1, 2, \ldots, N$, and $\sum_{k=1}^{N} p_k = 1$. The states of quantum systems, e.g., qudit or spin j states, where $j = 0, 1/2, 1, \ldots$, are described by the density $N \times N$ matrix [1–4] $\rho_{mm'} = \langle m|\hat{\rho}|m'\rangle$, where the spin projections m, m' take the values $-j, -j+1, \ldots, j-1, j$. The diagonal elements of the density matrix $p_m = \rho_{mm}$ can be considered as components of the probability vector $\mathbf{p} = (p_{-j}, p_{-j+1}, \ldots, p_{j-1}, p_j)$, where $1 \geq p_m \geq 0$ and $\sum_{m=-j}^{j} p_m = 1$. For several $M = 2, 3, \ldots$ random classical variables, the probability distributions are joint probability distributions $\mathcal{P}(n_1, n_2, \ldots, n_M) \geq 0$, such that

$$\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \ldots \sum_{n_M=1}^{N_M} \mathcal{P}(n_1, n_2, \ldots, n_M) = 1. \tag{1}$$

For several M quantum particles, e.g., for a composite system of qudits or composite system of M particles with spins $j_1, j_2, \ldots, j_M$, the state density operator $\hat{\rho}$ has the density matrix $\rho_{\mathbf{m},\mathbf{m}'} = \langle \mathbf{m}|\hat{\rho}|\mathbf{m}'\rangle$, where $\mathbf{m} = (m_1, m_2, \ldots, m_M)$, $\mathbf{m}' = (m_1', m_2', \ldots, m_M')$ are the vectors with components corresponding to spin j_k projections $m_k = -j_k, -j_k+1, \ldots, j_k$, $k = 1, 2, \ldots, M$. The diagonal elements of the density

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matrix $\rho_{\mathbf{mm}} = \langle \mathbf{m}|\hat{\rho}|\mathbf{m}\rangle$ give the joint probability distribution $\mathcal{P}(m_1, m_2, \ldots, m_M) \geq 0$ satisfying the normalization condition

$$\sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \ldots \sum_{m_M=-j_M}^{j_M} \mathcal{P}(m_1, m_2, \ldots, m_M) = 1. \tag{2}$$

The statistical properties of quantum observables were shown to be described in terms of the standard probability theory in [5].

Each probability distribution is characterized by the Shannon entropy [6]

$$H_{\mathbf{p}} = -\sum_{k=1}^{N} p_k \ln p_k \geq 0 \tag{3}$$

for one probability vector, and for the joint probability distribution $\mathcal{P}(n_1, n_2, \ldots, n_M)$ it s

$$H_{\mathcal{P}} = -\sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} \ldots \sum_{n_M=1}^{N_M} \mathcal{P}(n_1, n_2, \ldots, n_M) \ln \mathcal{P}(n_1, n_2, \ldots, n_M) \geq 0. \tag{4}$$

For quantum states, the entropy is defined in terms of the density operator $\hat{\rho}$ as the von Neumann entropy

$$S = -\mathrm{Tr}\hat{\rho} \ln \hat{\rho}. \tag{5}$$

If the density matrix of quantum state $\rho_{\mathbf{mm}'}$ is the Hermitian, i.e., $(\rho_{\mathbf{mm}'})^\dagger = \rho_{\mathbf{mm}'}$, the trace class, i.e., $\mathrm{Tr}\,\rho = 1$, has only nonnegative eigenvalues, i.e., $\hat{\rho} \geq 0$, and diagonalized, the entropy (5) can be expressed in the form of Shannon entropy determined by the eigenvalues of the density operators identified with the joint probability distribution $\mathcal{P}(n_1, n_2, \ldots, n_M)$. The entropies determined by classical probability vectors and joint probability distributions obey some inequalities. The entropies determined by the density operators also obey other inequalities [7–17]. The inequalities, called the subadditivity condition, the stronger subadditivity condition, and the strong subadditivity condition, are considered to be valid for composite classical and quantum systems.

Recently, the new representation of quantum mechanics, called the tomographic probability representation of quantum mechanics, was introduced [18] (see also reviews [19–21]). In this representation, the density operators are mapped onto standard probability distributions, called the quantum state tomograms. The connection between tomographic schemes and the star–product quantization procedure was investigated in [22–25]. The general geometrical relations of quantum tomographic probability distributions to other possible quasidistributions were found in [26]. Reviews of the tomographic representation of quantum and classical mechanics are presented in [20, 21, 27].

There exist important relations of entropies associated with the tomographic probability distributions and von Neumann entropies [15, 28]. These relations were used to introduce the notion of quantum tomographic discord [29, 30]. On the other hand, there exist mathematical inequalities for entropies associated with the nonnegative Hermitian matrices [7–17, 31].

Our aim in this article is to find the connection between the mathematical inequalities and new tomographic inequalities for entropies of physical system states. Also we extend the inequalities considered usually for bipartite systems to the case of systems without subsystems. To do this, we apply the qubit (qudit) portrait method [17,32–34] in the spirit of the application of the method to get subadditivity and strong subadditivity condition to qudit states.

This paper is organized as follows.

In Sec. 2, we demonstrate the method we elaborated on the example of two-qubit states. In Sec. 3, we study the qudit states for the case of $j = 3/2$, and in Sec. 4 we apply the method to get new inequalities for generic qudit states. In Sec. 5, we study some nonlinear maps of the density matrices and get the subadditivity condition for $j = 3/2$. In Sec. 6, we present conclusions and prospectives.

2. Two-Qubit Entropic Inequalities

In this section, we consider in detail examples of the system state in the four-dimensional Hilbert space. The density matrix of a system state in the four-dimensional Hilbert-space reads

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}, \qquad \rho_{jk} = \langle j|\hat{\rho}|k\rangle, \qquad j,k = 1,2,3,4. \tag{6}$$

The matrix has the properties $\rho_{jk} = \rho^*_{kj}$ and $\mathrm{Tr}\rho = 1$. The eigenvalues of the density operator $\hat{\rho}$ and its $4{\times}4$ matrix ρ are nonnegative numbers $(\rho_1, \rho_2, \rho_3, \rho_4) = \mathbf{p}$. These properties do not define what physical system the state with the density matrix (6) has. To clarify this issue, one must define the properties of a basis $|k\rangle$ $(k = 1,2,3,4)$ in the Hilbert space. For example, if this matrix corresponds to the two-qubit states, it can be rewritten in the form

$$\rho = \begin{pmatrix} \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} & \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}} & \rho_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}} & \rho_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}} \\ \rho_{\frac{1}{2}-\frac{1}{2}\frac{1}{2}\frac{1}{2}} & \rho_{\frac{1}{2}-\frac{1}{2}\frac{1}{2}-\frac{1}{2}} & \rho_{\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\frac{1}{2}} & \rho_{\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}} \\ \rho_{-\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} & \rho_{-\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}} & \rho_{-\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}} & \rho_{-\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}} \\ \rho_{-\frac{1}{2}-\frac{1}{2}\frac{1}{2}\frac{1}{2}} & \rho_{-\frac{1}{2}-\frac{1}{2}\frac{1}{2}-\frac{1}{2}} & \rho_{-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}\frac{1}{2}} & \rho_{-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}} \end{pmatrix}. \tag{7}$$

We used the notation for the basis vector $|k\rangle$ in the form $|m_1 m_2\rangle$. The vector is the eigenvector of the two operators $\hat{J}_{z1} = \hat{J}_z \otimes \hat{1}_2$ and $\hat{J}_{z2} = \hat{1}_2 \otimes \hat{J}_z$, where the operator $\hat{J}_z$ has the $2{\times}2$ matrix $J_z = \sigma_z/2$, and the matrix σ_z is the Pauli matrix, i.e., $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Thus, the $4{\times}4$ matrices of operators $\hat{J}_z \otimes \hat{1}_2$ and $\hat{1}_2 \otimes \hat{J}_z$ read

$$J_{z1} = J_z \otimes 1_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad J_{z2} = 1_2 \otimes J_z = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{8}$$

These matrices commute. This means that both observables $\hat{J}_{z1}$ and $\hat{J}_{z2}$ can be measured simultaneously. They have a common set of eigenvectors $|m_1 m_2\rangle$. Thus, one has $\hat{J}_{z1}|m_1 m_2\rangle = m_1|m_1 m_2\rangle$ and

$\hat{J}_{z2}|m_1m_2\rangle = m_2|m_1m_2\rangle$, where m_1 and m_2 take the values $\pm 1/2$. In fact, the possibility to consider the matrix ρ (6) as the density matrix of the two-qubit state is based on the possibility to make the following invertible map of natural numbers onto pairs of fractions

$$1 \Longleftrightarrow 1/2\ 1/2, \quad 2 \Longleftrightarrow 1/2\ -1/2, \quad 3 \Longleftrightarrow -1/2\ 1/2, \quad 4 \Longleftrightarrow -1/2\ -1/2. \tag{9}$$

The map (9) means that the first natural numbers $1, 2, 3$, and 4 can be coded by all the pairs of possible spin projections m and m' on an arbitrary quantization direction, e.g., the z axis in the system of two qubits. Since the matrices (6) and (7) are the same matrices, all their corresponding matrix elements are equal, e.g.,

$$\rho_{11} = \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}},\ \rho_{12} = \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}},\ \ldots,\ \rho_{44} = \rho_{-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}. \tag{10}$$

Representation of the density matrix (6) in the form (7) provides the possibility to apply the procedure of taking the partial trace of the density matrix, obvious for the two-qubit state. In fact, for two-qubit state, the density matrix can be denoted as the matrix $\rho \equiv \rho(1,2)$ of the operator $\hat{\rho}(1,2)$ of a bipartite system with matrix elements $\rho_{m_1\,m_2\,m_1'\,m_2'}$ given by (7). The partial trace procedure means the following positive map of the matrix (7) [and also matrix (6)]

$$\rho(1,2) \to \rho(1) = \mathrm{Tr}_2\rho(1,2), \qquad \rho(1,2) \to \rho(2) = \mathrm{Tr}_1\rho(1,2). \tag{11}$$

The explicit form of taking the partial trace is the map

$$\sum_{m_2=-1/2}^{1/2} \rho(1,2)_{m_1\,m_2\,m_1'\,m_2} = \rho(1)_{m_1\,m_1'} \tag{12}$$

and another map

$$\sum_{m_1=-1/2}^{1/2} \rho(1,2)_{m_1\,m_2\,m_1\,m_2'} = \rho(2)_{m_2\,m_2'}. \tag{13}$$

In view of (12) and (13), we obtain

$$\rho(1) = \begin{pmatrix} \rho(1)_{\frac{1}{2}\frac{1}{2}} & \rho(1)_{\frac{1}{2}-\frac{1}{2}} \\ \rho(1)_{-\frac{1}{2}\frac{1}{2}} & \rho(1)_{-\frac{1}{2}-\frac{1}{2}} \end{pmatrix}, \qquad \rho(2) = \begin{pmatrix} \rho(2)_{\frac{1}{2}\frac{1}{2}} & \rho(2)_{\frac{1}{2}-\frac{1}{2}} \\ \rho(2)_{-\frac{1}{2}\frac{1}{2}} & \rho(2)_{-\frac{1}{2}-\frac{1}{2}} \end{pmatrix}. \tag{14}$$

The matrix elements obtained read

$$\begin{aligned}
\rho(1)_{\frac{1}{2}\frac{1}{2}} &= \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} + \rho_{\frac{1}{2}-\frac{1}{2}\frac{1}{2}-\frac{1}{2}}, \\
\rho(1)_{\frac{1}{2}-\frac{1}{2}} &= \rho(1)^*_{-\frac{1}{2}\frac{1}{2}} = \rho_{\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}} + \rho_{\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}, \\
\rho(1)_{-\frac{1}{2}-\frac{1}{2}} &= \rho_{-\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}} + \rho_{-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}, \\
\rho(2)_{\frac{1}{2}\frac{1}{2}} &= \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}-\frac{1}{2}\frac{1}{2}}, \\
\rho(2)_{\frac{1}{2}-\frac{1}{2}} &= \rho(2)^*_{-\frac{1}{2}\frac{1}{2}} = \rho_{\frac{1}{2}\frac{1}{2}\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}, \\
\rho(2)_{-\frac{1}{2}-\frac{1}{2}} &= \rho_{\frac{1}{2}-\frac{1}{2}\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}-\frac{1}{2}}.
\end{aligned} \tag{15}$$

One can interpret the maps in the form of positive maps of 4×4 matrices

$$\rho(1,2) \to \begin{pmatrix} \rho(1) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho(1,2) \to \begin{pmatrix} \rho(2) & 0 \\ 0 & 0 \end{pmatrix}. \tag{16}$$

On the other hand, the maps obtained by the partial trace procedure can be applied directly to the matrix (6). In this case, the maps read

$$\rho \to \begin{pmatrix} \rho_1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho \to \begin{pmatrix} \rho_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{17}$$

where

$$\rho_1 = \begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{31}+\rho_{42} & \rho_{33}+\rho_{44} \end{pmatrix}, \qquad \rho_2 = \begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix}. \tag{18}$$

It is worth noting that all 24 permutations of the four numbers $1,2,3,4 \to 1_p, 2_p, 3_p, 4_p$, which provide permutations of the basis vectors $|j\rangle \to |j_p\rangle$ yield the positive maps of the initial matrix $\rho \to \rho_p$. The matrix elements of the matrix $(\rho_p)_{jk}$ become $\rho_{j_p k_p}$. Then the maps (17) and (18) become the positive maps of the initial matrix ρ onto the matrices $\rho_1^{(p)}$ and $\rho_2^{(p)}$, where these matrices have the matrix elements with index permutations $j \to j_p$ and $k \to k_p$. Due to an invertible coding of the numbers $1,2,3$, and 4 by pairs of spin projections m and m', the obvious positive maps of the two-qubit matrix given by (14) provide, after applying the number-permutation tool, new maps of the two-qubit density matrix, which are not obtained by a simple partial tracing.

The standard two-qubit entropic inequalities, which are the nonnegativity of von Neumann entropies of any qubit system and the subadditivity condition in the system of two qubits, read

$$-\mathrm{Tr}\,\rho(1)\ln\rho(1) \geq 0, \qquad -\mathrm{Tr}\,\rho(2)\ln\rho(2) \geq 0, \qquad -\mathrm{Tr}\,\rho(1,2)\ln\rho(1,2) \geq 0 \tag{19}$$

and

$$-\mathrm{Tr}\,\rho(1)\ln\rho(1) - \mathrm{Tr}\,\rho(2)\ln\rho(2) \geq -\mathrm{Tr}\,\rho(1,2)\ln\rho(1,2). \tag{20}$$

Here, the density matrices of the first qubit $\rho(1)$ and the second qubit $\rho(2)$ given by (14) with matrix elements (15) are obtained by partial tracing of the density matrix $\rho(1,2)$ given by (7) using the map of indices (10). On the other hand, the density matrix ρ given by (7) is identical to the matrix (16). Taking this fact into account, we write inequalities (19) and (20) in the form

$$-\mathrm{Tr}\begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{42}+\rho_{31} & \rho_{33}+\rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11}+\rho_{22} & \rho_{13}+\rho_{24} \\ \rho_{42}+\rho_{31} & \rho_{33}+\rho_{44} \end{pmatrix} \geq 0. \tag{21}$$

An analogous nonnegativity condition for the von Neumann entropy associated with the matrix ρ_2 given by (18) is also valid. An analog of the subadditivity condition (20) can be written for the matrix ρ given by (6); it appears as

$$-\mathrm{Tr}\begin{pmatrix}\rho_{11}&\rho_{12}&\rho_{13}&\rho_{14}\\ \rho_{21}&\rho_{22}&\rho_{23}&\rho_{24}\\ \rho_{31}&\rho_{32}&\rho_{33}&\rho_{34}\\ \rho_{41}&\rho_{42}&\rho_{43}&\rho_{44}\end{pmatrix}\ln\begin{pmatrix}\rho_{11}&\rho_{12}&\rho_{13}&\rho_{14}\\ \rho_{21}&\rho_{22}&\rho_{23}&\rho_{24}\\ \rho_{31}&\rho_{32}&\rho_{33}&\rho_{34}\\ \rho_{41}&\rho_{42}&\rho_{43}&\rho_{44}\end{pmatrix}\leq -\mathrm{Tr}\left[\begin{pmatrix}\rho_{11}+\rho_{22}&\rho_{13}+\rho_{24}\\ \rho_{42}+\rho_{31}&\rho_{33}+\rho_{44}\end{pmatrix}\right.$$
$$\left.\times\ln\begin{pmatrix}\rho_{11}+\rho_{22}&\rho_{13}+\rho_{24}\\ \rho_{42}+\rho_{31}&\rho_{33}+\rho_{44}\end{pmatrix}\right]-\mathrm{Tr}\begin{pmatrix}\rho_{11}+\rho_{33}&\rho_{12}+\rho_{24}\\ \rho_{21}+\rho_{43}&\rho_{22}+\rho_{44}\end{pmatrix}\ln\begin{pmatrix}\rho_{11}+\rho_{33}&\rho_{12}+\rho_{34}\\ \rho_{21}+\rho_{43}&\rho_{22}+\rho_{44}\end{pmatrix}. \tag{22}$$

There are 24 inequalities, which can be obtained from (22) for the matrix elements of the matrix ρ (6) by all permutations of the numbers $1, 2, 3$, and 4. Thus, we obtained entropic inequalities for arbitrary Hermitian nonnegative 4×4 matrix ρ with $\mathrm{Tr}\,\rho = 1$. We derived these inequalities using the identity of the mathematical structure of this matrix with the density matrix of the two-qubit system.

3. Entropic Inequalities for the Density Matrix of a Qudit with $j = 3/2$

We consider the density matrix of the qudit state corresponding to spin $j = 3/2$. The Hermitian nonnegative 4×4 matrix $\rho^{3/2}_{mm'}$ has the form

$$\rho^{3/2}=\begin{pmatrix}\rho_{\frac32\frac32}&\rho_{\frac32\frac12}&\rho_{\frac32-\frac12}&\rho_{\frac32-\frac32}\\ \rho_{\frac12\frac32}&\rho_{\frac12\frac12}&\rho_{\frac12-\frac12}&\rho_{\frac12\frac32}\\ \rho_{-\frac12\frac32}&\rho_{-\frac12\frac12}&\rho_{-\frac12-\frac12}&\rho_{-\frac12-\frac32}\\ \rho_{-\frac32\frac32}&\rho_{-\frac32\frac12}&\rho_{-\frac32-\frac12}&\rho_{-\frac32-\frac32}\end{pmatrix}. \tag{23}$$

One has the property $\mathrm{Tr}\,\rho^{3/2} = 1$. The indices m and m' in the matrix $\rho^{3/2}_{mm'} = \langle m|\hat\rho^{3/2}|m'\rangle$ are the spin projections taking values $3/2, 1/2, -1/2$, and $-3/2$.

Now we introduce the invertible map of four numbers

$$3/2\leftrightarrow1,\quad 1/2\leftrightarrow2,\quad -1/2\leftrightarrow3,\quad -3/2\leftrightarrow4.$$

Applying this map to matrix (23), we obtain the matrix $\rho^{3/2}$ in the form identical to the matrix (6). This means that all equalities and inequalities known for matrix (6) are valid for matrix (23). Thus, we arrive at new inequalities for the density matrix $\rho^{3/2}$ of the qudit state given (23) by making the map of indices, e.g., in (21) and (22).

We write the new inequalities explicitly. We have the inequality $-\mathrm{Tr}\,\rho^{3/2}\ln\rho^{3/2}\geq 0$; in the explicit form, it reads

$$-\mathrm{Tr}\begin{pmatrix}\rho_{\frac32\frac32}+\rho_{\frac12\frac12}&\rho_{\frac32-\frac12}+\rho_{\frac12-\frac32}\\ \rho_{-\frac12\frac32}+\rho_{-\frac32\frac12}&\rho_{-\frac12-\frac12}+\rho_{-\frac32-\frac32}\end{pmatrix}\ln\begin{pmatrix}\rho_{\frac32\frac32}+\rho_{\frac12\frac12}&\rho_{\frac32-\frac12}+\rho_{\frac12-\frac32}\\ \rho_{-\frac12\frac32}+\rho_{-\frac32\frac12}&\rho_{-\frac12-\frac12}+\rho_{-\frac32-\frac32}\end{pmatrix}\geq 0, \tag{24}$$

being analogous to the nonnegativity of the von Neumann entropy of the qubit-subsystem state, but the qudit with $j = 3/2$ does not have such a subsystem.

An analogous entropic nonnegativity condition can be written using the introduced map of indices in the matrix ρ_2 given by (18). The new subadditivity condition for qubit states with $j = 3/2$ has the explicit form

$$
\begin{aligned}
&-\mathrm{Tr}\begin{pmatrix}
\rho_{\frac{3}{2}\,\frac{3}{2}} & \rho_{\frac{3}{2}\,\frac{1}{2}} & \rho_{\frac{3}{2}\,-\frac{1}{2}} & \rho_{\frac{3}{2}\,-\frac{3}{2}} \\
\rho_{\frac{1}{2}\,\frac{3}{2}} & \rho_{\frac{1}{2}\,\frac{1}{/}2} & \rho_{\frac{1}{2}\,-\frac{1}{2}} & \rho_{\frac{1}{2}\,\frac{3}{2}} \\
\rho_{-\frac{1}{2}\,\frac{3}{2}} & \rho_{-\frac{1}{2}\,\frac{1}{2}} & \rho_{-\frac{1}{2}\,-\frac{1}{2}} & \rho_{-\frac{1}{2}\,-\frac{3}{2}} \\
\rho_{-\frac{3}{2}\,\frac{3}{2}} & \rho_{-\frac{3}{2}\,\frac{1}{2}} & \rho_{-\frac{3}{2}\,-\frac{1}{2}} & \rho_{-\frac{3}{2}\,-\frac{3}{2}}
\end{pmatrix}
\ln\begin{pmatrix}
\rho_{\frac{3}{2}\,\frac{3}{2}} & \rho_{\frac{3}{2}\,\frac{1}{2}} & \rho_{\frac{3}{2}\,-\frac{1}{2}} & \rho_{\frac{3}{2}\,-\frac{3}{2}} \\
\rho_{\frac{1}{2}\,\frac{3}{2}} & \rho_{\frac{1}{2}\,\frac{1}{2}} & \rho_{\frac{1}{2}\,-\frac{1}{2}} & \rho_{\frac{1}{2}\,\frac{3}{2}} \\
\rho_{-\frac{1}{2}\,\frac{3}{2}} & \rho_{-\frac{1}{2}\,\frac{1}{2}} & \rho_{-\frac{1}{2}\,-\frac{1}{2}} & \rho_{-\frac{1}{2}\,-\frac{3}{2}} \\
\rho_{-\frac{3}{2}\,\frac{3}{2}} & \rho_{-\frac{3}{2}\,\frac{1}{2}} & \rho_{-\frac{3}{2}\,-\frac{1}{2}} & \rho_{-\frac{3}{2}\,-\frac{3}{2}}
\end{pmatrix} \leq \\
&-\mathrm{Tr}\begin{pmatrix}
\rho_{\frac{3}{2}\,\frac{3}{2}} + \rho_{\frac{1}{2}\,\frac{1}{2}} & \rho_{\frac{3}{2}\,-\frac{1}{2}} + \rho_{\frac{1}{2}\,-\frac{3}{2}} \\
\rho_{-\frac{1}{2}\,\frac{3}{2}} + \rho_{-\frac{3}{2}\,\frac{1}{2}} & \rho_{-\frac{1}{2}\,-\frac{1}{2}} + \rho_{-\frac{3}{2}\,-\frac{3}{2}}
\end{pmatrix}
\ln\begin{pmatrix}
\rho_{\frac{3}{2}\,\frac{3}{2}} + \rho_{\frac{1}{2}\,\frac{1}{2}} & \rho_{\frac{3}{2}\,-\frac{1}{2}} + \rho_{\frac{1}{2}\,-\frac{3}{2}} \\
\rho_{-\frac{1}{2}\,\frac{3}{2}} + \rho_{-\frac{3}{2}\,\frac{1}{2}} & \rho_{-\frac{1}{2}\,-\frac{1}{2}} + \rho_{-\frac{3}{2}\,-\frac{3}{2}}
\end{pmatrix} \\
&-\mathrm{Tr}\begin{pmatrix}
\rho_{\frac{3}{2}\,\frac{3}{2}} + \rho_{-\frac{1}{2}\,-\frac{1}{2}} & \rho_{\frac{3}{2}\,\frac{1}{2}} + \rho_{-\frac{1}{2}\,-\frac{3}{2}} \\
\rho_{\frac{1}{2}\,\frac{3}{2}} + \rho_{-\frac{3}{2}\,-\frac{1}{2}} & \rho_{\frac{1}{2}\,\frac{1}{2}} + \rho_{-\frac{3}{2}\,-\frac{3}{2}}
\end{pmatrix}
\ln\begin{pmatrix}
\rho_{\frac{3}{2}\,\frac{3}{2}} + \rho_{-\frac{1}{2}\,-\frac{1}{2}} & \rho_{\frac{3}{2}\,\frac{1}{2}} + \rho_{-\frac{1}{2}\,-\frac{3}{2}} \\
\rho_{\frac{1}{2}\,\frac{3}{2}} + \rho_{-\frac{3}{2}\,-\frac{1}{2}} & \rho_{\frac{1}{2}\,\frac{1}{2}} + \rho_{-\frac{3}{2}\,-\frac{3}{2}}
\end{pmatrix}. \qquad (25)
\end{aligned}
$$

The subadditivity condition (25) takes place for the system (qudit with $j = 3/2$) which has no subsystems.

Other inequalities of such a form are obtained by arbitrary permutations in (25) of four numbers $3/2, 1/2, -1/2$, and $-3/2$.

Now we consider the system of two particles with $j = 1/2$.

The spin states are $|e_1\rangle = |11\rangle$, $|e_2\rangle = |10\rangle$, and $|e_3\rangle = |1{-}1\rangle$ for spin $j = 1$, and the state is $|e_4\rangle = |00\rangle$ for spin $j = 0$. This means that the density 4×4 matrix for this system has matrix elements ρ_{jk}, where j and k are equal to $1, 2, 3$, and 4, and these numbers are mapped onto pairs of numbers

$$1 \Leftrightarrow 1\ 1, \qquad 2 \Leftrightarrow 1\ 0, \qquad 3 \Leftrightarrow 1\ -1, \qquad 4 \Leftrightarrow 0\ 0.$$

Using this map, one can get the entropic subadditivity condition of the form (22) with the introduced substitutions of the indices. In fact, the introduced new inequality for spin-1 and spin-0 systems of two particles with spins $j = 1/2$ can be expressed in terms of the inequality written in the basis $|mm'\rangle$, if one uses the connection of two different bases through the Clebsch–Gordan coefficients, providing the unitary transform 4×4 matrix

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad F = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad (26)$$

with F being the quantum Fourier transform.

The basis $|e_n\rangle$ and basis $|mm'\rangle$ are related as

$$|e_1\rangle = \Big|\frac{1}{2}\ \frac{1}{2}\Big\rangle, \quad |e_2\rangle = \frac{1}{\sqrt{2}}\left(\Big|\frac{1}{2}\ \frac{1}{2}\Big\rangle + \Big|\frac{1}{2}\ -\frac{1}{2}\Big\rangle\right), \quad |e_3\rangle = \frac{1}{\sqrt{2}}\left(\Big|\frac{1}{2}\ \frac{1}{2}\Big\rangle - \Big|\frac{1}{2}\ -\frac{1}{2}\Big\rangle\right), \quad |e_4\rangle = \Big|-\frac{1}{2}\ -\frac{1}{2}\Big\rangle.$$

Thus, the matrix of the density operator $\langle e_n|\hat{\rho}|e_m\rangle = \rho^{(1)}_{nm}$ and the density matrix $\rho_{m_1m_2m_1'm_2} \equiv \rho^{(2)}_{m_1m_2m_1'm_2'}$ are connected through the basis transform expressed in terms of the unitary matrix C. Both matrices $\rho^{(1)}$ and $\rho^{(2)}$ satisfy the entropic inequality corresponding to the subadditivity condition. This property is a partial case of the obvious general property of two density matrices connected by an arbitrary unitary transform, i.e., $\rho^{(1)} = u\rho^{(2)}u^{+}$. Both matrices satisfy the entropic inequalities under discussion.

4. Improved Subadditivity Condition

The tomographic probability distribution for spin states introduced in [35–37] and developed in [38–43] provide the possibility to describe the states with the density matrix ρ of qudits by means of the tomogram

$$w(m, \mathbf{n}) = \langle m|u\rho u^+|m\rangle, \tag{27}$$

where u is the unitary matrix, and $m = -j, -j+1, \ldots, j,\ j = 0, 1/2, 1, \ldots$ are spin projections. The matrix u can also be considered as a matrix of the irreducible representation of the $SU(2)$ group. For several qudits, the tomogram of the composite system states reads

$$w(\mathbf{m}, n) = \langle \mathbf{m}|u\rho(1, 2, \ldots, N)u^+|\mathbf{m}\rangle, \qquad \mathbf{m} = (m_1, m_2, \ldots, m_N).$$

The components of the vector $\mathbf{m}$, $m_k = -j_k, -j_k + 1, \ldots, j_k$, correspond to a qudit with spin j_k. The tomogram $w(\mathbf{m}, u)$ is the joint probability distribution of random spin projections $m_1, m_2, \ldots, m_N$ depending on a fixed unitary matrix u. The matrix can be considered as the direct product $u = u_1 \otimes u_2 \otimes u_N$ of matrices of irreducible representations of the $SU(2)$ group.

There is the tomographic Shannon entropy [14, 16] corresponding to the tomogram $w(\mathbf{m}, u)$; it reads

$$H(u) = -\sum_{\mathbf{m}} w(\mathbf{m}, u) \ln w(\mathbf{m}, u). \tag{28}$$

The minimum of the entropy on the unitary group is equal to the von Neumann entropy [21, 26]

$$-\mathrm{Tr}\, \rho(1, 2, \ldots, N) \ln \rho(1, 2, \ldots, N) = \min_u H(u) = H(u_0). \tag{29}$$

For a bipartite system $\rho(1, 2)$, one has the entropic inequality, which is the subadditivity condition for tomogram $w(m_1, m_2, u)$; it reads

$$H_1(u) + H_2(u) \geq H(u), \tag{30}$$

where

$$H_k(u) = -\sum_{m_k} w_k(m_k, u) \ln w_k(m_k, u), \qquad k = 1, 2 \tag{31}$$

and

$$w_1(m_1, u) = \sum_{m_2} w(m_1, m_2, u), \qquad w_2(m_2, u) = \sum_{m_1} w(m_1, m_2, u). \tag{32}$$

The Shannon information is denoted as $I(u) = H_1(u) + H_2(u) - H(u)$. One has the quantum information inequality

$$I_q = S_1 + S_2 - S(1, 2) \geq 0, \tag{33}$$

where $S_k = -\mathrm{Tr}\, \rho(k) \ln \rho(k)$, $k = 1, 2$, $\rho_1 = \mathrm{Tr}_2 \rho(1, 2)$, $\rho_2 = \mathrm{Tr}_1 \rho(1, 2)$, and

$$S_1 = \min_{u_1} H_1(u_1) = H_1(u_{10}), \qquad S_2 = \min_{u_2} H_2(u_2) = H_2(u_{20}). \tag{34}$$

The Shannon entropies $H_k(u_k)$ satisfy the equality

$$H_1(u_1) \equiv H_1(u_1 \times u_2), \qquad H_2(u_2) \equiv H_2(u_1 \times u_2). \tag{35}$$

The unitary matrices u_1 and u_2 are unitary local transforms in the Hilbert spaces of the first and second subsystems, respectively. In [15], it was shown that

$$S_1 + S_2 \geq H(u_{10} \times u_{20}) \geq S(1,2), \tag{36}$$

where $S_1 = H_1(u_{10})$ and $S_2 = H_2(u_{20})$. The general inequality for entropies was found in [7]. In the tomographic picture of qudit states, we obtain the inequality in the form

$$I_q = S_1 + S_2 - S(1,2) \geq -\sum_{m_1} w_1(m_1,u_1)\ln[w_1(m_1,u_1)] - \sum_{m_2} w_2(m_2,u_2)\ln[w_2(m_2,u_2)] + \sum_{m_1 m_2} w(m_1,m_2,u_1\times u_2)\ln[w(m_1,m_2,u_1\times u_2)]. \tag{37}$$

This means that quantum information bounds the tomographic information for any local unitary transforms. This inequality provides the inequality associated with the tomographic quantum discord property [29,30]

$$S_1 + S_2 - S_{12} \geq S_1 + S_2 + \sum_{m_1 m_2} w(m_1,m_2,u_1\times u_2)\ln[w(m_1,m_2,u_1\times u_2)]. \tag{38}$$

It can be written in the form of spin-tomographic entropic inequality for the case where the unitary matrices of local transforms u_1 and u_2 are matrices of irreducible representations of the $SU(2)$ group. In this case, $w(m_1,m_2,u_1\times u_2) \equiv w(m_1,m_2,\mathbf{n_1},\mathbf{n_2})$ and

$$w_1(m_1,u_1) \equiv w_1(m_1,\mathbf{n_1}), \quad w_2(m_2,u_2) \equiv w_2(m_2,\mathbf{n_2}), \tag{39}$$

where $\mathbf{n_1}$ and $\mathbf{n_2}$ are unit vectors $\mathbf{n_1}^2 = \mathbf{n_2}^2 = 1$ determining the directions of the spin-projection axes.

The tomographic entropies $H_k(u_k)$ and $H(u)$ become the functions of the sphere S^2, i.e.,

$$H_1(u_1) \equiv H_1(\mathbf{n}_1), \qquad H_2(u_2) \equiv H_2(\mathbf{n}_2), \qquad H(u_1\times u_2) \equiv H(\mathbf{n}_1,\mathbf{n}_2). \tag{40}$$

Inequality (37) reads

$$S_1 + S_2 - S(1,2) \geq \langle I \rangle, \tag{41}$$

where

$$\langle I \rangle = \frac{1}{4\pi}\int H_1(\mathbf{n}_1)\,d\mathbf{n}_1 + \frac{1}{4\pi}\int H_2(\mathbf{n}_2)\,d\mathbf{n}_2 - \frac{1}{16\pi^2}\int H(\mathbf{n}_1,\mathbf{n}_2)\,d\mathbf{n}_1\,d\mathbf{n}_2 \tag{42}$$

is the averaged spin-tomographic information.

The difference between quantum information I_q and the maximum of the unitary tomographic information I_t

$$I_t = \max_{u_1\times u_2}[H_1(u_1) + H_2(u_2) - H(u_1\times u_2)], \tag{43}$$

i.e.,

$$I_q - I_t = \Delta I \geq 0 \tag{44}$$

is a characteristics of quantum correlations in the bipartite qudit system. One has the information inequality

$$I_q - \langle I \rangle \geq \Delta I \geq 0. \tag{45}$$

5. Nonlinear Positive Maps and New Inequalities for Qudits

We study now another inequality that follows from [7].

Let us construct nonlinear positive maps of the density 4×4 matrix ρ of the form

$$\rho \to \rho^{(1)} \to \frac{1}{\rho_{11}+\rho_{22}} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \qquad \rho \to \rho^{(2)} \to \frac{1}{\rho_{33}+\rho_{44}} \begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{pmatrix}, \tag{46}$$

along with the linear positive maps $\rho \to \rho_1$ and $\rho \to \rho_2$, where ρ_1 and ρ_2 are determined by (18). It turns out that there exists a new inequality valid for an arbitrary density 4×4 matrix. The inequality can be derived from the results of [7]. For the density 4×4 matrix with elements ρ_{jk}, $j,k = 1,2,3,4$; it reads

$$\begin{aligned} &\langle S_2 \rangle = \\ &-\mathrm{Tr}\begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \ln \begin{pmatrix} \dfrac{\rho_{11}}{\rho_{11}+\rho_{22}} & \dfrac{\rho_{12}}{\rho_{11}+\rho_{22}} \\ \dfrac{\rho_{21}}{\rho_{11}+\rho_{22}} & \dfrac{\rho_{22}}{\rho_{11}+\rho_{22}} \end{pmatrix} - \mathrm{Tr}\begin{pmatrix} \rho_{33} & \rho_{34} \\ \rho_{43} & \rho_{44} \end{pmatrix} \ln \begin{pmatrix} \dfrac{\rho_{33}}{\rho_{33}+\rho_{44}} & \dfrac{\rho_{34}}{\rho_{33}+\rho_{44}} \\ \dfrac{\rho_{43}}{\rho_{33}+\rho_{44}} & \dfrac{\rho_{44}}{\rho_{33}+\rho_{44}} \end{pmatrix} \\ &\leq -\mathrm{Tr}\begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix} \ln \begin{pmatrix} \rho_{11}+\rho_{33} & \rho_{12}+\rho_{34} \\ \rho_{21}+\rho_{43} & \rho_{22}+\rho_{44} \end{pmatrix}. \end{aligned} \tag{47}$$

Inequality (47) can be applied for density matrices of two-qubit states, for the density matrix of a qudit with $j = 3/2$, and for states of two qubits in the basis of states of spin $j = 1$ and $j = 0$. For this, we apply the corresponding notation of numbers 1, 2, 3, and 4 discussed in the previous sections. Inequality (47) means that there are two inequalities of the form [15]

$$S_1 + S_2 \geq H(u_{10} \times u_{20} \geq S(1,2) \tag{48}$$

and the inequality

$$S_1 + S_2 \geq S_1 + \langle S_2 \rangle \geq S(1,2). \tag{49}$$

Inequality (47) can be extended to obtain the inequalities by means of permutations of numbers 1, 2, 3, and 4. Analogous inequalities can be written for arbitrary composite qudit systems including only one qudit.

6. Conclusions

To conclude, we present the main results of our study.

We obtained new inequalities for the composite systems of qudits and for systems without subsystems. We derived the condition analogous to the subadditivity condition for a qudit with spin $j = 3/2$. Also we found the new inequality – an analog of the improved subadditivity condition for systems without subsystems. We presented the entropic inequality for states with $j = 3/2$ and $j = 1 \oplus j = 0$, as well as for two-qubit states, employing a unified method. The method can be used to formulate some generic entropic inequalities for arbitrary Hermitian nonnegative trace-class matrices. We pointed out that all the discussed inequalities take place for the matrices independently of the product structure of the Hilbert space. This means that all information entropic inequalities are valid for both composite and noncomposite quantum systems.

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