Journal of Russian Laser Research, Volume 34, Number 6, November, 2013

QUANTIZED MAGNETIC FLUX IN THE BOHR–SOMMERFELD MODEL

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Abstract

Based on the Bohr–Sommerfeld model, we investigate the quantization of magnetic flux through the electronic orbits together with its dependence on additional sources of magnetic fields. The additional magnetic field causes changes of the angular momentum and hence shifts of the energy of the atomic levels. We study this effect for the cases of the Zeeman effect, where the source is an external homogeneous magnetic field, and the hyperfine interaction, where the source is the field of the magnetic moment of the nucleus. We discuss a model for the handling of the different angular momentum contributions for which the energy shifts due to the Zeeman effect and the magnetic dipole contribution to the hyperfine interaction can be reproduced quite well. The meaning of "spin," however, changes within this approach drastically. The unusual Landé g-factor of the electron is discussed to be the result of a reduced ground-state angular momentum of the electron in combination with the field of the magnetic moment of the electron rather than an intrinsic property of the electron.

Keywords: Zeeman effect, hyperfine interaction, spin, magnetic flux quantization, hydrogen atom.

1. Introduction

The magnetic flux through the electronic orbits of the hydrogen atom was investigated by different methods within several atomic models; there are the Schrödinger model [1,2], the Dirac model [3], and the Rutherford–Bohr model [4] showing, in particular, that the magnetic flux through these orbits is quantized and has a pronounced spin dependence. The quantization of magnetic flux in units of $\Phi_0 = h/e$ was first recognized in the 1950s by London [5] and Onsager [6] by considering a supercurrent around a closed path. The quantization (in units of $\Phi_0/2$) was observed only ten years later by Doll and Näbauer [7] and, independently, by Deaver and Fairbank [8] while measuring the torque on superconducting rings (hollow cylinders) in external magnetic fields.

One method, which was used for studying the magnetic flux through the electronic orbits within the Schrödinger and Dirac models, uses the conversion of the area integral of the magnetic induction into a time-integral over the cyclotron period [9]. The source of the magnetic field was taken to be the magnetic moment of the nucleus (here proton) [1]. In [4], it was discussed that this approach fails to predict the magnetic flux through the orbits within the helium ion ${}^4He^+$. However, using a time-integrated version of the Faraday law of induction (see also [10–14]) it can be shown that, in the point-particle picture of the Rutherford–Bohr model, the magnetic flux through each electronic orbit, which fulfills the Bohr–Sommerfeld–Wilson (BSW) quantization rule, is an integer multiple of the magnetic flux quantum (h/e).

Manuscript submitted by the author in English on September 23, 2013.
1071-2836/13/3406-0553

By considering the magnetic flux from the magnetic moment of the nucleus as a disturbance, an energy shift of nearly 3/8 times the experimental value of the hyperfine splitting of the ground state of the hydrogen atom was shown to be the result of the additional magnetic flux.

Here, the method of magnetic flux quantization is applied to the more complicated but still classical model of the Bohr–Sommerfeld atom [15]. In the case of electrons, the time-integrated version of the Faraday law together with magnetic flux quantization is still equivalent to the BSW quantization rule in the case of elliptic orbits. The energy shifts due to small homogeneous external magnetic fields and the magnetic moment of the nucleus are investigated within the Bohr–Sommerfeld model of the atom. These shifts can be shown to be in good agreement with the well-known energy shifts according to the Zeeman effect, Paschen–Back effect, and the magnetic dipole contribution of the hyperfine coupling. The Zeeman effect was already associated to an additional magnetic field in the case of the Aharonov–Bohm effect [11, 16]. Also spin-orbit coupling was discussed as a special case of the Zeeman effect [17].

This paper is organized as follows.

In Sec. 2, the formalism is applied to the elliptic orbits of the Bohr–Sommerfeld model. In Sec. 3, small disturbances due to additional magnetic fields in a simplified version are discussed that, however, leads to a better understanding of the basic rules. Only within this section is the electron considered to have a magnetic moment but no "spin" angular momentum. In Sec. 4, the effects of external magnetic fields and the magnetic moments of the nucleus are discussed without that restriction.

The understanding of these effects in the Bohr–Sommerfeld model can be valuable for understanding the magnetic flux quantization in the Schrödinger and Dirac models. These probability-density-based models would need information on the structure of the magnetic field and are, therefore, much more complicated to study than the point-particle models. However, a recent study of a modified Bohr model of molecules gives sound results describing the interatomic potentials [18–21], where the Bohr model was related to the large-D limit of the Schrödinger equation by dimensional scaling methods.

2. Magnetic Flux through Elliptic Orbits

Closed electronic orbits fulfilling the Bohr–Sommerfeld–Wilson (BSW) quantization rule enclose a magnetic flux, which is an integer multiple of the magnetic flux quantum ($\Phi_0 = h/e$) [4]. The magnetic flux enclosed by the electronic orbit can be calculated by considering the adiabatic acceleration of the electron due to increase of the magnetic flux through its orbit by means of the Faraday law of induction (see, e.g., [12]). In contrast to the derivation within the framework of the Rutherford–Bohr model of the atom, not only one quantum number fulfills the BSW quantization rule but two, and, in the case of external fields, three quantum numbers have to be considered.

According to the Faraday law of induction, the time-derivative of the magnetic flux through a region Σ is opposite to the electromotive force (EMF) along the boundary $\partial\Sigma$ of that region

$$\oint_{\partial\Sigma} \vec{E} \cdot d\vec{s} =: \mathrm{EMF} = -\frac{d}{dt}\int_{\Sigma} \vec{B} \cdot d\vec{A} = -\frac{d}{dt}\Phi, \tag{1}$$

where Φ is the magnetic flux through Σ, and overhead arrows denote vector quantities. Performing the time integration of this equation and assuming an adiabatic acceleration of the electron with initially vanishing momentum, only the integration boundaries need to be considered, for an electron giving rise to

$$\oint_{\partial\Sigma} \vec{p} \cdot d\vec{s} = e \cdot (\Phi_f - \Phi_i) = e \cdot \Delta\Phi, \tag{2}$$

where Φ_i is the initial and Φ_f the final magnetic flux through Σ. The left-hand side is quantized for closed orbits according to the BSW quantization rule, and so is the right-hand side, which implies a quantization of the magnetic flux Φ_f through the region Σ for vanishing initial magnetic flux Φ_i. Postulating that the magnetic flux through the orbits is still quantized in the case of nonvanishing initial magnetic fluxes Φ_i, this equation expresses a modified version of the BSW quantization rule. Using the quantization condition $\Phi_f = nh/e$ for the final magnetic flux gives

$$\oint_{\partial\Sigma} \vec{p} \cdot d\mathbf{s} = nh - e\Phi_i. \tag{3}$$

The index "i" for initial flux will be suppressed from here, as only initial fluxes are considered in the following. The final flux is considered to be quantized. Here, energy shifts due to small initial magnetic fields will be studied by considering different initial magnetic fluxes associated to the different quantum numbers. In analogy to the derivation of the energy for elliptic orbits originally done by Sommerfeld [15, 22], the energy in the case of small disturbances can be derived by replacing the two generalized momenta J_φ and J_r as follows:

$$J_\varphi = \oint \frac{\partial S}{\partial \varphi} d\varphi = n_\varphi h \qquad \text{by} \qquad J_\varphi = \oint \frac{\partial S}{\partial \varphi} d\varphi = n_\varphi h - e\Phi_\varphi \tag{4}$$

and

$$J_r = \oint \frac{\partial S}{\partial r} dr = n_r h \qquad \text{by} \qquad J_r = \oint \frac{\partial S}{\partial r} dr = n_r h - e\Phi_r, \tag{5}$$

where S is the Hamilton principal function, and Φ_φ and Φ_r are the initial magnetic fluxes associated to the corresponding quantum numbers n_φ and n_r, respectively. For the binding energy of the orbit we find, using $\Phi = \Phi_\varphi + \Phi_r$, that

$$W = -\frac{m_e Z^2 e^4}{8\varepsilon_0^2} \frac{1}{(nh - e\Phi)^2} \approx -\frac{m_e Z^2 e^4}{8\varepsilon_0^2 n^2 h^2}(1 + 2e\Phi/nh), \tag{6}$$

where the approximation holds in the case of weak magnetic fluxes Φ compared to the magnetic flux quantum. The gross structure is given by the Bohr energy levels, and the deviations can be considered by proper initial magnetic fluxes. The energy shifts due to small initial magnetic fluxes are

$$\Delta W \approx \frac{m_e Z^2 e^5}{4\varepsilon_0^2 n^3 h^3}\Phi = 2R_\infty c \frac{eZ^2}{n^3}\Phi. \tag{7}$$

When considering the geometry of the orbits, the magnetic fluxes corresponding to different quantum numbers need to be examined individually. The modifications due to small perturbations can be taken into account by replacing

$$nh \quad \text{with} \quad nh - e\Phi, \quad n_\varphi h \quad \text{with} \quad n_\varphi h - e\Phi_\varphi \quad \text{and} \quad n_r h \quad \text{with} \quad n_r h - e\Phi_r. \tag{8}$$

In the corresponding equations, for the geometry of the ellipse the semi-major axis a changes from

$$a = \frac{n^2}{Z} a_0 \qquad \text{to} \qquad a = \frac{(n - e\Phi/h)^2}{Z} a_0, \tag{9}$$

and the semi-minor axis b from

$$b = \frac{nn_\varphi}{Z} a_0 \qquad \text{to} \qquad b = \frac{(n - e\Phi/h)(n_\varphi - e\Phi_\varphi/h)}{Z} a_0, \tag{10}$$

with the consequence that for elliptic orbits an initial magnetic flux Φ_φ alters the geometry of the ellipse in a different way than the initial magnetic flux Φ_r. Note that it is assumed that, for the effects discussed here, the Faraday law modifies only Φ_φ directly, and not Φ_r.

3. Simplified Approach: Neglecting "Spin" Angular Momentum

Although magnetic moment and "spin" angular momentum are not independent of each other, within this section, the "spin" angular momentum of the electron will be neglected for simplification. The modifications regarding the angular momentum of the electron will be discussed in the following section. The Zeeman effect and the hyperfine interaction can be understood in the flux quantum picture, where the "spin" angular momenta of the electron and atomic nucleus are neglected, but not their magnetic moments. The additional magnetic flux through atomic orbits will be calculated, and in a linear approximation the energy shifts due to the additional magnetic flux are deduced.

3.1. External Magnetic Field (Zeeman Effect)

Originally, the Zeeman effect describes the interaction of a homogeneous external magnetic field with the magnetic moment of the atom. Here, the Zeeman effect is taken to be the energy shift corresponding to the additional magnetic flux of an external homogeneous magnetic field through the electronic orbit. For small magnetic fields, where no change of the geometry of the atomic orbits has to be considered, the magnetic flux through the elliptic orbit is

$$\Phi_Z = \pi ab B \cos\alpha = \pi \frac{n^3 n_\varphi}{Z^2} a_0^2 B \cos\alpha, \tag{11}$$

where $a = n^2 a_0/Z$ and $b = nn_\varphi a_0/Z$ are the semi-major and semi-minor axes, πab is the size of the ellipse, and α is the angle between the normal vector of the orbital plane and the direction of the magnetic field B. For much higher magnetic fields, the change of the geometry of the atomic orbit has to be considered. The energy shift due to the external magnetic field is according to Eq. (7)

$$\Delta W \approx \frac{m_e Z^2 e^5}{4\varepsilon_0^2 n^3 h^3} \left(\pi \frac{n^3 n_\varphi}{Z^2} a_0^2 B \cos\alpha \right) = \mu_B n_\psi B, \tag{12}$$

where μ_B is the Bohr magneton and $n_\psi = n_\varphi \cos\alpha$ the magnetic quantum number according to the Bohr–Sommerfeld model. When interpreting n_ψ as the magnetic quantum number $m = n_\psi$, this equation describes the energy shift due to the normal (semi-classical) Zeeman effect.

3.2. Magnetic Dipole in the Focal Point of an Ellipse: Hyperfine Interaction

For the hyperfine interaction, the magnetic dipole contribution, where a magnetic dipole is in one of the focal points of the elliptical orbit, needs to be calculated analogously to a magnetic moment in the center of the circular orbit of the Bohr model [4]. A parametrization of the elliptic orbit is

$$r(\varphi) = \frac{p}{1 - \varepsilon \cos\varphi}, \tag{13}$$

where p is the focal parameter. Integration of the out-of-plane component of the magnetic field of a magnetic dipole with out-of-plane component $\mu_\perp$ within the orbital plane

$$B_\perp = -\frac{\mu_0}{4\pi}\frac{\mu_\perp}{r^3} \tag{14}$$

outside the boundary of the elliptic orbit but within the orbital plane gives the magnetic flux

$$\Phi_{\text{out}} = \int_0^{2\pi}\int_{r(\varphi)}^{\infty} B_\perp r\,dr\,d\varphi = -\frac{\mu_0}{4\pi}\mu_\perp\int_0^{2\pi}\frac{1}{r(\varphi)}d\varphi = -\frac{\mu_0}{2}\frac{\mu_\perp}{p}. \tag{15}$$

As magnetic flux lines are supposed to be closed, the magnetic flux through an infinite plane should be zero, and the flux through the elliptic orbit is $\Phi_{\text{in}} = -\Phi_{\text{out}}$. Hence, the additional magnetic flux for the geometry of the ellipse is

$$\Phi = \frac{\mu_0}{2}\frac{\mu_\perp}{p} = \frac{\mu_0}{2}\frac{\mu_\perp a}{b^2} = \frac{\mu_0}{2}\frac{Z\mu_\perp n^2 a_0}{n^2 n_\varphi^2 a_0^2} = \frac{\mu_0}{2}\frac{Z\mu_\perp}{n_\varphi^2 a_0}, \tag{16}$$

where the semi-major a and semi-minor b axes and their expressions depending on the quantum numbers n and n_φ are used instead of the focal parameter p.

Considering the magnetic moment $\vec{\mu}_c$ of the nucleus in one of the focal points of the elliptic orbit, we investigate the magnetic dipole contribution to the hyperfine interaction. The magnetic flux through the elliptic orbit is

$$\Phi_{hf} = \frac{\mu_0}{2}\frac{a\mu_c}{b^2}\cos\beta = \frac{\mu_0}{2}\frac{Z\mu_c}{n_\varphi^2 a_0}\cos\beta, \tag{17}$$

where β is the angle between the direction of the magnetic moment and the normal vector of the orbital plane. For small magnetic flux Φ_{hf}, the linear approximation of the energy is sufficient

$$\Delta W \approx \frac{m_e Z^2 e^5}{4\varepsilon_0^2 n^3 h^3}\left(\frac{\mu_0}{2}\frac{Z\mu_c}{n_\varphi^2 a_0}\cos\beta\right) = -\alpha^2 Z^3 h R_\infty c\,\frac{\mu_c\cos\beta}{n^3 n_\varphi^2}. \tag{18}$$

The correct hyperfine interval for the $1s$ orbit in the hydrogen atom can be found by considering two states, where the magnetic moment of the atomic nucleus is pointing first in a direction under an angle β with the normal vector of the orbital plane, and second in the opposite direction, where $n_\varphi = 1/2$ and $\cos\beta = 2/3$ is assumed (for experimental values, see, e.g., [23, 24]). A derivation of the angle between the direction of the magnetic moment and the normal vector of the elliptic plane will be discussed in the following section, as this can be attributed to the interplay of the different angular momenta contributions. The value $n_\varphi = 1/2$ reproduces the g-factor 2 for the electron (see next section) and means that the ground state is defined by the quantum numbers $n_r = n_\varphi = 1/2$. By assuming both quantum numbers n_φ and n_r to start from 1/2 with steps of one, the gross structure, where only the sum of both quantum numbers enters, will be equivalent to the gross structure of the Rutherford–Bohr model. Also the Zeeman level splitting is not affected by this assumption as the differences in n_φ are still considered to be integers. The ground state is characterized by a reduced orbital angular momentum ($n_\varphi = 1/2$, from here also called "spin" angular momentum), where the magnetic flux through the orbit can be considered to originate half from the orbital angular momentum of the electron and half from the magnetic field of the magnetic moment of the electron.

4. Interplay of Different Angular Momenta

Instead of interpreting the energy shifts of atomic levels due to the Zeeman effect, Paschen–Back effect, and the hyperfine level splitting as the additional energy of a magnetic moment within a magnetic field, these effects are here considered to be the result of the quantization of the magnetic flux through the atomic orbit in the case of a nonvanishing magnetic background field. Within the Bohr–Sommerfeld model, two contributions (orbital motion and "spin") to the magnetic flux through the electronic orbit of the atom will be considered. One of these contributions results purely from the orbital motion of the electron, and the other is due to a combination of the magnetic moment of the electron and an orbital motion. The atom is considered to be a symmetric top with nonprecessing total angular momentum. The angular momentum axis and the principal axis are, in general, not parallel.

The following points need to be considered for the description of the above-mentioned effects within the flux quantum picture.

1. **Different behavior of orbital and spin contribution:** Within the framework of the Bohr–Sommerfeld model, the electronic orbits are ellipses, and their sizes are defined by the quantum numbers n_r and n_φ; the orientation in the space is given by a third quantum number $n_\psi = n_\varphi \cos\alpha$, where α is the angle between the normal vector of the orbital plane and the direction of an external magnetic field. (It is assumed, that there is always at least a very small one.) Here two contributions will be distinguished. One contribution results purely from the motion of the electron around the nucleus (orbital contribution) and the associated quantities are labeled with the index l. This contribution can be described similar to the motion of the electron within the original Bohr–Sommerfeld model. The other contribution results partially from the magnetic moment ("spin") of the electron, where the associated quantities are labeled with the index s. This contribution is not present in the original Bohr–Sommerfeld model. This can be interpreted as a combination of the additional magnetic flux through the orbit due to the magnetic field of the magnetic moment of the electron, on the one hand, and, on the other hand, to an orbital motion (angular momentum) to stabilize the orbit. It is assumed, that the quantum numbers for the spin contribution are $n_r^s = n_\varphi^s = 1/2$ [see the previous section and rule (6)]. The combined effect will be described by the total quantum numbers given by $n = n^l + n^s$, $n_r = n_r^l + n_r^s$, $n_\varphi = n_\varphi^l + n_\varphi^s$, and so on, where the index j will also be used for the combination of the orbital and spin contributions. The two contributions behave independently of each other.

2. **Size of the atomic orbit:** For magnetic flux calculations, the size of the atomic orbits is needed. The orbits are of elliptic shape within the Bohr–Sommerfeld model with size A depending on the two quantum numbers n and n_φ
$$A = \pi ab = \pi \frac{n^3 n_\varphi}{Z^2} a_0^2, \tag{19}$$
where a is the semi-major and b the semi-minor axis. Here a small modification is necessary: Similar to the length of the angular momentum vectors in quantum mechanics, the length of the vector area (the size of the area) is assumed to be
$$|\vec{A}| = \pi \frac{n^3}{Z^2} \sqrt{n_\varphi(n_\varphi + 1)} a_0^2, \tag{20}$$

where the quantum number n_φ is replaced by $\sqrt{n_\varphi(n_\varphi+1)}$ in the semi-classical model. A discussion of the reasons for the replacement is not intended, but in the probability-density-based models, this might be explained by the difference between the mean average and maximum values of the radius of the orbital distribution.

3. **Projection of vector areas:** It is necessary to determine the size of the projection of a vector area into the direction of another vector area $\vec{A}_1 \cdot \vec{A}_2/|\vec{A}_2|$. Here this is done for the example of the two vector areas $\vec{A}_l$ and $\vec{A}_j$. The vector product will be calculated by squaring the expression $\vec{A}_l = \vec{A}_j - \vec{A}_s$, which is equivalent to the postulation of a linear summation of vector areas

$$\vec{A}_l \cdot \frac{\vec{A}_j}{|\vec{A}_j|} = \frac{|\vec{A}_j|^2 - |\vec{A}_s|^2 + |\vec{A}_l|^2}{2|\vec{A}_j|}. \tag{21}$$

Inserting the sizes of the vector areas as described in rule (2) gives

$$\vec{A}_l \cdot \frac{\vec{A}_j}{|\vec{A}_j|} = \frac{\pi n^3 a_0^2}{2Z^2} \frac{n_\varphi^j(n_\varphi^j+1) - n_\varphi^s(n_\varphi^s+1) + n_\varphi^l(n_\varphi^l+1)}{\sqrt{n_\varphi^j(n_\varphi^j+1)}}. \tag{22}$$

Analogously, one finds for the projection of $\vec{A}_s$ into the direction of $\vec{A}_j$

$$\vec{A}_s \cdot \frac{\vec{A}_j}{|\vec{A}_j|} = \frac{\pi n^3 a_0^2}{2Z^2} \frac{n_\varphi^j(n_\varphi^j+1) + n_\varphi^s(n_\varphi^s+1) - n_\varphi^l(n_\varphi^l+1)}{\sqrt{n_\varphi^j(n_\varphi^j+1)}}. \tag{23}$$

4. **Projection of angular momenta:** In general, the angular momentum vector and the vector area are not parallel. Here it is proposed that the projection of the angular momentum on the direction of its corresponding vector area is

$$\left(\frac{\vec{A}_j \cdot \vec{j}}{|\vec{A}_j|}\right) = n_\varphi^j \hbar = \left(n_\varphi^l \pm \frac{1}{2}\right)\hbar, \tag{24}$$

where $n_\varphi^j = j$ and $n_\varphi^l = l$ are identified.

5. **External magnetic fields:** The magnetic flux Φ of a homogeneous external magnetic field $\vec{B}$ through an orbital area with vector area $\vec{A}$ is

$$\vec{A} \cdot \vec{B} = \pi \frac{n^3 n_\varphi}{Z^2} a_0^2 B \cos\alpha, \tag{25}$$

with $n_\varphi \cos\alpha = n_\psi$, where the "classical" size of the vector area [see rule (2)] and the definition of the Sommerfeld quantum number n_ψ are used. Here, this is explained by the deviation of the vector area from the direction of angular momentum. An averaging effect occurs, resulting in a smaller value for the effective area seen from the magnetic field.

6. **Spin rule (g-factor):** The orbital motion caused by the spin of the electron has to be considered by postulating a spin rule. For the ground state already discussed, the quantum numbers for the

spin contribution are $n^s_\varphi = n^s_r = 1/2$. The anomalous gyromagnetic factor for the electron can be explained by assuming that the ratio between the radial and orbital contributions remains always the same for the two spin quantum numbers and their additional magnetic fluxes

$$n^s_\varphi = n^s_r \qquad \text{and} \qquad \Phi^s_\varphi = \Phi^s_r. \tag{26}$$

This condition ensures, that in the case of increasing magnetic flux Φ_φ, which is assumed to be modified by the Faraday law for the effects discussed here and not Φ_r, the increase in the spin contribution $\Phi^s = \Phi^s_\varphi + \Phi^s_r$ is twice as large as other contributions not fulfilling the spin rule, like the orbital contribution. This assumption leads to a g-factor of 2.

Using these rules, we study several effects in more detail.

4.1. Zeeman Effect

We consider the energy shift of atomic levels due to small magnetic fields as the energy shift due to the additional magnetic flux of the external magnetic field through the atomic orbit. Because of spin–orbit coupling for weak external magnetic fields, the spin and orbital parts are not independent of each other, and only the projections of the spin vector area $\vec{A}_s$ and the orbital vector area $\vec{A}_l$ in the direction of the total vector area need to be considered. Keeping in mind the rule (6) of the equivalence of the two spin quantum numbers n^s_φ and n^s_r and their fluxes, a factor of 2 has to be applied to the spin contribution, resulting in the additional magnetic flux

$$\Phi_Z \propto (2\vec{A}_s + \vec{A}_l) \cdot \vec{B}. \tag{27}$$

Due to the coupling of the spin and orbital contributions, the projections of these vectors in the direction of the combined vector area $\vec{A}_j$ enter the equation of magnetic flux

$$\Phi_Z = A_{\text{proj}} B_{\text{proj}} = \frac{(2\vec{A}_s + \vec{A}_l) \cdot \vec{A}_j}{|\vec{A}_j|} \frac{\vec{A}_j \cdot \vec{B}}{|\vec{A}_j|}. \tag{28}$$

The projection of the vector areas in the direction of the other vector areas is given in the previous section [see rule (3)]. Here, only the case of weak magnetic fields is considered, where the deformation of the geometry is negligible. Hence, the effective area is

$$\frac{(2\vec{A}_s + \vec{A}_l) \cdot \vec{A}_j}{|\vec{A}_j|} = \frac{\pi n^3 a_0^2}{2Z^2} \, \frac{3n^j_\varphi(n^j_\varphi + 1) + n^s_\varphi(n^s_\varphi + 1) - n^l_\varphi(n^l_\varphi + 1)}{\sqrt{n^j_\varphi(n^j_\varphi + 1)}}. \tag{29}$$

However, the vector area, being parallel to the principal axis of the top, at which the atom is considered, and not parallel to the direction of the angular momentum, is rotating around the direction of the magnetic field. As the full angular momentum is assumed to be constant in the space, the angular momentum of the nucleus and the angular momentum of the orbiting electron are circulating around the direction of the full angular momentum. The projection of the magnetic field vector $\vec{B}$ in the direction of the area vector $\vec{A}_j$ gives [see rules (2) and (5)]

$$\frac{\vec{A}_j \cdot \vec{B}}{|\vec{A}_j|} \frac{n^j_\varphi \cos\alpha_j B}{\sqrt{n^j_\varphi(n^j_\varphi + 1)}} = \frac{m_j B}{\sqrt{j(j+1)}}, \tag{30}$$

where n_ψ^j and n_φ^j are identified by m_j and j, respectively. Combining these equations, we see that the additional magnetic flux due to the external magnetic field is

$$\Phi_Z = \frac{\pi n^3 a_0^2}{Z^2} \underbrace{\left(1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)}\right)}_{g_j} m_j B, \tag{31}$$

where n_φ^j with j and n_φ^l are identified with l and n_φ^s, with s. The expression in the brackets is identical to the Landé factor g_j. For the energy shifts, one finds

$$\Delta W = \frac{m_e Z^2 e^4}{4\varepsilon_0^2 n^3 h^3} e\Delta\Phi = \frac{m_e Z^2 e^4}{4\varepsilon_0^2 n^3 h^3} e\pi \frac{n^3}{Z^2} a_0^2 g_j m_j B = \mu_B g_j m_j B, \tag{32}$$

which is the usual expression for the energy shifts of the Zeeman effect due to external magnetic fields.

4.2. Paschen–Back Effect

If the magnetic field is strong enough, the orbital angular momentum and the "spin" angular momentum are not coupled to the total angular momentum due to the spin–orbit coupling as in the case of weak external magnetic fields, but act independently. For the calculation of the magnetic flux, the time-averaged vector areas for the orbital contribution $\vec{A}_l$ and for the spin contribution $\vec{A}_s$ need to be considered. Due to the equivalence of the spin quantum numbers n_φ^s and n_r^s and their magnetic fluxes [see model property (6)], a factor of 2 has to be considered for the spin contribution. The initial magnetic flux in the case of the Paschen–Back effect becomes

$$\Phi_{\rm PB} = (2\vec{A}_s + \vec{A}_l) \cdot \vec{B}. \tag{33}$$

The magnitude of the time-averaged vector areas is proportional to the corresponding quantum numbers, resulting for the magnetic flux in [see rule (5)]

$$\Phi_{\rm PB} = \left(2n_\varphi^s n^3 \frac{\pi a_0^2}{Z^2} \cos\alpha_s + n_\varphi^l n^3 \frac{\pi a_0^2}{Z^2} \cos\alpha_l\right) B, \tag{34}$$

where α_s and α_l are the angles between the magnetic field and the vector areas of the spin and angular momentum contributions, respectively. Using the quantization of the orientation in the space $n_\psi = n_\varphi \cos\alpha$, we see that the initial magnetic flux in the case of the Paschen–Back effects becomes

$$\Phi_{\rm PB} = \left(2n_\psi^s + n_\psi^l\right) n^3 \frac{\pi a_0^2}{Z^2} B, \tag{35}$$

and the corresponding shift in energy with respect to the undisturbed orbit is

$$\Delta W_{PB} = \mu_B \left(2n_\psi^s + n_\psi^l\right) B = \mu_B (2m_s + m_l) B, \tag{36}$$

where the quantum numbers n_ψ^s and n_ψ^l are identified by the magnetic quantum numbers m_s and m_l, respectively.

4.3. Hyperfine Interaction

We describe the hyperfine interaction as the change in energy resulting from the additional magnetic flux of the magnetic dipole of the nucleus through the electron orbit. The magnetic flux through an elliptic orbit with the focal parameter p from a magnetic dipole $\mu_\perp$ orthogonal to the orbital plane in one of the focal points of the ellipse reads

$$\Phi = \frac{\mu_0}{2}\frac{\mu_\perp}{p}. \tag{37}$$

Simplified, the effective magnetic moment is given by the projection of the magnetic moment of the nucleus $\vec{\mu}_I$ into the direction of the normal vector of the orbital plane $\vec{A}_j/|\vec{A}_j|$. In a simplified version, the magnetic flux is

$$\Phi_{\rm hfs}^{\rm simple} = \frac{\mu_0}{2}\frac{\vec{A}_j\cdot\vec{\mu}_I}{|\vec{A}_j|}\frac{1}{p} = \frac{\mu_0}{2}\frac{\vec{A}_j}{|\vec{A}_j|}\cdot\vec{\mu}_I\frac{a}{b^2} = \frac{\mu_0}{2}\frac{Z}{a_0}\frac{\vec{A}_j\cdot\vec{\mu}_I}{(n_\varphi^j)^2|\vec{A}_j|} \quad \text{(simplified)}, \tag{38}$$

where the expressions of the semi-major and semi-minor axes are employed. However, the involved angular momenta, the spin angular momentum of the electron $\vec{s}$, the orbital angular momentum $\vec{l}$, and the angular momentum of the nucleus $\vec{I}$ define at the end the vector areas of the different contributions and the direction of the magnetic moment of the nucleus. Expecting the time-averaged normal vector of the electron orbit to be $\vec{j} = \vec{s} + \vec{l}$, we replace both vectors $\vec{A}_j$ and $\vec{\mu}_I$ by the projection of each of these vectors into the direction of $\vec{j}$

$$\Phi_{\rm hfs} = \frac{\mu_0}{2}\frac{Z}{(n_\varphi^j)^2 a_0}\left(\frac{\vec{A}_j}{|\vec{A}_j|}\cdot\frac{\vec{j}}{|\vec{j}|}\right)\frac{\vec{j}\cdot\vec{\mu}_I}{|\vec{j}|}. \tag{39}$$

The projection of the angular momentum in the direction of the corresponding vector area was postulated in rule (4) and gives

$$\frac{\vec{A}_j\cdot\vec{j}}{(n_\varphi^j)^2|\vec{A}_j|} = \frac{(n_\varphi^l \pm 1/2)\hbar}{(n_\varphi^l \pm 1/2)^2} = \frac{g_s\hbar}{(2n_\varphi^l \pm 1)}, \tag{40}$$

where $g_s = 2$ and $n_\varphi^j = n_\varphi^l + n_\varphi^s = n_\varphi^l \pm 1/2$. With $\vec{\mu}_I = g_I\mu_K\vec{I}/\hbar$ and $\mu_0/(2a_0\hbar^2) = \pi\alpha^2/(2m_e\mu_B^2)$, the additional magnetic flux through the electronic orbit is

$$\Phi_{hfs} = \frac{\mu_0}{2}\frac{Z}{a_0}\frac{g_s\hbar}{(2l\pm 1)}\frac{g_I\mu_K\vec{j}\cdot\vec{I}}{j(j+1)\hbar^3} = \alpha^2 Z\frac{\pi}{2m_e}\frac{g_s g_I\mu_K\vec{I}\cdot\vec{j}}{\mu_B^2 j(j+1)(2l\pm 1)}. \tag{41}$$

With $\vec{I}\cdot\vec{j} = \hbar^2/2[F(F+1) - I(I+1) - j(j+1)]$, $\mu_e = (g_s/2)(e\hbar/2m_e)$, and $\mu_{\rm nuc} = g_I\mu_K I$, the additional magnetic flux caused by the magnetic dipole results in

$$\Phi_{\rm hfs} = \alpha^2 Z\frac{h}{2e}\frac{[F(F+1) - I(I+1) - j(j+1)]\mu_e\mu_{\rm nuc}}{\mu_B^2 j(j+1)(2l\pm 1)I}. \tag{42}$$

The energy shift, according to Eq. (7), of the hyperfine levels amounts to

$$\Delta W_{\rm hfs} \approx 2R_\infty c\frac{eZ^2}{n^3}\Delta\Phi_{\rm hfs} = \frac{A_{nlj}}{2}[F(F+1) - I(I+1) - j(j+1)], \tag{43}$$

with

$$A_{nlj} = 2\alpha^2 Z^3 R_\infty hc \frac{\mu_e \mu_{\rm nuc}}{\mu_B^2 n^3 j(j+1)(2l \pm 1) I}. \tag{44}$$

This expression differs from the usual expression for the hyperfine level shifts [23], if neglecting the reduced mass correction, the relativistic correction factor, and the off-diagonal terms only by the term $(2l \pm 1)$, which is $(2l+1)$ in [23].

5. Conclusions

We investigated the quantization of magnetic flux through atomic orbits in more detail for the Bohr–Sommerfeld model. Neglecting the angular momentum of the constituents, one can, in principle, explain effects like the Zeeman effect and hyperfine splitting of atomic levels. Taking the angular momenta into account, one can explain the Zeeman effect, the Paschen–Back effect, and the hyperfine splitting of atomic levels with high accuracy. As a consequence, the "spin" needs to be seen from a different point of view. The unusual properties of the "spin" are a result of the magnetic moment of the electron: The quantized magnetic flux through the electron orbit comes partly from the magnetic flux caused by the magnetic moment of the electron and partly from the angular momentum (orbital motion) of the electron, which stabilizes the orbit, resulting in the g-factor of 2 for the electron. We proposed the rules accounting for the interplay of the different angular momentum contributions for explaining the energy-level shifts of several effects, which also contain corrections for the classical assumption of the electron to be a point particle. It could be interesting to investigate a density-based model, like the Schrödinger equation and the Dirac-equation-based models, with respect to energy shifts caused by additional magnetic flux through electronic orbits. However, in these theories, the full vector field for the magnetic field has to be considered.

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