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DYNAMIC PROPERTIES OF WEHRL INFORMATION ENTROPY AND WEHRL PHASE DISTRIBUTION FOR A MOVING FOUR-LEVEL ATOM

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Abstract

We study the dynamics of the von Neumann entropy, Wehrl entropy, and Wehrl phase distribution for a single four-level ladder-type atom interacting with a one-mode cavity field taking into account the atomic motion. We obtain the exact solution of the model using the Schrödinger equation under specific initial conditions. Also we investigate the quantum and classical quantifiers of this system in the nonresonant case. We examine the effects of detuning and the atomic motion parameter on the entropies and their density operators. We observe an interesting monotonic relation between the different physical quantities in the case of nonmoving and moving atoms during the time evolution. We show that both the detuning and the atomic motion play important roles in the evolution of the Wehrl entropy, its marginal distributions, entanglement, and atomic populations.

Keywords: Wehrl entropy, Wehrl phase distribution, von Neumann entropy, atomic motion, atomic populations, entanglement.

1. Introduction

The Jaynes–Cummings model (JCM) [1] is a commonly used model in the field of quantum optics, which describes the interaction of an atom with the electromagnetic field. A general formalism of the interaction of a two-level atom and a single-mode cavity field in arbitrary forms of nonlinearities has been proposed [2]. Also the interaction between the two-level atom and one- and two-mode cavity fields was studied [3–5]. Some time ago, the interaction between three-level different configurations (Ξ, Λ, and V) and one- and two-mode fields was investigated [6, 7]. Recently, some authors have paid attention to studying the four-level systems under various configurations interacting with the cavity modes. Many

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schemes of the four-level atomic systems were studied and also visualized [8, 9]. Moreover, a system consisting of the four-level atom interacting with the single-mode cavity field, when the atom was initially prepared in the momentum eigenstate, was studied [10]. The interaction between the four-level N-type and double Ξ-type atoms with multimode cavity field in the presence of nonlinearities and in the case of one- and two-mode cavity fields was investigated [11].

It is well known that the information entropies are widely used to quantify the entanglement in quantum information [12–16]. In this regard, the von Neumann entropy (NE) [12], linear entropy (LE), and Shannon information entropy (SE) are frequently employed in treating entanglement in quantum systems. It is worth mentioning that the SE involves only the diagonal elements of the density matrix and in some cases gives information similar to that obtained from the NE and LE. On the other hand, there is an additional entropic quantity, namely, the field Wehrl entropy (FWE) [17, 18]. This physical quantity has been successfully applied in the description of different properties of quantum optical fields such as the phase–space uncertainty [19], decoherence, etc. [20, 21].

The Wehrl entropy (WE) is used in treating the dynamics of the quantum systems [17, 18, 22]. This measure has been successfully applied in describing different properties of the quantum optical fields such as the phase–space uncertainty and decoherence [19, 20]. The problem of measuring nonlocal correlations (entanglement) in the phase space with application of the WE has been discussed [23], where it was shown that the degree of the intermode correlation strongly depends on the photon-number difference in the two-mode Fock state. On the other hand, the effect of phase damping of the classical correlations measured by the WE and Wehrl phase distribution (Wehrl PD) has been investigated [24]. Furthermore, we note that the WE-time evolution in the case of the JCM has been thoroughly investigated [21, 25, 26], and it is more appropriate for distinguishing among the states than the von Neumann entropy [22]. It is known that the WE yields helpful information on the atomic-inversion processes, and the WE studies of a single-trapped ion interacting with laser fields with different configurations were the subject of interesting research [27]. Now we also know that both (i) the fluctuations of the laser phase and (ii) the initial-state setting play important roles in the evolution of quantifiers like the Husimi Q function and the WE and Wehrl PD [27]. On the other hand, the WE has been used as a good measure of the entanglement in a semiconductor cavity QED containing a quantum well in comparison with the concurrence [28].

In this paper, we study the interaction between a four-level ladder-type atomic system and a quantized radiation field in the rotating wave approximation. Here, the field is a single-mode one and the interaction is affected by a three-photon process. We present the wave function of the atom–field system and discuss different phenomena related to this problem if the detuning parameters are taken into account. Based on analytical and numerical analysis, we discuss the influence of both the detuning and the atomic motion on the degree of entanglement, WE, Wehrl PD, and population probabilities in the process of time evolution when the atom starts in the superposition of its two upper states and the field is in the coherent state. The results of our study may be important for quantum information processing.

The paper is structured as follows.

In Sec. 2, we present the model that describes the four-level ladder-type atomic one-mode system by introducing the wave function of the atom–field system and the reduced density operator of the field. In Sec. 3, we describe the numerical results obtained for the dynamics of the field entropy, WE, and Wehrl PD, in view of the effect of different parameters on different quantifiers of the system under consideration. Finally, we summarize our main results in Sec. 4.

2. Model and Wave Function

In this section, we present the model that allow us to describe the four-level ladder-type atomic system interacting with a quantized radiation field in the rotating-wave approximation.

Let us consider a single four-level cascade-type atom, ω_j ($j = 1, 2, 3, 4$) with $\omega_1 > \omega_2 > \omega_3 > \omega_4$, as the transition energy between the four levels. This atom interacts with a single-mode cavity field described by the creation (annihilation) operator $\widehat{a}^\dagger(\widehat{a})$ and frequency Ω. In the rotating-wave approximation (RWA), the total Hamiltonian $\widehat{H}$ for the considered system can be written in the form

$$\widehat{H} = \widehat{H}_{A-F} + \widehat{H}_{\rm in}, \tag{1}$$

where $\widehat{H}_{A-F}$ is the Hamiltonian for the free field and the noninteracting atom, while $\widehat{H}_{\rm in}$ is the interaction part of the total Hamiltonian. We divide $\widehat{H}$ as follows:

$$\widehat{H}_{A-F} = \sum_j \omega_j \widehat{\sigma}_{j,j} + \Omega \widehat{a}^\dagger \widehat{a}, \tag{2}$$

where we assume that $\hbar = 1$, as is usually the case in quantum mechanics, and $\widehat{\sigma}_{j,j} = |j\rangle\langle j|$ are the population operators. The interaction between the atomic system and the field is considered to be affected through four-photon processes that accomplish the transitions. In the nonresonant case, the interaction Hamiltonian of this system is given by

$$\widehat{H}_{\rm in} = \sum_{j=1}^{3} \lambda_j g(z) \left[\widehat{a} e^{-i\Delta_j t} \widehat{S}_{j,j+1} + \text{h.c.}\right], \tag{3}$$

where

$$\Delta_j = \Omega + \omega_{j+1} - \omega_j \tag{4}$$

is the detuning parameter, λ_j is the atom–field coupling constant, and $g(z)$ is the shape function of the cavity field mode. We restrict our study to the atomic motion along the z axis. In this case,

$$g(z) \longrightarrow g(vt) = \sin(p\pi vt/L), \tag{5}$$

where v denotes the atomic motion velocity and p represents the number of half-wave length of the field mode inside the cavity of the length L. By a proper choice of the atomic motion velocity $v = \lambda_j L/\pi$ with $g(z) = \sin(p\lambda_j t)$, we solve the model of the considered atomic system and write the atom–field wave function of this system at an arbitrary time t in the form

$$|\Psi(t)\rangle = \sum_n Q_n \left[\sum_{k=1}^{4} \Psi_k(n,t)|n+k-1\rangle\rangle \otimes |k\rangle\right], \tag{6}$$

where $Q_n = \exp\left(\alpha^2/2\right)\alpha^n/\sqrt{n!}$, $|\alpha|^2$ is the initial mean photon number for the mode, $\Psi_k(n,t)$ is the probability-amplitude coefficients, which have the initial atomic state $|\Psi(0)\rangle$, and $|n\rangle$ is the Fock state of the mode field.

Using the time-dependent Schrödinger equation in the interaction picture

$$i\,\frac{d}{dt}|\Psi(t)\rangle = \widehat{H}_{\rm in}|\Psi(t)\rangle$$

and the action of $\widehat{a}$ and $\widehat{a}^\dagger$ on the state $|n\rangle$, we obtain the following system of ordinary differential equations:

$$i\frac{d}{dt}\Lambda = M\Lambda, \tag{7}$$

where

$$\Lambda = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 0 & F_1 & 0 & 0 \\ F_1^* & 0 & F_2 & 0 \\ 0 & F_2^* & 0 & F_3 \\ 0 & 0 & F_3^* & 0 \end{pmatrix}, \tag{8}$$

such that

$$F_j = \lambda_j g(vt)\sqrt{n+j}\exp(-i\Delta_j t), \tag{9}$$

and F_j^* is the conjugate of F_j. We turn our attention to solving the system of equations (7), where the exact solution can be obtained when $\lambda_j g(vt) = \lambda_j$.

We assume that at time $t = 0$ the atom is initially prepared in the superposition of its two upper states $|1\rangle$ and $|2\rangle$. Thus, the initial wave function can be decomposed into its atomic and field parts of the form

$$|\Psi(t=0)\rangle = \sum_n Q_n\big(\cos(\theta/2)|1\rangle + \sin(\theta/2)|2\rangle\big)|n\rangle. \tag{10}$$

The four eigenvalues of the matrix M in Eq. (8) can be obtained when the parameters of these systems are given. It is important to mention that the wave function of the assumed atomic system, with the atomic motion taken into account, is obtained by computer simulations. Now, with the wave function given, we can discuss the dynamic properties of the considered system. In the following section, we investigate numerically the effect of the initial atomic position and atomic motion on the dynamic behavior of the von Neumann entropy $S_F(t)$, Wehrl entropy $S_W(t)$, and the Wehrl phase distribution $S_\Theta(t)$ in terms of the field density matrix and as a function of the populations $\rho_{jj}(t)$, $j = 1, 2, 3, 4$.

3. Quantum and Classical Quantifiers

In this section, we present the overall state of the different quantum and classical quantifies for the system under consideration and their values during the time evolution.

3.1. Von Neumann Entropy

Due to the crucial role of entanglement in quantum information processes, the study of entanglement has attracted a lot of interest in recent years. Among various studies on entanglement, the first question that may be asked is how to know that a quantum state is entangled. For a pure bipartite state, the Schmidt decomposition [29] can be used to judge whether the state is entangled, and the degree of entanglement can be quantified by the partial von Neumann entropy [30]. Thus, in principle, the problem of entanglement for pure states of a bipartite system has been completely solved.

Firstly, the von Neumann entropy for the three-level atom and one-mode field has been investigated [31]. Also the entropy behavior and statistical properties of the field interacting with a three-level

Ξ-type atom has been explored [32]. The standard von Neumann definition of the quantum-mechanical entropy [12] reads

$$S_{\mathrm{vN}} = -\mathrm{Tr}\,(\rho \ln \rho)\,. \tag{11}$$

This is zero for all pure states, i.e., for states that satisfy the condition $\hat{\rho}^2 = \hat{\rho}$, where $\hat{\rho}$ is the density operator describing a given quantum state. For this reason, this entropy cannot distinguish between various pure states, and it is rather a measure of the purity of quantum states. For the system under consideration, the von Neumann entropy is given as [32, 33]

$$S_{A[F]} = -\mathrm{Tr}_{A[F]}\left\{\hat{\rho}_{A[F]} \ln\left[\hat{\rho}_{A[F]}\right]\right\} = -\sum_j \zeta_j \ln \zeta_j, \tag{12}$$

where $S_{A[F]}$ is the reduced density operator for the subsystem $A[F]$. It is obtained by performing a partial trace over subsystem $F[A]$ of the density matrix for the combined system $\hat{\rho}_{AF}$. The ζ_j are the eigenvalues of the reduced density matrix $\hat{\rho}_{A[F]}$.

Since the trace is invariant under the similarity transformation, the field density matrix $\hat{\rho}_F$ can be written as

$$\hat{\rho}_F(t) = \left[\sum_{m=0}\sum_{n=0}\sum_{k=1}^{4} Q_n Q_m^* \Psi_k(n,t)\Psi_k^*(m,t)|n+k-1\rangle\langle m+k-1|\right]. \tag{13}$$

The von Neumann entropy varies from $S_{A[F]} = 0$ for an unentangled state to $S_{A[F]} = 1$ for a maximum entangled state as for the well-known case of EPR states.

3.2. Wehrl Space Entropy

The Wehrl entropy is a useful tool to investigate dynamic properties of quantum systems that contains all information on the dynamics of the optical field, being completely equivalent to the density operator; it can be interpreted as an information measure for such a joint measurement [21]. The Wehrl entropy clearly distinguishes coherent states, and it can be used as a measure of the statistical properties of the optical fields. It is conjectured that the minimum of entropy is obtained for coherent states, which are localized in the phase space, as allowed by the Heisenberg uncertainty principle. In this way, it presents a good measure of the strength of the coherent component in an optical field, i.e., it can be used to observe squeezing in the quantum fluctuations of the quadrature operators when the field is initially in the coherent state where it measures how much "coherence" a given state has. As such, it can be used to classify quantum fields with respect to their statistical properties; this measure has many advantages compared to other quantities such as the Wigner function, Glauber–Sudarshan P representation, etc., which can take negative values. Here, the variation of the Wehrl entropy gives some insight into important details of the input optical field and the output state entanglement by studying the influences of the parameters involved in the ECSs on the WE.

Any quantum state described by a density matrix ρ can be represented by the Husimi quasidistribution function $Q_\beta(t) = \dfrac{1}{\pi}\langle\beta|\rho_F(t)|\beta\rangle$, where $|\beta\rangle$ is the coherent state.

The Wehrl entropy of any quantum state $\hat{\rho}$ can be written in terms of the Husimi Q function as [17,18]

$$S_W = -\int_\Omega Q_\beta(t) \ln Q_\beta(t)\, d\nu. \tag{14}$$

According to Eq. (14), S_ρ is the field Wehrl space entropy [24]

$$S_W(t) = -\int_0^{2\pi} \left\{ \int_0^{\infty} \left[\mathbf{Q}_\beta(t) \ln \mathbf{Q}_\beta(t)\right] |\beta|\mathbf{d}|\beta| \right\} d\Theta. \tag{15}$$

3.3. Wehrl Phase Entropy

The Wehrl PD is defined to be the phase density of the WE. In this way, the Wehrl PD has been extensively developed and shown to be a successful indicator of both noise (phase–space uncertainty) and phase randomization [22]. The dynamics of the Wehrl PD in the case of the absence of the Stark shift effect has been discussed [24]. On the other hand, the evolution of the Wehrl PD in the presence of the Stark shift effect has been investigated [34]; however, the decoherence effect was ignored. In this section, we discuss the dynamics of the Wehrl PD in the presence of the Stark shift parameter under the decoherence. The Wehrl PD is defined in terms of the Husimi Q function as follows:

$$\mathbf{S}_\Theta(t) = -\int_0^{\infty} \mathbf{Q}_\beta(t) \ln \mathbf{Q}_\beta(t)|\beta|\mathbf{d}|\beta|. \tag{16}$$

4. Numerical Results and Discussion

In this section, we demonstrate the quantum and classical quantifier dynamics of a single four-level ladder-type atom interacting with a single-mode cavity field under the action of the atomic motion parameter for various initial atomic positions. We are interested in the relation between the dynamics of the quantum entanglement S_F, Wehrl entropy S_W, and the Wehrl phase distribution S_θ, in particular, the changes in the behavior and dynamics of different kinds of quantifiers.

Based on Eqs. (12), (15), and (16), we present the main results for different parameters that influence the time evolution of the S_F, S_W, ρ_{ii}, and S_θ.

Figures 1 and 2 show the different quantum and classical quantifiers for the density matrix $\rho(t)$ versus λt under different initial conditions when the atomic motion effect is neglected. In Fig. 1 a, by choosing $\theta = 0$ (upper state), one observes that the quantum-entanglement behavior has a different order, being a function of the scaled time λt. The von Neumann entropy increases from zero and undergoes an initial change with slow oscillatory behavior tending to a stabilization and with very rapid oscillatory behavior as the scaled time becomes significantly large, which indicates that the atom is trapped by the cavity field. In Fig. 2 a, the von Neumann entropy is shown for $\theta = \pi/4$. In this case, the curve is found to be similar, but the oscillations become less fast with small amplitudes, and the maximum value of S_F stabilizes for significantly high values of the scaled time. On the other hand, we find that S_W has a similar behavior as the entanglement, exhibiting the statistical properties of the field dynamics, where the S_W parameter increases until it reaches a maximum providing that the field becomes more quantum-mechanical during the time for the different initial atomic positions (see Figs. 1 b and 2 b). Moreover, as we expected, some agreement between the behavior of S_F and S_W is observed since both quantities are defined through the reduced field density operator. These illustrate that S_W can be used as a measure of entanglement for the system under consideration. Within the framework of the population probabilities, as we can see in Figs. 1 c and 2 c, the populations exhibit the same behavior with different amplitudes during the time evolution providing the collapse and revival phenomena.

In order to observe the influence of the atomic motion effect on the behavior of the quantum and classical quantifiers, we show in Figs. 3 and 4 the different quantities as functions of the scaled time

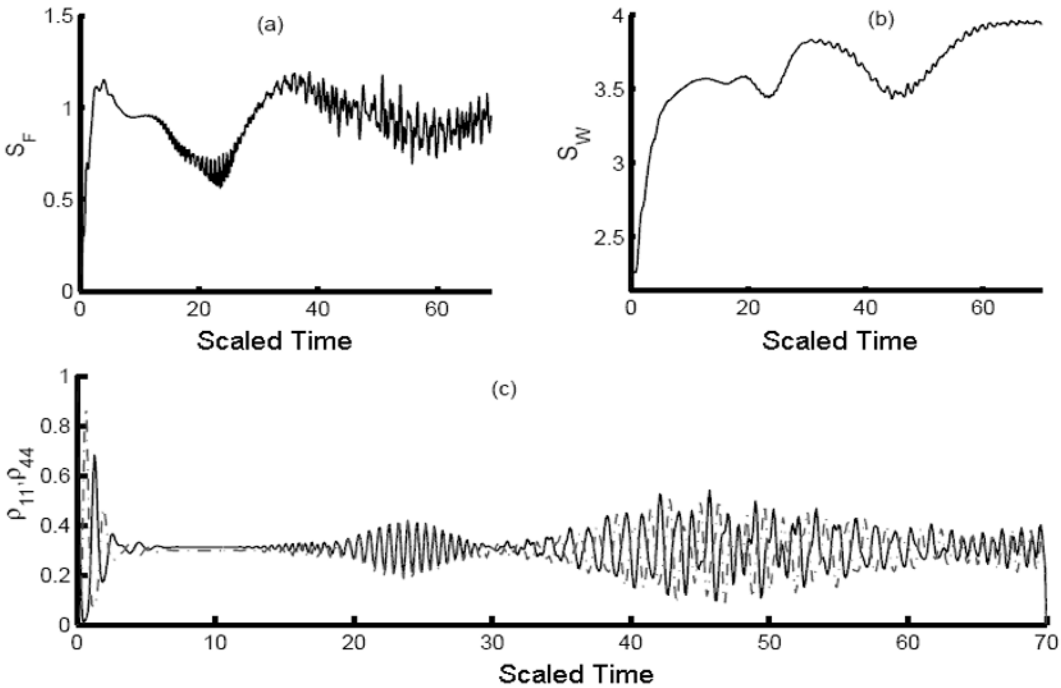

Fig. 1. The time evolution of the field entropy S_F (a), Wehrl entropy S_W (b), and atomic population probabilities (c) ρ_{11} (solid curve) and ρ_{44} (dash-dotted curve) of the four-level atom interacting with a single-mode field; the atom is initially in the upper state at $\theta = 0$ and $\alpha = 4$, when the atomic motion parameter is neglected.

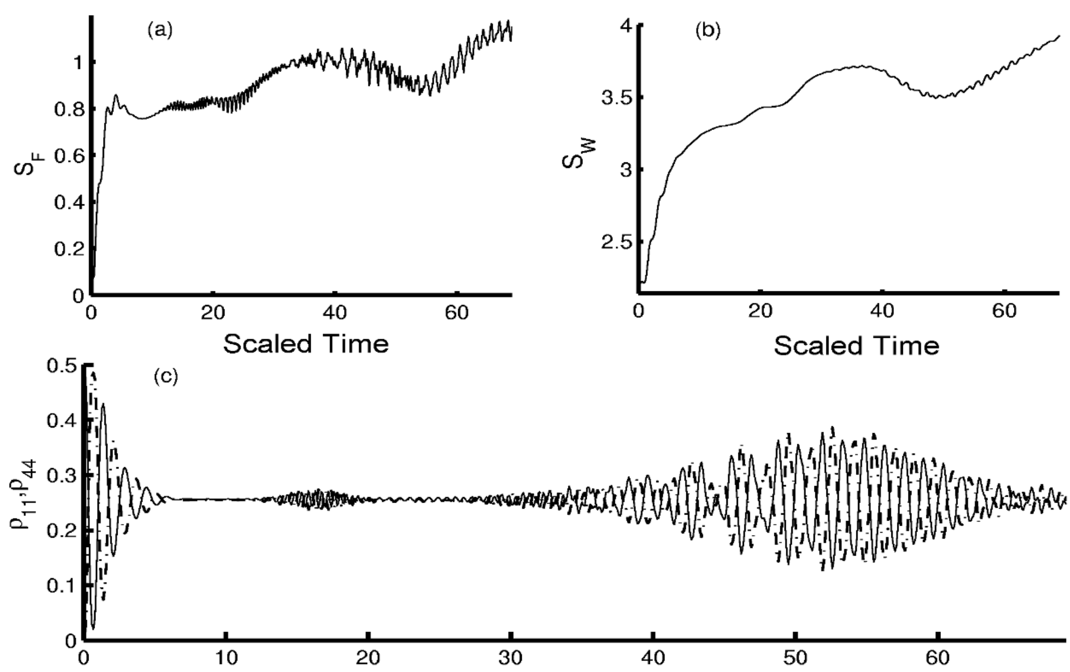

Fig. 2. The time evolution of the field entropy S_F (a), Wehrl entropy S_W (b), and atomic population probabilities (c) ρ_{11} (solid curve) and (d) ρ_{44} (dash-dotted curve) of the four-level atom interacting with a single-mode field; the atom is initially in the superposition of two upper states $|1\rangle$ and $|2\rangle$ at $\theta = \pi/4$ and $\alpha = 4$, when the atomic motion parameter is neglected.

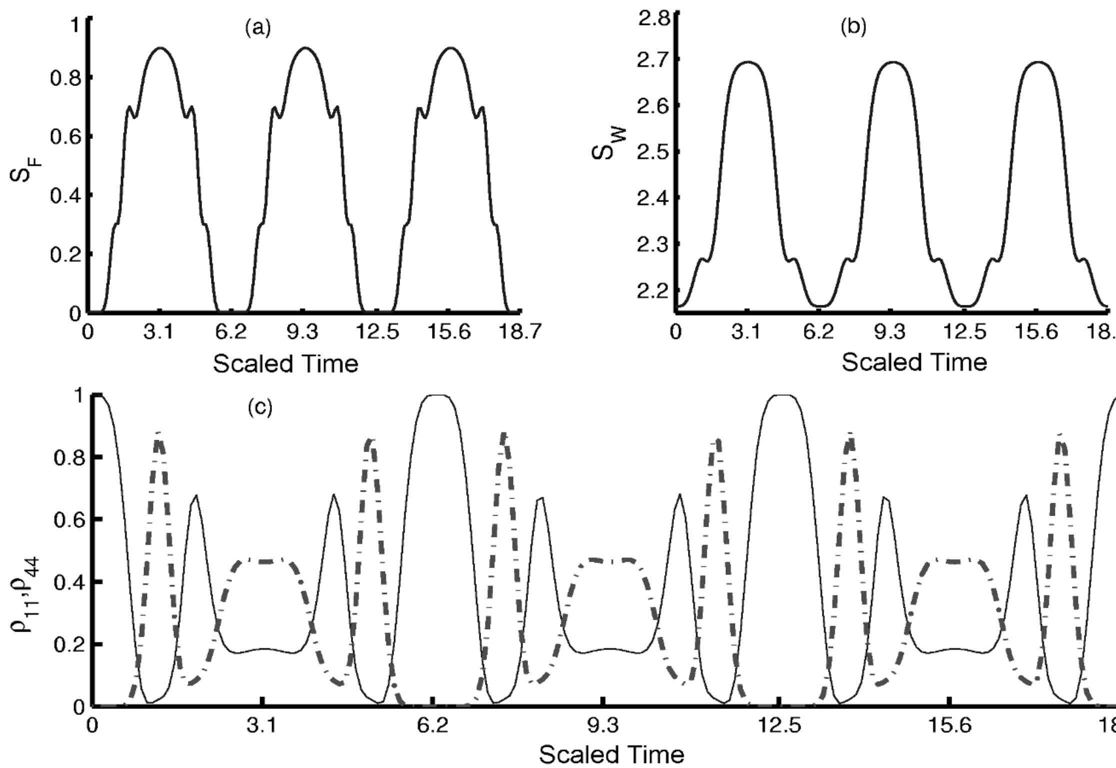

Fig. 3. The same as in Fig. 1 but the atomic-motion parameter is considered as $p = 1$.

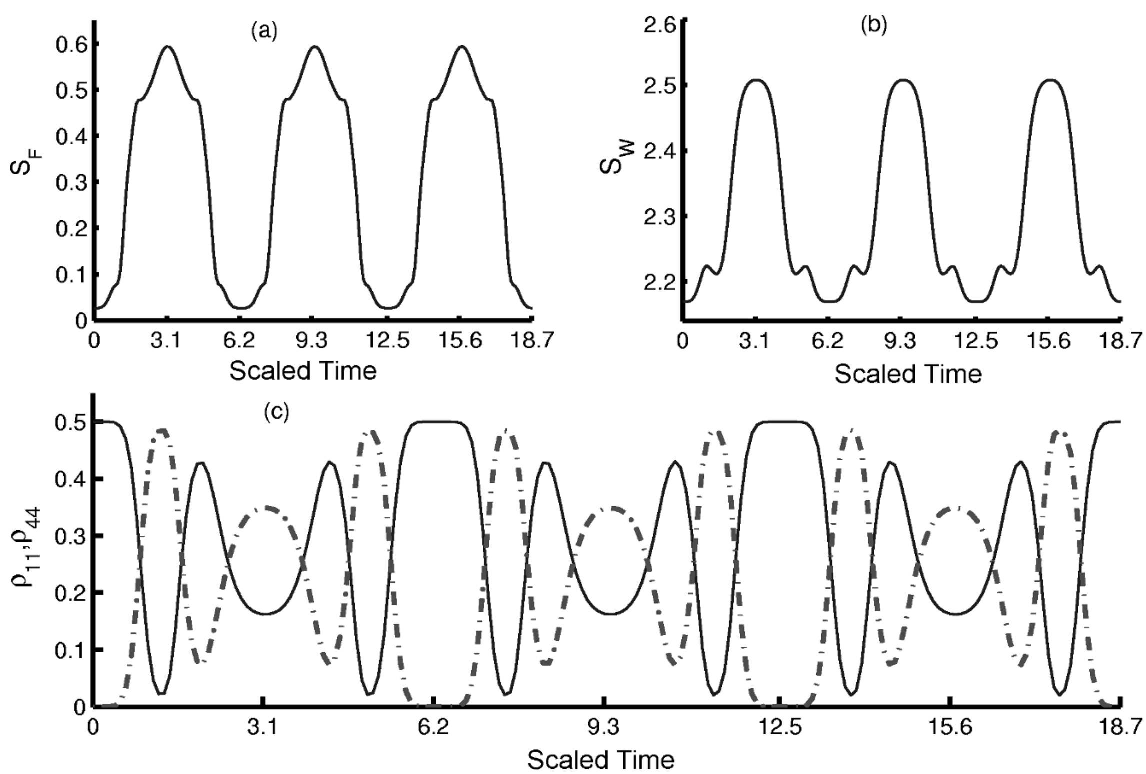

Fig. 4. The same as in Fig. 2 but the atomic motion parameter is considered as $p = 1$.

for various values of the initial atomic position. Then, we consider a more practical situation of the interaction between the atom and the cavity field. In this considered case, the above situation completely changes under the effect of atomic motion. As we can see in Figs. 3 a and b and 4 a and b, the amount of the entanglement and Werhl entropy exhibit a periodic oscillatory behavior in the process of time evolution. In each periodicity, both quantities increase to a maximum at the point $gt = (2m+1)/2\pi$, and thereafter they decrease to a minimum where the scaled time is closed to $2m\pi$. Moreover, the amount of entanglement exhibits the sudden-death and sudden-birth phenomena depending on the value of the atomic position parameter θ. In fact, for other values of θ, the curve keeps the same shape but with different values, of course. On the other hand, we find that the populations exhibit a periodicity of the dynamics with the vanishing time given by $\lambda t = 2m\pi$. We also note that the amplitude of the oscillations may decrease depending on the initial atomic position, and the maximum value of the populations cannot exceed unity (see Figs. 3 c and 4 c).

Finally, we consider the evolution of $\mathbf{S}_\Theta$ in terms of the scaled time λt and atomic phase space parameter Θ. In the absence of the atomic motion effect, we find in Fig. 5 that the Werhl phase entropy exhibits a random oscillatory behavior during the time evolution depending on the values of the phase Θ. Moreover, we can observe in Fig. 6, under the atomic motion effect, that $\mathbf{S}_\Theta$ shows a periodic behavior during the time evolution. In fact, we find an interesting correlation between $\mathbf{S}_\Theta$ and S_F for different ranges of the scaled time. In the case of $\theta = \pi/4$, we can see an indirect relation between $\mathbf{S}_\Theta$ and S_F exhibiting an inverse monotonic change with minimum values for $\mathbf{S}_\Theta$ accompanied by maximum values of the von Neumann entropy at points $m\pi$, while for $\theta = 0$ we have a direct monotonic dependence. On the other hand, the dependence on the phase-space parameter shows that Θ leads to an increase in the amplitude of $\mathbf{S}_\Theta$, which illustrates that the phase parameter is able to enhance the amount of phase information.

From the above result, the effect on the Wehrl phase distribution is found to be different, which indicates that the $\mathbf{S}_\Theta$ cannot be used as an indicator of the entanglement for the system under consideration.

5. Conclusions

In this paper, we present the result of a detailed study of the different quantum and classical quantifier dynamics of more general atom states, including four-level atom interacting with a single-mode field cavity in the presence and absence of the atomic motion effect for various values of the initial atomic position. As a measure, we used the von Neumann entropy S_F, Werhl entropy S_W, and Werhl phase entropy S_Θ and investigated their behavior in terms of different parameters involved in the atom–field-system state. We demonstrated an interesting monotonic relationship between the entropies during the time evolution. Our results show that some features such as the maximum long-life entanglement and the short temporal disentanglement can be obtained by a proper choice of the initial atomic position and atomic motion parameters for the systems under consideration and are very useful in different problems of quantum information processing.

We consider two distinct cases of the dynamics of S_F, S_W, and S_Θ for the field–atom interaction, when the atomic motion effect is neglected and taken into account as well. For both cases, we find that S_F and S_W are a good measure of entanglement of the system exhibiting the same behavior during the time evolution. If the atomic motion parameter is not taken into account, the entanglement tends to exhibit a random behavior with increasing time; meanwhile the Werhl entropy increases, and the field becomes a more quantum-mechanical one. On the other hand, the quantum and classical quantifiers

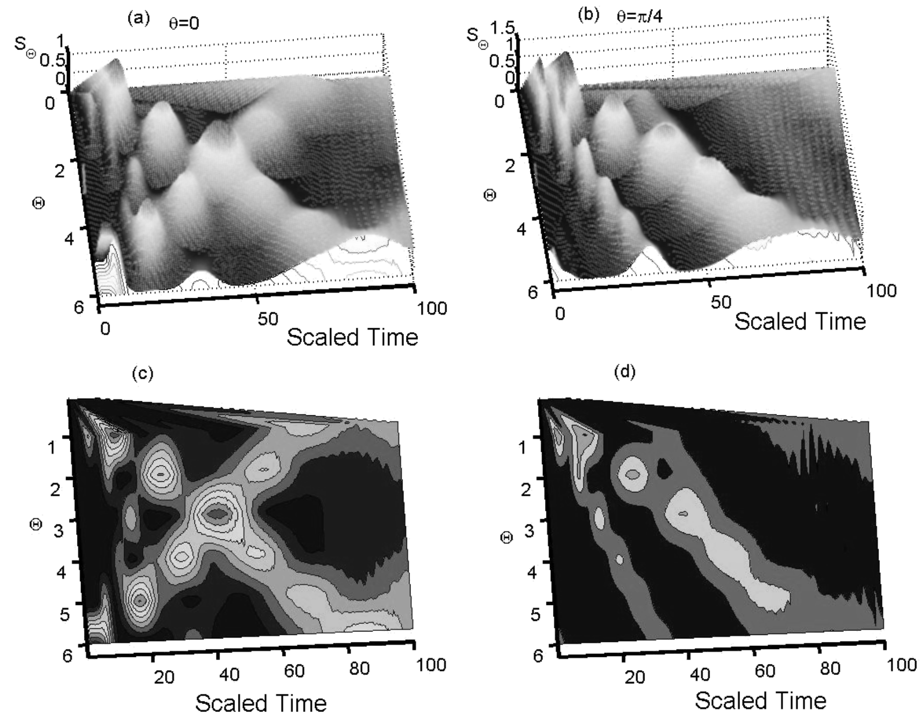

Fig. 5. The time evolution of the Wehrl phase distribution S_Θ as a function of the scaled time and the phase-space parameter of the four-level atom interacting with a single-mode field; the atom is initially in the upper state at $\theta = 0$ (a and c) and the superposition state $\theta = \pi/4$ (b and d) for $\alpha = 4$, when the atomic motion parameter is neglected.

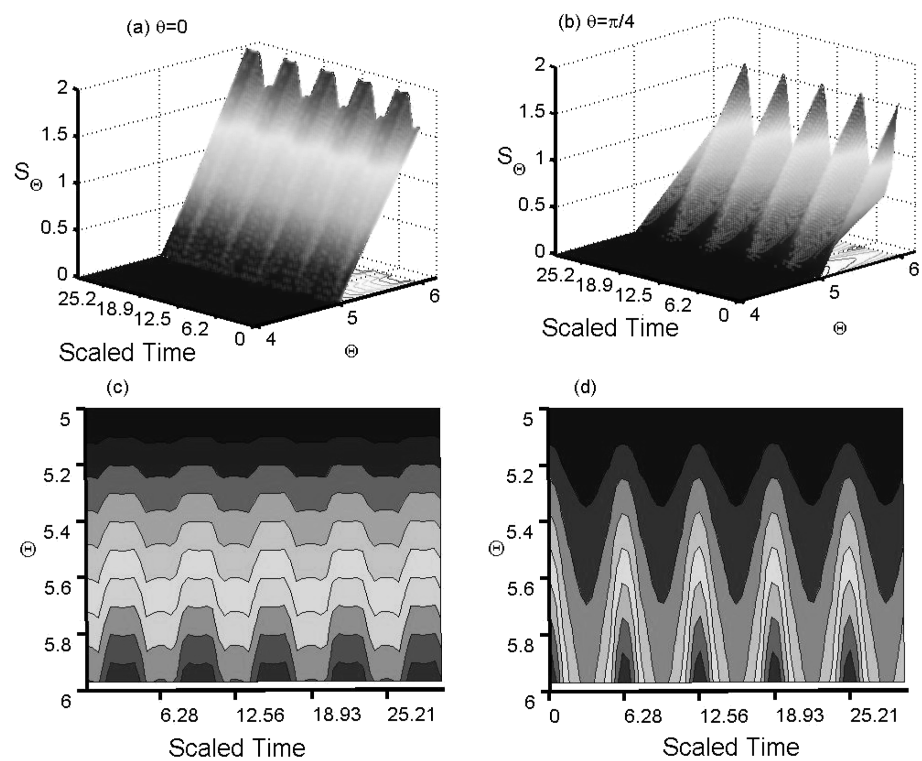

Fig. 6. The same as in Fig. 5 but the atomic motion parameter is considered as $p = 1$.

are very sensitive to the atomic motion phenomenon and their behavior complectly changes — the S_F and S_W have an oscillatory periodic behavior exhibiting the sudden death and sudden birth phenomena depending on the initial atomic position. Moreover, the Werhl phase entropy S_Θ is very sensitive to the atomic motion parameter and exhibits a monotonic correlation between S_Θ and S_F, but it depends on the initial atomic position.

Realistic quantum systems are not closed, which causes the rapid destruction of crucial quantum properties. Therefore, due to the unavoidable interaction between a quantum system and its environment, understanding the dynamics of different entropies may stimulate great interest. An important future investigation will be the study of the effect of decoherence on the evolution of quantum and classical quantifiers under both Markovian and non-Markovian environments in the case of finite temperatures.

Acknowledgments

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