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DYNAMIC INVARIANTS AND THE BERRY PHASE FOR GENERALIZED DRIVEN HARMONIC OSCILLATORS

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Abstract

We present quadratic dynamic invariants and evaluate the Berry phase for the time-dependent Schrödinger equation with the most general variable quadratic Hamiltonian.

Keywords: time-dependent Schrödinger equation, generalized harmonic oscillators, Green function, dynamic invariants, Berry phase, Ermakov-type system.

1. Introduction

In the previous paper [1], the exact wave functions for generalized (driven) harmonic oscillators [2–9] have been constructed in terms of Hermite polynomials by transforming the time-dependent Schrödinger equation into an autonomous form [10]. Relationships with certain Ermakov and Riccati-type systems have been investigated. A goal of this paper is to find the corresponding dynamic invariants and evaluate the Berry phase [2, 11–13] for quantum systems with general variable quadratic Hamiltonians as an extension of the works [5, 14–23] (see also the references therein).

2. Generalized Driven Harmonic Oscillators

We consider the one-dimensional time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H\psi, \tag{1}$$

where the variable Hamiltonian $H = Q(p, x)$ is an arbitrary quadratic of two operators $p = -i\partial/\partial x$ and x, namely,

$$i\psi_t = -a(t)\psi_{xx} + b(t)x^2\psi - ic(t)x\psi_x - id(t)\psi - f(t)x\psi + ig(t)\psi_x \tag{2}$$

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(a, b, c, d, f, and g are suitable real-valued functions of time only). We shall refer to these quantum systems as the generalized (driven) harmonic oscillators. A general approach and known elementary solutions can be found in [1, 3, 6, 7, 9, 24–33]. In addition, a case related to Airy functions is discussed in [34] and [35] deals with another special case of transcendental solutions.

In this paper, we use the following result established in [1].

Lemma 1 *The substitution*

$$\psi = \frac{e^{i\left[\alpha(t)x^2+\delta(t)x+\kappa(t)\right]}}{\sqrt{\mu(t)}}\,\chi(\xi,\tau), \qquad \xi = \beta(t)x + \varepsilon(t), \quad \tau = \gamma(t) \tag{3}$$

transforms the nonautonomous and inhomogeneous Schrödinger equation (2) into the autonomous form

$$-i\chi_\tau = -\chi_{\xi\xi} + c_0\xi^2\chi \qquad (c_0 = 0, 1), \tag{4}$$

provided that

$$\frac{d\alpha}{dt} + b + 2c\alpha + 4a\alpha^2 = c_0 a\beta^4, \tag{5}$$

$$\frac{d\beta}{dt} + (c + 4a\alpha)\beta = 0, \tag{6}$$

$$\frac{d\gamma}{dt} + a\beta^2 = 0, \tag{7}$$

$$\frac{d\delta}{dt} + (c + 4a\alpha)\delta = f + 2g\alpha + 2c_0 a\beta^3\varepsilon, \tag{8}$$

$$\frac{d\varepsilon}{dt} = (g - 2a\delta)\beta, \tag{9}$$

$$\frac{d\kappa}{dt} = g\delta - a\delta^2 + c_0 a\beta^2\varepsilon^2. \tag{10}$$

Here

$$\alpha = \frac{1}{4a}\frac{\mu'}{\mu} - \frac{d}{2a}. \tag{11}$$

The substitution (11) reduces the inhomogeneous equation (5) to the second-order ordinary differential equation

$$\mu'' - \tau(t)\mu' + 4\sigma(t)\mu = c_0(2a)^2\beta^4\mu, \tag{12}$$

that has the familiar time-varying coefficients

$$\tau(t) = \frac{a'}{a} - 2c + 4d, \qquad \sigma(t) = ab - cd + d^2 + \frac{d}{2}\left(\frac{a'}{a} - \frac{d'}{d}\right). \tag{13}$$

If $c_0 = 0$, Eq. (5) is called the Riccati nonlinear differential equation [36, 37], and the system (5)–(10) is referred to as a Riccati-type system. (Similar terminology is used in [38, 39] for the corresponding parabolic equation.) If $c_0 = 1$, Eq. (12) can be reduced to a generalized version of the Ermakov nonlinear differential equation (see, for example, [20, 25, 40, 41] and the references therein regarding the Ermakov equation), and we refer to the corresponding system (5)–(10) with $c_0 \neq 0$ as the Ermakov-type system.

Throughout this paper, we use the notation from [1] where a more detailed bibliography on the quadratic systems can be found.

Using the standard oscillator wave functions for Eq. (4) at $c_0 = 1$ (for example, [42–44]) results in the solution

$$\psi_n(x,t) = \frac{e^{i(\alpha x^2+\delta x+\kappa)+i(2n+1)\gamma}}{\sqrt{2^n n! \mu \sqrt{\pi}}}\, e^{-(\beta x+\varepsilon)^2/2}\, H_n(\beta x+\varepsilon), \tag{14}$$

where $H_n(x)$ are the Hermite polynomials [45] and the general real-valued solution of the Ermakov-type system (5)–(10) is available in [1] (Lemma 3, Eqs. (42)–(48)).

The Green function of generalized harmonic oscillators has been constructed in [3]. (See also important previous works [7, 9, 10, 46, 47] and the references therein for more details.)

The corresponding Cauchy initial value problem can be solved (formally) by the superposition principle,

$$\psi(x,t) = \int_{-\infty}^{\infty} G(x,y,t)\psi(y,0)\, dy \tag{15}$$

for some suitable initial data $\psi(x,0) = \varphi(x)$ (see [3, 20, 33] for further details). The corresponding eigenfunction expansion can be written in terms of the wave functions (14) as follows:

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n\, \psi_n(x,t), \tag{16}$$

where the time-independent coefficients are given by

$$c_n = \frac{\displaystyle\int_{-\infty}^{\infty} \psi_n^*(x,0)\psi(x,0)\, dx}{\displaystyle\int_{-\infty}^{\infty} |\psi_n(x,0)|^2\, dx}. \tag{17}$$

This expansion complements the integral form of solution (15).

The maximum symmetry group of the autonomous Schrödinger equation (4) is studied in [48, 49] (see also [50] and the references therein).

3. Dynamic Invariants for Generalized Driven Harmonic Oscillators

The concept of dynamic invariants for generalized harmonic oscillators has been recently revisited in [20, 25] (see [27, 46, 47, 51, 52] and the references therein for classical works). In this paper, we would like to point out a simple extension of the quadratic dynamic invariant to the case of driven oscillators

$$E(t) = \frac{\lambda(t)}{2}\left[\widehat{a}(t)\widehat{a}^\dagger(t) + \widehat{a}^\dagger(t)\widehat{a}(t)\right] = \frac{\lambda(t)}{2}\left[\frac{(p-2\alpha x-\delta)^2}{\beta^2} + (\beta x+\varepsilon)^2\right], \qquad \frac{d}{dt}\langle E\rangle = 0. \tag{18}$$

(See also [8, 14, 15, 19].) Here, $\lambda(t) = \exp\left[-\int_0^t [c(s) - 2d(s)]\, ds\right]$ and the corresponding time-dependent annihilation $\widehat{a}(t)$ and creation $\widehat{a}^\dagger(t)$ operators are explicitly given by

$$\widehat{a}(t) = \frac{1}{\sqrt{2}}\left(\beta x + \varepsilon + i\frac{p-2\alpha x-\delta}{\beta}\right), \qquad \widehat{a}^\dagger(t) = \frac{1}{\sqrt{2}}\left(\beta x + \varepsilon - i\frac{p-2\alpha x-\delta}{\beta}\right), \tag{19}$$

with $p = i^{-1}\partial/\partial x$ in terms of solutions of the Ermakov-type system (5)–(10). These operators satisfy the canonical commutation relation

$$\widehat{a}(t)\widehat{a}^{\dagger}(t) - \widehat{a}^{\dagger}(t)\widehat{a}(t) = 1. \tag{20}$$

The oscillator-type spectrum of the dynamic invariant E can be obtained in a standard way using the Heisenberg–Weyl algebra of the rasing and lowering operators (a second quantization [53], the Fock states)

$$\widehat{a}(t)\Psi_n(x,t) = \sqrt{n}\,\Psi_{n-1}(x,t), \quad \widehat{a}^{\dagger}(t)\Psi_n(x,t) = \sqrt{n+1}\,\Psi_{n+1}(x,t), \tag{21}$$

$$E(t)\Psi_n(x,t) = \lambda(t)\,(n+1/2)\,\Psi_n(x,t). \tag{22}$$

The corresponding orthogonal time-dependent eigenfunctions are given by

$$\Psi_n(x,t) = \frac{e^{i\left(\alpha x^2+\delta x+\kappa\right)-(\beta x+\varepsilon)^2/2}}{\sqrt{2^n n!\mu\sqrt{\pi}}}\, H_n\left(\beta x+\varepsilon\right), \qquad \langle\Psi_m,\Psi_n\rangle = \delta_{mn}\lambda^{-1}, \tag{23}$$

provided that $\beta\,(0)\,\mu\,(0) = 1$ when $\beta\mu = \lambda$ [1]) in terms of Hermite polynomials [45] and

$$\psi_n(x,t) = e^{i(2n+1)\gamma(t)}\Psi_n(x,t) \tag{24}$$

is the relation to the wave functions (14) with

$$\varphi_n(t) = -\,(2n+1)\,\gamma(t), \tag{25}$$

being the Lewis phase [5,15,53].

The dynamic invariant operator derivative identity [20,25]

$$\frac{dE}{dt} = \frac{\partial E}{\partial t} + i^{-1}\left(EH - H^{\dagger}E\right) = 0 \tag{26}$$

can be verified in the following way. Introducing new linear momentum and coordinate operators in the form

$$P = \frac{\lambda}{\beta}\left(p - 2\alpha x - \delta\right), \qquad Q = \lambda\left(\beta x+\varepsilon\right), \tag{27}$$

when $[Q,P] = i\lambda^2$ (a generalized canonical transform), one can derive the simple differentiation rules

$$\frac{dP}{dt} = -2c_0 a\beta^2 Q, \qquad \frac{dQ}{dt} = 2a\beta^2 P. \tag{28}$$

(It is worth noting that, if $c_0 = 0$, the operator P becomes the linear invariant of Dodonov, Malkin, Man'ko, and Trifonov [8,27,47,52] for generalized driven harmonic oscillators.)

Then

$$E = \frac{\lambda^{-1}}{2}\left(P^2 + c_0 Q^2\right) \qquad (c_0 = 0,1) \tag{29}$$

and it is useful to realize that E is just the original Hamiltonian H after the canonical transform [5]. The required operator identity (26) can be formally derived with the aid of product rule (3.7) of [20] (quantum calculus)

$$2\frac{dE}{dt} = \frac{d}{dt}\left(\lambda^{-1}P^2\right) + c_0\frac{d}{dt}\left(\lambda^{-1}Q^2\right) = \lambda^{-1}\left(\frac{dP}{dt}P + P\frac{dP}{dt}\right) + c_0\lambda^{-1}\left(\frac{dQ}{dt}Q + Q\frac{dQ}{dt}\right) \tag{30}$$

and by (28)

$$\lambda \frac{dE}{dt} = c_0 a \beta^2 \left(-QP - PQ + PQ + QP \right) = 0, \tag{31}$$

which completes the proof.

Remark 1 *The kernel*

$$K\left(x, y, t\right) = \frac{1}{\sqrt{\mu}} e^{i\left(\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \kappa\right)} \tag{32}$$

is a particular solution of the Schrödinger equation (2) for any solution of the Riccati-type system (5)–(11) with $c_0 = 0$ *[3]. A direct calculation shows that this kernel is an eigenfunction*

$$\beta^{-1}\left(p - 2\alpha x - \delta\right) K\left(x, y, t\right) = yK\left(x, y, t\right) \tag{33}$$

of the linear dynamic invariant [20].

4. Evaluation of the Berry Phase

The holonomic effect in quantum mechanics known as the Berry phase [2,11] had received considerable attention over the years (see, for example, [4, 5, 12–14, 16–19, 21–23, 54–57] and the references therein). The solution of the time-dependent Schrödinger equation (2) has the form (16) with the oscillator-type wave functions $\psi_n(x, t)$ presented by (14) [1]

$$\psi_n(x, t) = e^{-i\varphi_n(t)}\ \Psi_n(x, t), \tag{34}$$

where $\varphi_n(t)$ is the Lewis (or dynamic) phase and $\Psi_n(x, t)$ is the eigenfunction of quadratic invariant (22). (In the self-adjoint case, one chooses $c = 2d$ when $\lambda = 1$.)

Then

$$i \int_{\mathbb{R}} \psi_n^* \frac{\partial \psi_n}{\partial t}\ dx = \lambda^{-1} \frac{d\varphi_n}{dt} + i \int_{\mathbb{R}} \Psi_n^* \frac{\partial \Psi_n}{\partial t}\ dx \tag{35}$$

and the Berry phase θ_n is given by

$$\lambda^{-1} \frac{d\theta_n}{dt} = \operatorname{Re}\left(i \int_{\mathbb{R}} \Psi_n^* \frac{\partial \Psi_n}{\partial t}\ dx \right) = \operatorname{Re}\left(i \left\langle \Psi_n, \frac{\partial}{\partial t} \Psi_n \right\rangle \right). \tag{36}$$

Here, the eigenfunction Ψ_n is a γ-free part [5] of the wave function (14), namely,

$$\Psi_n = \lambda^{-1/2} e^{i\left(\alpha x^2 + \delta x + \kappa\right)} \Phi_n(x, t), \tag{37}$$

and Φ_n is, in fact, the real-valued stationary orthonormal wave function for the simple harmonic oscillator with respect to the new variable $\xi = \beta x + \varepsilon$; see (23) and the normalization condition

$$\int_{-\infty}^{\infty} \Phi_n^2\ dx = 1. \tag{38}$$

The integral (36) can be evaluated as in [5, 17]

$$\begin{aligned} \lambda \left\langle \Psi_n, \frac{\partial \Psi_n}{\partial t} \right\rangle &= i \left\langle \Phi_n, \left(\frac{d\alpha}{dt} x^2 + \frac{d\delta}{dt} x + \frac{d\kappa}{dt} \right) \Phi_n \right\rangle + \frac{1}{2}\left(c - 2d\right) + \left\langle \Phi_n, \frac{\partial \Phi_n}{\partial t} \right\rangle \\ &= i \frac{d\alpha}{dt} \left\langle \Phi_n, x^2 \Phi_n \right\rangle + i \frac{d\delta}{dt} \left\langle \Phi_n, x \Phi_n \right\rangle + i \frac{d\kappa}{dt} \left\langle \Phi_n, \Phi_n \right\rangle + \frac{1}{2}\left(c - 2d\right) + \left\langle \Phi_n, \frac{\partial \Phi_n}{\partial t} \right\rangle, \end{aligned}$$

where the last term is zero due to the normalization condition (38). Moreover,

$$\left\langle \Phi_n, x^2\Phi_n\right\rangle = \beta^{-3}\int_{-\infty}^{\infty}\left(\xi^2+\varepsilon^2\right)\Phi_n^2\, d\xi = \beta^{-2}\left(\varepsilon^2+n+1/2\right),$$
$$\left\langle \Phi_n, x\Phi_n\right\rangle = -\varepsilon\beta^{-2}\int_{-\infty}^{\infty}\Phi_n^2\, d\xi = -\varepsilon\beta^{-1}$$

with the help of

$$\beta^{-1}\int_{-\infty}^{\infty}\xi\Phi_n^2\, d\xi = 0, \qquad \beta^{-1}\int_{-\infty}^{\infty}\xi^2\Phi_n^2\, d\xi = n+\frac{1}{2}. \tag{39}$$

As a result,

$$\frac{d\theta_n}{dt} = -\beta^{-2}\left(\varepsilon^2+n+\frac{1}{2}\right)\frac{d\alpha}{dt}+\varepsilon\beta^{-1}\frac{d\delta}{dt}-\frac{d\kappa}{dt} \tag{40}$$

and the phase θ_n can be obtain by integrating (40). Our observation reveals the connection of the Berry phase with the Ermakov-type system (5)–(11), whose general solution is found in [1].

When $c-2d=f=g=0$, one may choose $\delta=\varepsilon=\kappa=0$, and our expression (40) simplifies to

$$\frac{d\theta_n}{dt} = -\mu^2\left(n+\frac{1}{2}\right)\frac{d\alpha}{dt} = -\frac{1}{4a}\left(n+\frac{1}{2}\right)\left[\mu''\mu-\left(\mu'\right)^2-\frac{a'}{a}\mu'\mu+2d\left(\frac{a'}{a}-\frac{d'}{d}\right)\mu^2\right] \tag{41}$$

with the help of (11). The function μ is a solution of the Ermakov equation (12)–(13) with $c_0=1$ and $\beta=\mu^{-1}$. This result is consistent with [5, 14], where the original expression of [18] has been corrected.

5. An Alternative Derivation of the Berry Phase

In view of (1) and (34)–(36), we obtain

$$\lambda^{-1}\left(\frac{d\theta_n}{dt}+\frac{d\varphi_n}{dt}\right) = \mathrm{Re}\left\langle\psi_n, H\psi_n\right\rangle = \mathrm{Re}\left\langle\Psi_n, H\Psi_n\right\rangle, \tag{42}$$

because the Hamiltonian in (1)–(2) does not involve time differentiation. Here,

$$H = ap^2+bx^2+\frac{c}{2}\left(px+xp\right)+\frac{i}{2}\left(c-2d\right)-fx-gp \tag{43}$$

and the position and linear momentum operators are given by

$$x = \frac{1}{\beta}\left[\frac{1}{\sqrt{2}}\left(\widehat{a}+\widehat{a}^\dagger\right)-\varepsilon\right], \qquad p = \frac{\beta}{i\sqrt{2}}\left(\widehat{a}-\widehat{a}^\dagger\right)+\frac{\sqrt{2}\alpha}{\beta}\left(\widehat{a}+\widehat{a}^\dagger\right)+\delta-\frac{2\alpha\varepsilon}{\beta} \tag{44}$$

in terms of the creation and annihilation operators (19). After the substitution, the Hamiltonian takes the form

$$\begin{aligned} H = & \left[\frac{a}{2}\left(\frac{4\alpha^2}{\beta^2}-\beta^2\right)+\frac{b+2c\alpha}{\beta^2}-\frac{i}{2}\left(c+4a\alpha\right)\right]\widehat{a}^{\,2}+\left[\frac{a}{2}\left(\frac{4\alpha^2}{\beta^2}-\beta^2\right)+\frac{b+2c\alpha}{\beta^2}+\frac{i}{2}\left(c+4a\alpha\right)\right]\widehat{a}^{\dagger\,2} \\ & +\frac{1}{2}\left[a\left(\beta^2+\frac{4\alpha^2}{\beta^2}\right)+\frac{b+2c\alpha}{\beta^2}\right]\left(\widehat{a}\widehat{a}^{\dagger}+\widehat{a}^{\dagger}\widehat{a}\right)+\frac{i}{2}\left(c-2d\right)+\sqrt{2}\left[\frac{4a\alpha+c}{2\beta}\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)-\frac{\varepsilon}{\beta^2}\left(b+c\alpha\right)\right. \\ & \left.-\frac{f+2g\alpha}{2\beta}+\frac{i}{2}\left(\beta\left(g-2a\delta\right)+\varepsilon\left(c+4a\alpha\right)\right)\right]\widehat{a}+\sqrt{2}\left[\frac{4a\alpha+c}{2\beta}\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)-\frac{\varepsilon}{\beta^2}\left(b+c\alpha\right)-\frac{f+2g\alpha}{2\beta}\right. \\ & \left.-\frac{i}{2}\left(\beta\left(g-2a\delta\right)+\varepsilon\left(c+4a\alpha\right)\right)\right]\widehat{a}^{\dagger}+a\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)^2+\frac{\varepsilon}{\beta}\left(f+\frac{b\varepsilon}{\beta}\right)-\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)\left(g+\frac{c\varepsilon}{\beta}\right). \end{aligned} \tag{45}$$

Here, $J_+ = \widehat{a}^{\dagger\,2}/2$, $J_- = \widehat{a}^{\,2}/2$, $J_0 = \left(\widehat{a}\widehat{a}^{\dagger}+\widehat{a}^{\dagger}\widehat{a}\right)/4$ are the generators of a noncompact $SU\,(1,1)$ algebra

$$[J_0, J_\pm] = \pm J_\pm, \qquad [J_+, J_-] = -2J_0 \tag{46}$$

and, therefore, use can be made of the group properties of the corresponding discrete positive series $\mathcal{D}_+^j$ for further investigation of the Berry phase. (This is a "standard procedure" for quadratic Hamiltonians — more details can be found in [6, 23, 32, 45, 46, 54, 55, 58] and elsewhere.) Together, the linears and bilinears in $\widehat{a}$ and $\widehat{a}^{\dagger}$ realize the semidirect sum of the $SU(1,1)$ and the Heisenberg algebra (20) (see [50] for more details).

Thus

$$\begin{aligned} \lambda \mathrm{Re}\left\langle \Psi_n, H\Psi_n\right\rangle = & \left(n+\frac{1}{2}\right)\left[a\left(\beta^2+\frac{4\alpha^2}{\beta^2}\right)+\frac{b+2c\alpha}{\beta^2}\right]+a\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)^2+\frac{\varepsilon}{\beta}\left(f+\frac{b\varepsilon}{\beta}\right) \\ & -\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)\left(g+\frac{c\varepsilon}{\beta}\right) \end{aligned} \tag{47}$$

by (21)–(22).

Finally, from (25) and (42) we arrive at a different formula for the Berry phase

$$\begin{aligned} \frac{d\theta_n}{dt} = & \left(n+\frac{1}{2}\right)\left[a\left(\frac{4\alpha^2}{\beta^2}-\beta^2\right)+\frac{b+2c\alpha}{\beta^2}\right]+a\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)^2+\frac{\varepsilon}{\beta}\left(f+\frac{b\varepsilon}{\beta}\right) \\ & -\left(\delta-\frac{2\alpha\varepsilon}{\beta}\right)\left(g+\frac{c\varepsilon}{\beta}\right), \end{aligned} \tag{48}$$

which is consistent with the previous expression (40) for any solution of the Ermakov-type system (5)–(10) ($c_0 = 1$).

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References

1. N. Lanfear, R. M. Lopez, and S. K. Suslov, *J. Russ. Laser Res.*, **32**, 352 (2011) [arXiv:11002.5119v2 [math-ph] 20 Jul 2011].
2. M. V. Berry, *J. Phys. A: Math. Gen.*, **18**, 15 (1985).
3. R. Cordero-Soto, R. M. Lopez, E. Suazo, and S. K. Suslov, *Lett. Math. Phys.*, **84**, 159 (2008).
4. J. H. Hannay, *J. Phys. A: Math. Gen.*, **18**, 221 (1985).
5. P. G. L. Leach, *J. Phys. A: Math. Gen.*, **23**, 2695 (1990).
6. C. F. Lo, *Eur. Phys. Lett.*, **24**, 319 (1993).
7. K. B. Wolf, *SIAM J. Appl. Math.*, **40**, 419 (1981).
8. J-B. Xu and X-Ch. Gao, *Phys. Scr.*, **54**, 137 (1996).
9. K-H. Yeon, K-K. Lee, Ch-I. Um, et al., *Phys. Rev. A*, **48**, 2716 (1993).
10. A. V. Zhukov, *Phys. Lett. A*, **256**, 325 (1999).
11. M. V. Berry, *Proc. Roy. Soc. London*, **A392**, 45 (1984).
12. B. Simon, *Phys. Rev. Lett.*, **51**, 2167 (1983).
13. F. Wilczek and A. Zee, *Phys. Rev. Lett.*, **52**, 2111 (1984).
14. M. H. Engineer and G. Ghosh, *J. Phys. A: Math. Gen.*, **21**, L95 (1988).
15. X-Ch. Gao, J-B. Xu, and T-Zh. Qian, *Ann. Phys.*, **204**, 235 (1990).
16. D. H. Kobe, *J. Phys. A: Math. Gen.*, **23**, 4249 (1990).
17. D. H. Kobe, *J. Phys. A: Math. Gen.*, **24**, 2763 (1991).
18. D. A. Morales, *J. Phys. A: Math. Gen.*, **21**, L889 (1988).
19. D. B. Monteoliva, H. J. Korsch, and J. A. Núñez, *J. Phys. A: Math. Gen.*, **27**, 6897 (1994).
20. S. K. Suslov, *Phys. Scr.*, **81**, 055006 (2010) [arXiv:1002.0144v6 [math-ph] 11 Mar 2010].
21. V. V. Dodonov and V. I. Man'ko, "Adiabatic invariants, correlated states and Berry's phase" in: B. Markovski and S. I. Vinitsky (Eds.), *Topological Phases in Quantum Theory, Proceedings of the International Seminar, Dubna, SU, September 1988*, World Scientific, Singapore (1989), p. 74.
22. S. S. Mizrahi, *Phys. Lett. A*, **138**, 465 (1989).
23. J. M. Cerveró and J. D. Lejarreta, *J. Phys. A: Math. Gen.*, **22**, L663 (1989).
24. R. Cordero-Soto, E. Suazo, and S. K. Suslov, *J. Phys. Math.*, **1**, S090603 (2009).
25. R. Cordero-Soto, E. Suazo, and S. K. Suslov, *Ann. Phys.*, **325**, 1884 (2010).
26. R. Cordero-Soto and S. K. Suslov, *Theor. Math. Phys.*, **162**, 286 (2010) [arXiv:0808.3149v9 [math-ph] 8 Mar 2009].
27. V. V. Dodonov, I. A. Malkin, and V. I. Man'ko, *Int. J. Theor. Phys.*, **14**, 37 (1975).
28. R. P. Feynman, "The principle of least action in quantum mechanics," Ph.D. Thesis, Princeton University, USA (1942) [reprinted in: L. M. Brown (Ed.), *Feynman's Thesis – A New Approach to Quantum Theory*, World Scientific, Singapore (2005), p. 1].
29. R. P. Feynman, *Rev. Mod. Phys.*, **20**, 367 (1948) [reprinted in: L. M. Brown (Ed.), *Feynman's Thesis – A New Approach to Quantum Theory*, World Scientific, Singapore (2005), p. 71].
30. R. P. Feynman and A. R. Hibbs, *Quantum Mechanics and Path Integrals*, McGraw–Hill, New York (1965).
31. R. M. Lopez and S. K. Suslov, *Rev. Mex. Fís.*, **55**, 195 (2009) [arXiv:0707.1902v8 [math-ph] 27 Dec 2007].
32. M. Meiler, R. Cordero-Soto and S. K. Suslov, *J. Math. Phys.*, **49**, 072102 (2008) [arXiv: 0711.0559v4 [math-ph] 5 Dec 2007].

33. E. Suazo and S. K. Suslov, “Cauchy problem for Schrödinger equation with variable quadratic Hamiltonians” (under preparation).
34. N. Lanfear and S. K. Suslov, “The time-dependent Schrödinger equation, Riccati equation, and Airy functions,” arXiv:0903.3608v5 [math-ph] 22 Apr 2009.
35. R. Cordero-Soto and S. K. Suslov, *J. Phys. A: Math. Theor.*, **44**, 015101 (2011) [arXiv:1006.3362v3 [math-ph] 2 Jul 2010].
36. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, Cambridge (1944).
37. E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed., Cambridge University Press, Cambridge (1927).
38. E. Suazo, S. K. Suslov, and J. M. Vega-Guzmán, *N.Y. J. Math.*, **17a**, 225 (2011).
39. E. Suazo, S. K. Suslov, and J. M. Vega-Guzmán, “The Riccati system and a diffusion-type equation,” arXiv:1102.4630v1 [math-ph] 22 Feb 2011.
40. V. P. Ermakov, “Second-order differential equations. Conditions of complete integrability” [in Russian], *Izvestiya Universiteta Kiev*, Series III, **9**, 1 (1880) [English translation: *Appl. Anal. Discrete Math.*, **2**, 123 (2008)].
41. P. G. L. Leach and K. Andriopoulos, *Appl. Anal. Discrete Math.*, **2**, 146 (2008).
42. S. Flügge, *Practical Quantum Mechanics*, Springer Verlag, Berlin (1999).
43. L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory*, Pergamon Press, Oxford (1977).
44. E. Merzbacher, *Quantum Mechanics*, 3rd ed., John Wiley & Sons, New York (1998).
45. A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer Verlag, Berlin, New York (1991).
46. I. A. Malkin and V. I. Man’ko, *Dynamic Symmetries and Coherent States of Quantum Systems*, Nauka, Moscow, (1979) [in Russian].
47. V. V. Dodonov and V. I. Man’ko, “Invariants and correlated states of nonstationary quantum systems” [in Russian], in: *Invariants and the Evolution of Nonstationary Quantum Systems*, *Proceedings of the P. N. Lebedev Physical Institute*, Nauka, Moscow, Vol. 183, p. 71 (1987) [English translation: Nova Science, Commack, New York (1989), p. 103].
48. U. Niederer, *Helv. Phys. Acta*, **45**, 802 (1972).
49. U. Niederer, *Helv. Phys. Acta*, **46**, 191 (1973).
50. L. Vinet and A. Zhedanov, *J. Phys. A: Math. Theor.*, **44**, 355201 (2011).
51. V. V. Dodonov, *J. Phys. A: Math. Gen.*, **33**, 7721 (2000).
52. I. A. Malkin, V. I. Man’ko, and D. A. Trifonov, *J. Math. Phys.*, **14**, 576 (1973).
53. H. R. Lewis, Jr. and W. B. Riesenfeld, *J. Math. Phys.*, **10**, 1458 (1969).
54. C. C. Gerry, *Phys. Rev. A*, **39**, 3204 (1989).
55. L. Vinet, *Phys. Rev. D*, **37**, 2369 (1988).
56. D. Xiao, M-Ch. Chang, and Q. Niu, *Rev. Mod. Phys.*, **82**, 1959 (2010).
57. S. I. Vinitskiĭ, V. L. Derbov, V. M. Dubovik, et al., *Uspekhi Fiz. Nauk*, **160**, 1 (1990) [*Sov. Phys. - Uspekhi*, **33**, 403 (1990)].
58. Yu. F. Smirnov and K. V. Shitikova, *Sov. J. Particles Nuclei*, **8**, 344 (1977).