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OPTICAL PROPAGATOR OF QUANTUM SYSTEMS IN THE PROBABILITY REPRESENTATION

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Abstract

The evolution equation for the propagator of the quantum system in the optical probability representation (optical propagator) is obtained. The relations between the optical and quantum propagators for the Schrödinger equation and the optical propagator of an arbitrary quadratic system are found explicitly.

Keywords: propagator, optical tomogram, Green function, quantum state evolution.

1. Introduction

Since the beginning of quantum mechanics there were attempts at understanding the notion of quantum states in terms of a classical approach [1–5]. Within the framework of this approach, a number of the so-called quasiprobability distribution functions, such as the Wigner function [6], the Husimi function [7], and the Glauber–Sudarshan function [8, 9], later on unified into a one-parametric family [10], were suggested. But due to the Heisenberg principle, in contrast to the classical probabilities, all these quasiprobabilities do not describe distributions of measurable variables.

A formulation of quantum mechanics that is very similar to the classical stochastic approach has been presented by Moyal [11], but the evolution equation suggested by Moyal was an equation for the quasiprobability distribution function (the Wigner function) and not for the probability.

Recently in [12, 13], it was suggested to consider quantum dynamics as a classical stochastic process described namely by the probability distribution – so-called marginal distribution function (MDF) (which was discussed in a general context in [10]) associated to the position coordinate X, taking values in an ensemble of reference frames in the phase space. Such a classical probability distribution is shown to describe completely quantum states [14–16]. A detailed analysis of the tomographic-probability picture of quantum mechanics can be found in recent review [17].

Within this approach, the notion of measuring a quantum state provides the usual optical tomography approach [18–20] and its extension called the symplectic-tomography formalism [14–16]. Both allow one to link explicitly the MDF and the Wigner function or the density matrix in other representations. In this way, starting from the evolution equation for the density matrix, the evolution equation of a Fokker–Planck type for the symplectic marginal distribution function (symplectic tomogram) is obtained [12, 13]. The quantum evolution equation and the energy level equation have been found for the optical tomogram

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in an explicit form in [21]. Such equations allow one to define independently the marginal distribution function. Thus it may be assumed as a starting point for an alternative but equivalent formulation of quantum mechanics in what we call the probability representation or the classical-like description of quantum mechanics.

In [22–25], the symplectic propagator for a wide class of Hamiltonians and its relation to the quantum propagator for the density matrix was found, which establishes an important bridge between the probability representation of quantum mechanics and other formulations such as the path-integral approach.

In the present work, we extend the optical probability description of quantum mechanics and verify the formalism by applying it to a nontrivial case of physical interest, which is the forced parametric oscillator. From the viewpoint of general formalism, we find new equations that connect the optical propagator of a quantum system with its integrals of motion.

This paper is organized as follows.

In Sec. 2, we review the optical tomography of quantum states. In Sec. 3, we find general relations for the optical propagator and present the evolution equation for the optical propagator with the initial conditions. In Sec. 4, we obtain the optical propagator of an arbitrary quadratic system and check the general relations by direct substitution of found propagators for simple quantum systems.

2. Optical Tomographic Representation of Quantum States

In this section, we give a short review of the tomographic representation of quantum mechanics using the so-called optical tomogram [18–20]. For simplicity, we consider the case of one degree of freedom with dimensionless variables, because the generalization of all our calculations and results to the case of any arbitrary number of dimensional degrees of freedom is obvious.

If $\hat{\rho}$ is the density matrix of the quantum state, the optical tomogram is defined as follows:

$$w(X,\theta) = \mathrm{Tr}\{\hat{\rho}\delta(X - \hat{q}\cos\theta - \hat{p}\sin\theta)\} = \langle X,\theta|\hat{\rho}|X,\theta\rangle, \tag{1}$$

where $|X,\theta\rangle$ is an eigenvector of the Hermitian operator $\hat{q}\cos\theta + \hat{p}\sin\theta$ for the eigenvalue X

$$\langle q|X,\theta\rangle = (2\pi|\sin\theta|)^{-1/2}\exp\left(i\frac{Xq - \frac{q^2}{2}\cos\theta}{\sin\theta}\right). \tag{2}$$

In terms of the Wigner function [6], the tomogram $w(X,\theta)$ is expressed as

$$w(X,\theta) = \frac{1}{(2\pi)^2}\int W(q,p)e^{i\eta(X - q\cos\theta - p\sin\theta)}dq\,dp\,d\eta. \tag{3}$$

This relation can be reversed using the symmetry property of the optical tomogram

$$w(X,\theta,t) = w((-1)^k X, \theta + \pi k, t), \qquad k = 0,\ \pm 1,\ \pm 2, \cdots \tag{4}$$

After some algebra, we arrive at

$$W(q,p) = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\theta \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} w(X,\theta)|\eta|\ e^{i\eta(X - q\cos\theta - p\sin\theta)}\mathrm{d}\eta\ \mathrm{d}X. \tag{5}$$

From (3), using the relations between the Wigner function and the density matrix $\rho(q,q')$ in the coordinate representation

$$W(q,p) = \int \rho(q+u/2, q-u/2)e^{-ipu}du, \tag{6}$$

$$\rho(q,q') = \frac{1}{2\pi}\int W\left(\frac{q+q'}{2},p\right)e^{ip(q-q')}dp, \tag{7}$$

we can write the relations between the optical tomogram and the density matrix in the coordinate representation as follows:

$$w(X,\theta) = \frac{1}{2\pi}\int \rho\left(q+\frac{u\sin\theta}{2}, q-\frac{u\sin\theta}{2}\right)e^{-iu(X-q\cos\theta)}du\ dq, \tag{8}$$

$$\rho(q,q') = \frac{1}{2\pi}\int_0^{\pi} \mathrm{d}\theta \int_{-\infty}^{+\infty} w(X,\theta)|\eta| \exp\left\{i\eta\left(X-\frac{q+q'}{2}\cos\theta\right)\right\}\delta(q-q'-\eta\sin\theta)\ \mathrm{d}\theta\ d\eta\ dX. \tag{9}$$

Thus, the tomogram $w(X,\theta)$ contains all information on the quantum state.

The quadrature statistics can be obtained from the optical tomogram

$$\langle X^n\rangle(\theta) = \int X^n w(X,\theta)\ dX. \tag{10}$$

3. Quantum Evolution and Optical Propagator

The quantum evolution equation for the optical tomogram has been found in explicit form in [21]. In the one-dimensional case, for the Hamiltonian $\hat{H} = (\hat{p}^2/2) + U(\hat{q})$ this equation reads

$$\frac{\partial}{\partial t}w(X,\theta,t) = \left[\cos^2\theta\frac{\partial}{\partial\theta} - \frac{1}{2}\sin 2\theta\left\{1+X\frac{\partial}{\partial X}\right\}\right]w(X,\theta,t)$$
$$+2\left[\mathrm{Im}\ U\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial X}\right]^{-1} + X\cos\theta + i\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right]w(X,\theta,t). \tag{11}$$

The evolution of the probability distribution $w(X,\theta)$ can be represented as the following integral relationship:

$$w(X,\theta,t) = \int_0^{\pi}\mathrm{d}\theta\int_{-\infty}^{\infty}\mathrm{d}X\ \Pi(X,\theta,t;X',\theta',t')w(X',\theta',t'), \qquad t\geq t', \tag{12}$$

where $\Pi(X,\theta,t;X',\theta',t')$ is the Green function of Eq. (11). We call this function the optical propagator. In the probability terminology, it can be interpreted as the classical probability density of the system's transition from the initial position X' in the ensemble of reference frames of the classical phase space to the position X.

Due to (4), the optical propagator satisfies the symmetry property

$$\Pi\left(X,\theta,t;X',\theta',t'\right) = \Pi\left((-1)^k X, \theta+\pi k, t; X',\theta',t'\right), \qquad t\geq t'. \tag{13}$$

Taking into account the physical meaning, the propagator satisfies the nonlinear integral relationship in the case of N consecutive moments of time t_k ($k = \overline{1,\ N}$) between $t_1 = t_{in}$ and $t_f = t_N$

$$\Pi(X_N,\theta_N,t_f;X_1,\theta_1,t_{in}) = \int \left\{ \prod_{k=1}^{N-1} \Pi(X_{k+1},\theta_{k+1},t_{k+1};X_k,\theta_k,t_k) \right\} \prod_{k=2}^{N-1} dX_k\, d\theta_k. \tag{14}$$

If one takes in this relation $t_f - t_{in} = N\tau$, $t_k = t_{in} + k\tau$, in the limit case $\tau \to 0$, $N \to \infty$, the expression for the optical propagator can be obtained in terms of the functional integral.

We establish the connection between the optical propagator and the quantum propagator (Green function) for the density matrix $\rho(q,q',t)$.

For the pure state with the wave function $\Psi(q,t)$, we have

$$\rho(q,q',t) = \Psi(q,t)\Psi^*(q',t).$$

Since the wave function at the time moment t is related to the wave function at the time moment t' by the Green function $G(q,q',t,t')$ of the Schrödinger equation

$$\Psi(q,t) = \int G(q,\tilde{q},t,t')\Psi(\tilde{q},t')d\tilde{q}, \qquad t \geq t',$$

for the density matrix we can write

$$\rho(q,q',t) = \int K(q,q',t;\tilde{q},\tilde{q}',t')\rho(\tilde{q},\tilde{q}',t')\, d\tilde{q}\, d\tilde{q}', \qquad t \geq t',$$

where $K(q,q',t;\tilde{q},\tilde{q}',t') = G(q,\tilde{q},t,t')G^*(q',\tilde{q}',t,t')$ is the quantum propagator for the density matrix. Obviously, the quantum propagator satisfies the initial condition

$$K(q,q',t';\tilde{q},\tilde{q}',t') = \delta(q-\tilde{q})\delta(q'-\tilde{q}'). \tag{15}$$

Usually the quantum propagator is taken to be zero at $t < t'$:

$$K(q,q',t;\tilde{q},\tilde{q}',t') = 0, \qquad t < t'. \tag{16}$$

In view of the relation between the density matrix and the optical tomogram (9), we obtain

$$K(q,q',t;\tilde{q},\tilde{q}',t') = \frac{1}{(2\pi)^2} \int \Pi\left(X,\theta,t;\frac{1}{2}(\tilde{q}+\tilde{q}')\cos\theta' + p'\sin\theta',\theta',t'\right)\ e^{-ip'(\tilde{q}-\tilde{q}')}$$
$$\times e^{i\eta[X-((q+q')\cos\theta)/2]}|\eta|\delta(q-q'-\eta\sin\theta)\, d\eta\, dp'\, dX\, d\theta\, d\theta'. \tag{17}$$

Thus, if we know the propagator for the optical probability distribution $w(X,\theta)$, we can find the quantum propagator for the density matrix of the quantum state.

Formula (17) can be reversed as follows:

$$\Pi(X,\theta,t;X',\theta',t') = \frac{1}{(2\pi)^2} \int K\left(q + \frac{k\sin\theta}{2}, q - \frac{k\sin\theta}{2}, t; \tilde{q}' + \eta\sin\theta', \tilde{q}', t'\right)$$
$$\times \exp\left\{i\eta\left(X' - \frac{2\tilde{q}+\eta\sin\theta'}{2}\cos\theta'\right)\right\} \exp\left\{-ik(X - q\cos\theta)\right\} |\eta|\, dk\, d\eta\, dq\, d\tilde{q}'. \tag{18}$$

After substituting the initial condition (15) for the quantum propagator into expression (18), we can obtain the initial condition for the optical propagator, namely,

$$\Pi(X,\theta,t';X',\theta',t') = \delta(X\cos(\theta-\theta')-X')\delta(\sin(\theta-\theta')). \tag{19}$$

Obviously, this condition satisfies the symmetry property (13).

In view of (12), taking into account the initial condition (19), and defining

$$\Pi(X,\theta,t;X',\theta',t') = 0, \quad \text{at} \quad t<t',$$

from (11) we can obtain that the optical propagator satisfies the equation

$$\begin{aligned}&\frac{\partial}{\partial t}\Pi(X,\theta,t;X',\theta',t') - \left[\cos^2\theta\frac{\partial}{\partial\theta} - \frac{1}{2}\sin 2\theta\left\{1+X\frac{\partial}{\partial X}\right\}\right]\Pi(X,\theta,t;X',\theta',t')\\ &-2\left[\operatorname{Im} U\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial X}\right]^{-1} + X\cos\theta + i\frac{\sin\theta}{2}\frac{\partial}{\partial X}\right\}\right]\Pi(X,\theta,t;X',\theta',t')\\ &= \delta(X\cos(\theta-\theta')-X')\delta(\sin(\theta-\theta'))\delta(t-t').\end{aligned} \tag{20}$$

It is worth noting that relations (17) are (18) are the extensions of known results [26] for the symplectic tomogram

$$M(X,\mu,\nu,t) = \frac{1}{(2\pi)^2}\int W(q,p,t)e^{ik(X-\mu q-\nu p)}\,dk\,dq\,dp, \tag{21}$$

in the case where the symplectic propagator is defined as

$$M(X,\mu,\nu,t) = \int \Pi(X,\mu,\nu,t;X',\mu',\nu';t')M(X',\mu',\nu',t')\,dX'\,d\mu'\,d\nu'. \tag{22}$$

The relation between the symplectic tomogram and the quantum propagator reads

$$\begin{aligned}K(q,q',t;\tilde q,\tilde q',t') \;&=\; \frac{1}{(2\pi)^2}\int\frac{1}{|\nu'_\sigma|}\exp\left\{iY - i\mu\frac{q+q'}{2} - iY'\frac{\tilde q-\tilde q'}{\nu'} + i\mu'\frac{\tilde q^2-\tilde q'^2}{2\nu'}\right\}\\ &\times\Pi(Y,\mu,q-q',t;Y',\mu',\nu';t')\,d\mu\,d\mu'\,dY\,dY'\,d\nu'.\end{aligned} \tag{23}$$

For the inverse relation we have

$$\begin{aligned}\Pi(X,\mu,\nu,t;X',\mu',\nu';t') \;&=\; \frac{1}{(2\pi)^2|\nu|}\int\exp\left\{-i\frac{q-q'}{\nu}\left(X-\mu\frac{q+q'}{2}\right) + iX' - i\mu'\frac{2\tilde q-\nu'}{2}\right\}\\ &\times K(q,q',t;\tilde q,\tilde q-\nu',t')\,dq\,dq'\,d\tilde q.\end{aligned} \tag{24}$$

Using the homogeneity property of the symplectic tomogram

$$M(\lambda X,\lambda\mu,\lambda\nu,t) = |\lambda|^{-1}M(X,\mu,\nu,t), \tag{25}$$

we can rewrite this relation as follows:

$$\begin{aligned}&\Pi(X,\mu,\nu,t;X',\mu',\nu';t') = \frac{1}{(2\pi)^2|\nu|}\int\exp\left\{i\frac{\nu_1}{\nu}\left(X'-\mu'\frac{2\tilde q-\nu_1}{2}\right)\right\}\\ &\times\exp\left\{-i\frac{q-q'}{\nu}\left(X-\mu\frac{q+q'}{2}\right)\right\}|\nu_1|\delta(\nu-\nu')K(q,q',t;\tilde q,\tilde q-\nu_1,t')\,dq\,dq'\,d\tilde q\,d\nu_1.\end{aligned} \tag{26}$$

Also the property (25) enables us to find the relation between the symplectic and optical propagators:

$$\Pi_{\rm opt}(X,\theta,t;X',\theta',t') = \int\Pi_{\rm s}(X,\cos\theta,\sin\theta,t;rX',r\cos\theta',r\sin\theta';t')|r|\,dr. \tag{27}$$

4. Optical Propagator for Quadratic Systems

As an example, we consider the system with quadratic Hermitian Hamiltonian

$$\hat{H} = \frac{1}{2}(\hat{\mathbf{Q}}B\hat{\mathbf{Q}}) + \mathbf{C}\hat{\mathbf{Q}},$$

where $\hat{\mathbf{Q}} = (\hat{p}, \hat{q})$ is a vector operator, B is a symmetric $2{\times}2$ matrix, and $\mathbf{C}$ is a real 2-vector dependent on time. As known [27, 28], the system has linear integrals of motion:

$$\hat{\mathbf{I}}(t) = \Lambda(t)\hat{\mathbf{Q}} + \boldsymbol{\Delta}(t), \tag{28}$$

where the real symplectic $2{\times}2$ matrix $\Lambda(t)$ and the real vector $\boldsymbol{\Delta}(t)$ satisfy the equations

$$\dot{\Lambda} = i\Lambda B\sigma_y, \qquad \dot{\boldsymbol{\Delta}} = i\boldsymbol{\Delta}\sigma_y\mathbf{C},$$

with the initial conditions $\Lambda(0) = 1$ and $\boldsymbol{\Delta}(0) = 0$. The integrals of motion are defined by the following equation:

$$\partial_t\hat{I}(t) + i[\hat{H}, \hat{I}(t)] = 0. \tag{29}$$

For the quadratic systems under consideration, any integrals of motion can be expressed as functions of two operators: $\hat{A}(t) = \hat{U}\hat{a}\hat{U}^{-1}$ and $\hat{A}^\dagger(t) = \hat{U}\hat{a}^\dagger\hat{U}^{-1}$, where $\hat{U}$ is the evolution operator and

$$\hat{A}(t) = \frac{i}{\sqrt{2}}(\varepsilon(t)\hat{p} - \dot{\varepsilon}(t)\hat{q}) + \beta(t), \qquad \hat{A}^\dagger(t) = -\frac{i}{\sqrt{2}}(\varepsilon^*(t)\hat{p} - \dot{\varepsilon}^*(t)\hat{q}) + \beta^*(t), \tag{30}$$

with ε satisfying the equations

$$\ddot{\varepsilon} + \omega^2(t)\varepsilon = 0, \qquad \dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*\varepsilon = 2i, \tag{31}$$

with the initial conditions $\varepsilon(0) = 1$ and $\dot{\varepsilon}(0) = i$. The function $\beta(t)$ is defined from (29) as

$$\beta(t) = -\frac{i}{\sqrt{2}}\int_0^t \mathrm{d}t'\varepsilon(t')f(t'), \tag{32}$$

and the integrals of motion (28) are expressed from $\hat{A}$ and $\hat{A}^\dagger$ as follows:

$$\hat{\mathbf{I}}(t) = \begin{pmatrix} \hat{I}_p \\ \hat{I}_q \end{pmatrix} = \begin{pmatrix} \frac{\hat{A}-\hat{A}^\dagger}{i\sqrt{2}} \\ \frac{\hat{A}+\hat{A}^\dagger}{\sqrt{2}} \end{pmatrix}, \qquad \hat{\mathbf{I}}(t=0) = \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}. \tag{33}$$

The matrix Λ and the vector $\boldsymbol{\Delta}$ read

$$\Lambda = \frac{1}{2}\begin{pmatrix} \varepsilon + \varepsilon^* & -(\dot{\varepsilon} + \dot{\varepsilon}^*) \\ i(\varepsilon - \varepsilon^*) & -i(\dot{\varepsilon} - \dot{\varepsilon}^*) \end{pmatrix}, \qquad \boldsymbol{\Delta} = \frac{1}{\sqrt{2}}\begin{pmatrix} i(\beta - \beta^*) \\ \beta + \beta^* \end{pmatrix}.$$

The knowledge of integrals of motion (33) allows one to find the Green function (or quantum propagator) for the Schrödinger equation of the system (see [27]).

Now we expand the method of integrals of motion in order to find the optical propagator. Taking into account the initial condition (19), we can write the system of equations (we put $t'=0$)

$$\tilde{q}\Pi(X,\theta,X',\theta',t=0)=\tilde{q}'\Pi(X,\theta,X'\theta',t=0),\qquad \tilde{p}\Pi(X,\theta,X',\theta',t=0)=-\tilde{p}'\Pi(X,\theta,X'\theta',t=0),\tag{34}$$

where $\tilde{p}$ and $\tilde{q}$ are the quadrature operators in the optical tomographic representation [21],

$$\begin{aligned}\tilde{q} &= \sin\theta\left[\frac{\partial}{\partial X}\right]^{-1}\frac{\partial}{\partial\theta}+X\cos\theta+\frac{i}{2}\sin\theta\frac{\partial}{\partial X},\\ \tilde{p} &= -\cos\theta\left[\frac{\partial}{\partial X}\right]^{-1}\frac{\partial}{\partial\theta}+X\sin\theta-\frac{i}{2}\cos\theta\frac{\partial}{\partial X}.\end{aligned}\tag{35}$$

Here, the primes mean the action to the primed variables of the propagators.

During the time evolution (or formal action of the evolution operator to nonprimed variables), the system (34) evolves to the following one:

$$\tilde{\mathbf{I}}_q(t)\Pi(X,\theta;X',\theta',t) = \tilde{q}'\Pi(X,\theta;X',\theta',t),\tag{36}$$

$$\tilde{\mathbf{I}}_p(t)\Pi(X,\theta;X',\theta',t) = -\tilde{p}'\Pi(X,\theta;X',\theta',t),\tag{37}$$

where the operators $\tilde{\mathbf{I}}_p$ and $\tilde{\mathbf{I}}_q$ in the optical tomographic representation correspond to the operators (33):

$$\begin{aligned}\tilde{\mathbf{I}}_q &= \frac{1}{4}[\cos\theta(\varepsilon-\varepsilon^*)+\sin\theta(\dot\varepsilon-\dot\varepsilon^*)]\frac{\partial}{\partial X}+\frac{i}{2}\left[\frac{\partial}{\partial X}\right]^{-1}\left[(\dot\varepsilon-\dot\varepsilon^*)\left(-\sin\theta\frac{\partial}{\partial\theta}-\cos\theta\left(1+X\frac{\partial}{\partial X}\right)\right)\right.\\ &\quad\left.-(\varepsilon-\varepsilon^*)\left(\cos\theta\frac{\partial}{\partial\theta}-\sin\theta\left(1+X\frac{\partial}{\partial X}\right)\right)\right]+\frac{\beta+\beta^*}{\sqrt{2}},\\ \tilde{\mathbf{I}}_p &= -\frac{i}{4}[\cos\theta(\varepsilon+\varepsilon^*)+\sin\theta(\dot\varepsilon+\dot\varepsilon^*)]\frac{\partial}{\partial X}+\frac{1}{2}\left[\frac{\partial}{\partial X}\right]^{-1}\left[(\dot\varepsilon+\dot\varepsilon^*)\left(-\sin\theta\frac{\partial}{\partial\theta}-\cos\theta\left(1+X\frac{\partial}{\partial X}\right)\right)\right.\\ &\quad\left.-(\varepsilon+\varepsilon^*)\left(\cos\theta\frac{\partial}{\partial\theta}-\sin\theta\left(1+X\frac{\partial}{\partial X}\right)\right)\right]+\frac{\beta-\beta^*}{\sqrt{2}}.\end{aligned}$$

One can solve the system of equations (36), (37) by the following method.

First, from (36) the dependence on nonprimed variables is completely obtained, considering the primed variables as parameters, then from (37) the dependence on primed variables is found. The time-dependent factor is obtained by the substitution of the propagator to the dynamic equation (20) with the initial condition (19).

In our case, Eq. (20) has the form

$$\begin{aligned}&\dot\Pi-(\cos^2\theta+\omega^2(t)\sin^2\theta)\partial_\theta\Pi+(1-\omega^2(t))\sin\theta\cos\theta(1+X\partial_X)\Pi+f(t)\sin\theta\partial_X\Pi\\ &=\delta(t)\delta(X\cos(\theta-\theta')-X')\delta(\sin(\theta-\theta')).\end{aligned}\tag{38}$$

The described procedure enables us to find the optical propagator for our quantum system, but we can also find it from the results of [26] where the symplectic propagator of the parametric-driven quadratic system was found,

$$\Pi(X,\mu,\nu;X',\mu',\nu')=\delta(X-X'+\mathcal{N}\Lambda^{-1}\Delta)\delta(\mathcal{N}'-\mathcal{N}\Lambda^{-1})\Theta(t),\tag{39}$$

where $\mathcal{N}$ and $\mathcal{N}'$ are vectors $\mathcal{N} = (\nu, \mu)$ and $\mathcal{N}' = (\nu', \mu')$, and $\Theta(t)$ is a Heaviside step function

$$\Theta(t) = \begin{cases} 1, & \text{at} \quad t \geq 0, \\ 0, & \text{at} \quad t < 0. \end{cases}$$

With the help of (27), we can write

$$\Pi(X, \theta; X', \theta', t) = \delta\{\mathcal{N}_2'(X + \mathcal{N}\Lambda^{-1}\boldsymbol{\Delta}) - X'\mathcal{N}\Lambda_{II}^{-1}\}\ \delta\{\mathcal{N}_1' - \mathcal{N}_2\mathcal{N}\Lambda_I^{-1}/\mathcal{N}\Lambda_{II}^{-1}\}\Theta(t), \tag{40}$$

where the vectors $\mathcal{N} = (\sin\theta, \cos\theta)$ and $\mathcal{N}' = (\sin\theta', \cos\theta')$, now Λ_I^{-1} and Λ_{II}^{-1} are the first and second columns of the matrix Λ^{-1}, inverse to Λ, and the indices 1 and 2 are related to the first and second coordinates of the vectors. In view of the time-dependent functions $\varepsilon(t)$ and $\varepsilon^*(t)$, the propagator $\Pi(x, \theta; x', \theta', t)$ is expressed as follows:

$$\begin{aligned}
&\Pi(X, \theta; X', \theta', t) = \delta[X\cos\theta' - X'(\sin\theta(\dot{\varepsilon}^* + \dot{\varepsilon}) + \cos\theta(\varepsilon^* - \varepsilon))/2 \\
&+ \cos\theta'(\sin\theta(\dot{\varepsilon}^*\beta + \dot{\varepsilon}\dot{\beta}) + \cos\theta(\varepsilon^*\beta + \varepsilon\beta^*))/\sqrt{2}] \\
&\times\delta\left[i\cos\theta'\frac{\sin\theta(\dot{\varepsilon}^* - \dot{\varepsilon}) + \cos\theta(\varepsilon^* - \varepsilon)}{\sin\theta(\dot{\varepsilon}^* + \dot{\varepsilon}) + \cos\theta(\varepsilon^* + \varepsilon)} - \sin\theta'\right]\Theta(t).
\end{aligned}$$

Thus if the integrals of motion are known, e.g., the matrix $\Lambda(t)$ and the vector $\boldsymbol{\Delta}(t)$ are known, then according to formula (40), the optical propagator is known. In the particular case for a free motion (with the Hamiltonian $\hat{H} = \hat{p}^2/2$), we have [27, 28]

$$p_o(t) = p, \qquad q_o(t) = q - pt, \qquad \boldsymbol{\Delta}(t) = 0, \qquad \Lambda(t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}$$

and the propagator of free motion reads

$$\Pi_{\rm f}(X, \theta; X', \theta', t) = \delta(X\cos\theta' - X'\cos\theta)\delta(\cos\theta'(t + \tan\theta) - \sin\theta')\Theta(t). \tag{41}$$

For the harmonic oscillator with the Hamiltonian $\hat{H} = (\hat{p}^2/2) + (\hat{q}^2/2)$,

$$\Lambda(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix},$$

from (40) we obtain

$$\Pi_{\rm os}(X, \theta; X', \theta', t) = \delta\left(X\cos(\theta - \theta' + t) - X'\right)\delta(\sin(\theta - \theta' + t))\Theta(t). \tag{42}$$

The evolution equation, in this case, is very simple,

$$(\partial_t - \partial_\theta)\Pi(X, \theta; X', \theta', t) = \delta(t)\delta(X\cos(\theta - \theta') - X')\delta(\sin(\theta - \theta')), \tag{43}$$

or for the optical distribution

$$(\partial_t - \partial_\theta)w_{\rm os}(X, \theta, t) = 0.$$

It is easily to check that (41) and (42) satisfy the nonlinear expression (14), and substitution of (41) and (42) into (17) provides known results for quantum propagators

$$\begin{aligned}
K_{\rm f}(q, q', \tilde{q}, \tilde{q}', t) &= \frac{1}{2\pi t}\exp\left[\frac{i}{2t}(q - \tilde{q})^2 - \frac{i}{2t}(q' - \tilde{q}')^2\right]\Theta(t), \\
K_{\rm os}(q, q', \tilde{q}, \tilde{q}', t) &= \frac{1}{2\pi|\sin t|}\exp\left[\frac{i}{2}(q^2 - q'^2 + \tilde{q}^2 - \tilde{q}'^2)\cot t - \frac{i}{\sin t}(q\tilde{q} - q'\tilde{q}')\right]\Theta(t).
\end{aligned}$$

Substituting these expressions to (18) gives us (41) and (42), respectively. Thus, we have checked the expressions obtained in the present paper.

Note that during the procedure of dimension restoration, the Planck's constant $\hbar$ appears neither in the dynamic equation for the propagator and the marginal distribution of quadratic system nor in the general expression for the propagator; in this sense, the dynamic equation and the propagator are classical (if, of course, ω and f do not depend on $\hbar$). However, the initial condition for the dynamic equation of marginal distribution can contain the Planck's constant, i.e., to be quantum.

5. Conclusions

To summarize, we point out the main results of this study.

We obtained the relations between the propagators for the optical tomogram and for the density matrix of quantum systems. We presented the evolution equation for the optical propagator along with the initial conditions. We showed the correspondence between the optical and symplectic propagators. As an example, we derived the optical propagators of arbitrary quadratic systems and checked our general expressions by direct substitution. The importance of the expressions written here for optical tomographic representation is connected with the fact that namely these tomograms are measured in the experiments on photon states where their characteristics are studied. The propagators and the evolution equation found in our work provide the possibility of monitoring the system quantum states in the process of time evolution.

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