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TENSOR PRODUCT REPRESENTATION OF QUBIT AND ITS TRANSFORM UNDER LINEAR-OPERATOR ACTION

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Abstract

The general one-particle spin states are considered. It is shown that information contained in a spin-1/2 state can be recorded in an equivalent form with the help of three mixed completely decoherent qubit states. The density matrix of such a system has the form of the tensor product of three diagonal matrices. The linear operator defined in the space of one-particle spin states generates some transform of the tensor products of the diagonal matrices. We construct this transform in the explicit form.

Keywords: density matrix, spin state, quantum computer, quantum tomography, entangled states, information.

1. Introduction

In modern quantum information processing and quantum computation, the pure entangled states are used. The density matrices of these states are of a general form, and their off-diagonal elements are not equal to zero. These states are used for processing and keeping information [1, 2]. Both mixed and pure states are unstable. Under various external influences, they become more mixed and the off-diagonal elements of their density matrices decrease and can become equal to zero. This process of decoherentization results in losses of information and can disturb the quantum computing process.

For this reason, it is reasonable to look for such a model of quantum computing that uses only mixed states with diagonal density matrices. Such states are more stable with respect to decoherentization processes. The possibility to use such states for design of quantum computers was discussed in [3, 6]. To connect this model with known methods of quantum computing, it is necessary to perform two operations. First, one must determine the connection between information contained in one-particle spin states and present this information with the help of mixed states with diagonal density matrices. Second, since there exist linear operators acting in the space of one-particle spin states, it is necessary to show how these operators act in the space of those mixed states that contain the same information.

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It is worth noting that in quantum mechanics the states of quantum systems can be described not only by the density matrices or wave functions. There exists the probability representation of quantum states [4, 5] where the states are associated with fair probability distributions called state tomograms. The state tomograms (or tomographic-probability distributions) are related to the density matrices by invertible transforms; due to this, information contained in the density matrix can be coded by the tomogram, which depends on extra parameters. The number of parameters is sufficient for describing with the help of tomograms not only the modulus squared of the probability distribution (which is also the probability distribution) but the phase of the wave function as well.

Formally the tomographic-probability distributions are diagonal matrix elements of the density matrices calculated in unitary transformed reference frames in the Hilbert space of the quantum-system states. The diagonal matrix elements of the density matrix given in several reference frames determine completely the off-diagonal matrix elements in the initial nonrotated basis. Thus, the problem formulated is closely related to the tomographic-probability representation of quantum states and their properties.

In this study, we focus on the most important for the quantum information case of qudit states, the density matrix of which depends on three parameters. It is clear that, if one has three probability distributions (spin "up" and spin "down") corresponding to the spin-1/2 tomogram, it is possible to describe by means of these three distributions an arbitrary spin-state density matrix. So one can model three probability distributions associated with spin tomograms with three diagonal density matrices related either to three qubits or to three two-level atoms.

We discuss in our paper these problems.

We consider general one-particle spin states and show that there exists a tensor product of three mixed one-particle spin states with diagonal density matrices that contain the same information. The connection between the states is obtained in the explicit form.

We also construct a linear operator that acts at the tensor product of density matrices and is an exact analog of the operator that acts in the space of the original density matrix. We show a concrete example of this construction.

2. Tensor Representation of One-Particle Spin State

We consider a one-particle spin state described by the density matrix

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}. \tag{1}$$

Our aim is to construct another state (generally speaking, the multiparticle's state) that contains the same information as the state ρ. For this, we define a vector $\vec{L}$. The components of this vector are elements of matrix (1). The correspondence between matrices and vectors whose components are the matrix elements was investigated in [7, 8]

$$\vec{L} = \begin{pmatrix} \rho_{11} \\ \rho_{12} \\ \rho_{21} \\ \rho_{22} \end{pmatrix} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{pmatrix}. \tag{2}$$

From vector $\vec{L}$, one can go to vector $\vec{M}$. It has six components, which are linear combinations of the components of the vector $\vec{L}$

$$\vec{M} = A\vec{L} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \\ m_5 \\ m_6 \end{pmatrix}, \tag{3}$$

where $A = \|\alpha_{ik}\| \quad \{i = 1, \ldots, 6, k = 1, \ldots, 4\}$ is the 6×4 rectangular matrix

$$m_i = \sum_{k=1}^{4} \alpha_{ik} l_k. \tag{4}$$

There exists an inverse transform

$$\vec{L} = B\vec{M}, \qquad l_k = \sum_{i=1}^{6} \beta_{ki} m_i, \tag{5}$$

where $B = \|\beta_{ki}\|, \quad \{i = 1, \ldots, 6, k = 1, \ldots, 4\}$ is the 4×6 rectangular matrix.

The components m_i of vector (3) must satisfy the following conditions:

$$m_i \geq 0, \;\; i = 1, \ldots, 6, \qquad m_1 + m_2 = m_3 + m_4 = m_5 + m_6 = 1. \tag{6}$$

This means that the pairs (m_1, m_2), (m_3, m_4), and (m_5, m_6) can be interpreted as the standard probability distributions for the three classical systems with two outputs each. There are many rectangular matrices A and B satisfying the conditions (4)–(6). Some examples will be presented below.

Now we consider the following problem.

Given an unitary transform U of a matrix ρ

$$U\rho U^{-1} = \tilde{\rho}. \tag{7}$$

The example of the unitary transform is the simplest example of the positive map of the density operator that yields another density operator. A general approach to positive maps is given in [9] (see, also [10]). One needs to find the transform T of a vector $\vec{M}$, which corresponds to the transform of the matrix ρ. This transform is defined by the matrix T

$$\tilde{\vec{M}} = T\vec{M}. \tag{8}$$

One has to find the explicit form of this matrix. To do this, we employ the method developed in [7, 8].

3. Construction of the Transforms

Transform (7) of the density matrix (1) generates a corresponding transform of vector (2)

$$\tilde{\vec{L}} = W\vec{L}, \tag{9}$$

where W is a superoperator corresponding to the matrix U. It has the form of the tensor product of the matrix U and its complex conjugate, i.e.,

$$W = \begin{pmatrix} u_{11}u_{11}^* & u_{11}u_{12}^* & u_{12}u_{11}^* & u_{12}u_{12}^* \\ u_{11}u_{21}^* & u_{11}u_{22}^* & u_{12}u_{21}^* & u_{12}u_{22}^* \\ u_{21}u_{11}^* & u_{21}u_{12}^* & u_{22}u_{11}^* & u_{22}u_{12}^* \\ u_{21}u_{21}^* & u_{21}u_{22}^* & u_{22}u_{21}^* & u_{22}u_{22}^* \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \otimes \begin{pmatrix} u_{11}^* & u_{12}^* \\ u_{21}^* & u_{22}^* \end{pmatrix}. \tag{10}$$

In view of (10), one can find transform (8) of vector (3) as follows:

$$\tilde{\vec{M}} = AWB\vec{M} = T\vec{M}. \tag{11}$$

To obtain the explicit form of matrices A and B, one can use the tomographic approach for describing the quantum states. Within the frame of this approach, a quantum state is described by a number of measured tomographic probabilities instead of the wave functions and density matrices, which usually correspond to diagonal elements of the density matrix. In spite of the fact that it is impossible to reconstruct the whole density matrix knowing only its diagonal elements, one can do this with the help of a set of diagonal elements measured in different reference frames. The tomography approach for describing the spin states was developed in [11–17], where it was shown that, to find a minimum sufficient set of data, it is necessary to measure diagonal elements of the density matrix in three different reference frames. As these reference frames, one can use the initial reference frame defined by the axes X, Y, and Z, and two other ones obtained from the initial one by two different rotations. In such a way, one can get six diagonal elements describing a one-particle spin state. In view of of these diagonal elements, one can construct components of the vector $\vec{M}$ (3).

The rotation of the coordinate system is described by the three Euler's angles (ϕ, θ, ψ). The diagonal elements ρ_{ii} of the density matrix are transformed as follows:

$$\begin{aligned} \rho'_{ii} &= D_{i1}(\phi,\theta,\psi)\rho_{11}D^*_{i1}(\phi,\theta,\psi) + D_{i1}(\phi,\theta,\psi)\rho_{12}D^*_{i2}(\phi,\theta,\psi) \\ &\quad + D_{i2}(\phi,\theta,\psi)\rho_{21}D^*_{i1}(\phi,\theta,\psi) + D_{i2}(\phi,\theta,\psi)\rho_{22}D^*_{i2}(\phi,\theta,\psi), \end{aligned} \tag{12}$$

where $D_{ij}(\phi,\theta,\psi)$ are the Wigner D functions of the form

$$D = \begin{pmatrix} \cos\theta/2\, e^{i(\phi+\psi)/2} & \sin\theta/2\, e^{i(-\phi+\psi)/2} \\ -\sin\theta/2\, e^{i(\phi-\psi)/2} & \cos\theta/2\, e^{i(-\phi-\psi)/2} \end{pmatrix}. \tag{13}$$

After performing transform (12), the diagonal matrix elements of a one-particle spin state take the form

$$\begin{aligned} \rho'_{11}(\phi,\theta) &= \cos^2\frac{\theta}{2}\,\rho_{11} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}\,\rho_{12} + \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}\,\rho_{21} + \sin^2\frac{\theta}{2}\,\rho_{22}, \\ \rho'_{22}(\phi,\theta) &= \sin^2\frac{\theta}{2}\,\rho_{11} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{i\phi}\,\rho_{12} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}e^{-i\phi}\,\rho_{21} + \cos^2\frac{\theta}{2}\,\rho_{22}. \end{aligned} \tag{14}$$

Now one can make two rotations — the first one is produced by the Euler's angles $(0, \theta, 0)$, and the second one is produced by the angles $(0, \theta, \pi/2)$. In this case, the vector $\vec{M}$ reads

$$\vec{M}(\phi,\theta) = \begin{pmatrix} \rho_{11} \\ \rho_{22} \\ \cos^2\theta/2\,\rho_{11} + \cos\theta/2\sin\theta/2(\rho_{12}+\rho_{21}) + \sin^2\theta/2\,\rho_{22} \\ \sin^2\theta/2\,\rho_{11} - \cos\theta/2\sin\theta/2(\rho_{12}+\rho_{21}) + \cos^2\theta/2\,\rho_{22} \\ \cos^2\theta/2\,\rho_{11} + i\cos\theta/2\sin\theta/2(\rho_{12}-\rho_{21}) + \sin^2\theta/2\,\rho_{22} \\ \sin^2\theta/2\,\rho_{11} - i\cos\theta/2\sin\theta/2(\rho_{12}-\rho_{21}) + \cos^2\theta/2\,\rho_{22} \end{pmatrix}. \tag{15}$$

The components of vector (15) satisfy the conditions (6).

The transform of vector $\vec{L}$ into the vector $\vec{M}(\phi,\theta)$ is produced by the rectangular matrix A. It contains four columns and six rows and reads as follows:

$$\vec{M}(\phi,\theta) = A\vec{L}, \tag{16}$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \cos^2\theta/2 & \sin^2\theta/2 & \cos\theta/2\sin\theta/2 & \cos\theta/2\sin\theta/2 \\ \sin^2\theta/2 & \sin^2\theta/2 & -\cos\theta/2\sin\theta/2 & -\cos\theta/2\sin\theta/2 \\ \cos^2\theta/2 & \sin^2\theta/2 & i\cos\theta/2\sin\theta/2 & -i\cos\theta/2\sin\theta/2 \\ \cos^2\theta/2 & \sin^2\theta/2 & -i\cos\theta/2\sin\theta/2 & i\cos\theta/2\sin\theta/2 \end{pmatrix}. \tag{17}$$

The inverse transform of the vector $\vec{M}(\phi,\theta)$ into the vector $\vec{L}$ is produced by the rectangular matrix B, which contains six columns and four rows and reads

$$\vec{L} = B\vec{M}(\phi,\theta), \tag{18}$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\dfrac{(1-i)\cos\theta}{2\sin\theta} & \dfrac{(1-i)\cos\theta}{2\sin\theta} & \dfrac{1}{2\sin\theta} & -\dfrac{1}{2\sin\theta} & -\dfrac{i}{2\sin\theta} & \dfrac{i}{2\sin\theta} \\ -\dfrac{(1+i)\cos\theta}{2\sin\theta} & \dfrac{(1+i)\cos\theta}{2\sin\theta} & \dfrac{1}{2\sin\theta} & -\dfrac{1}{2\sin\theta} & \dfrac{i}{2\sin\theta} & -\dfrac{i}{2\sin\theta} \end{pmatrix}. \tag{19}$$

It worth mentioning that, in addition to representation (15) for vector $\vec{M}$, there exist other possibilities for constructing its components, and different matrices A and B correspond to a pair of the vectors $\vec{L}$ and $\vec{M}$.

The main object in quantum information is qubit, i.e., the one-particle spin state. It can be realized as a two-level system. One can present a lot of examples of such systems — atoms with two low-oscillation

levels, electron and nuclear spins, electromagnetic fields in the resonator, photon crystals, and so on. All such objects can be used for creating quantum computers, but it is very difficult to employ for such purpose systems with a number of levels greater than 2. For this reason, one must not interpret the components of vector $\vec{M}$ (3) as diagonal elements of the six-level system density matrix, but consider these components as diagonal elements of three density matrices of three two-level systems, for example, three atoms. A physical object that contains three two-level systems is described by the density matrix $\bar{\rho}$, which has the form of the tensor product of three density matrices corresponding to each two-level system. In our case, all the matrices are diagonal and their elements are the components of vector $\vec{M}$ (3)

$$\bar{\rho} = \rho^1 \otimes \rho^2 \otimes \rho^3 = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \otimes \begin{pmatrix} m_3 & 0 \\ 0 & m_4 \end{pmatrix} \otimes \begin{pmatrix} m_5 & 0 \\ 0 & m_6 \end{pmatrix}. \tag{20}$$

The matrix $\bar{\rho}$ is the diagonal matrix with matrix elements on its diagonal written as

$$\begin{aligned} &r_1 = m_1 m_3 m_5, \qquad r_2 = m_1 m_3 m_6, \qquad r_3 = m_1 m_4 m_5, \\ &r_4 = m_1 m_4 m_6, \qquad r_5 = m_2 m_3 m_5, \\ &r_6 = m_2 m_3 m_6, \qquad r_7 = m_2 m_4 m_5, \qquad r_8 = m_2 m_4 m_6, \end{aligned} \tag{21}$$

while all off-diagonal elements are equal to zero.

The robust method for preparing an arbitrary two-dimensional diagonal state was developed in [18].

Let us construct a vector $\vec{r}$ with components r_i, $i = 1, \ldots, 8$ being the matrix elements of tensor product (20), where r_i satisfy the following relations:

$$\begin{aligned} &\sum_{i=1}^{8} r_i = 1, \\ &\frac{r_1}{r_2} = \frac{r_3}{r_4} = \frac{r_5}{r_6} = \frac{r_7}{r_8} = \frac{r_1 + r_3 + r_5 + r_7}{r_2 + r_4 + r_6 + r_8}, \\ &\frac{r_1}{r_3} = \frac{r_2}{r_4} = \frac{r_5}{r_7} = \frac{r_6}{r_8} = \frac{r_1 + r_2 + r_5 + r_6}{r_3 + r_4 + r_7 + r_8}, \\ &\frac{r_1}{r_5} = \frac{r_2}{r_6} = \frac{r_3}{r_7} = \frac{r_4}{r_8} = \frac{r_1 + r_2 + r_3 + r_4}{r_5 + r_6 + r_7 + r_8}. \end{aligned} \tag{22}$$

Relations (22) are necessary and sufficient conditions for the diagonal matrix, with matrix elements r_i, to be presented as tensor product (20).

The components m_i of vector $\vec{M}$ (3) are the linear combinations of r_k, namely,

$$\begin{aligned} &m_1 = r_1 + r_2 + r_3 + r_4, \qquad m_2 = r_5 + r_6 + r_7 + r_8, \\ &m_3 = r_1 + r_2 + r_5 + r_6, \qquad m_4 = r_3 + r_4 + r_5 + r_7, \\ &m_5 = r_1 + r_3 + r_5 + r_7, \qquad m_6 = r_2 + r_4 + r_6 + r_8. \end{aligned} \tag{23}$$

Our goal is to find a linear transform Γ of vector $\vec{r}$ (21) into a vector $\tilde{\vec{r}}$ that corresponds to the unitary transform (7) of the density matrix (1)

$$\tilde{\vec{r}} = \Gamma \vec{r}. \tag{24}$$

The unitary transform (7) of the density matrix (1) generates a transform (11) of the vector $\vec{M}$. The components m_i of the vector $\vec{M}$ are transformed as follows:

$$\tilde{m}_i = \sum_{j=1}^{6} t_{ij} m_j, \tag{25}$$

where t_{ij} are elements of the matrix T.

With the help of (25), one can find the transform of r_k corresponding to the unitary transform (7). Then in view of (21), one obtains

$$\tilde{m}_i \tilde{m}_k \tilde{m}_p = \left(\sum_{j}^{6} t_{ij} m_j \right) \left(\sum_{l}^{6} t_{kl} m_l \right) \left(\sum_{q}^{6} t_{pq} m_q \right). \tag{26}$$

There are 56 different products of three quantities m_i on the right-hand side of equality (26); this means that the transform of quantities r_k is nonlinear. Our goal is to find its explicit form. For this, we consider a vector $\vec{R}$

$$\vec{R} = \begin{pmatrix} R_1 = r_1 \\ R_2 = r_2 \\ \cdots \\ R_8 = r_8 \\ R_9 \\ \cdots \\ R_{56} \end{pmatrix} = \begin{pmatrix} m_1 m_3 m_5 \\ m_1 m_3 m_6 \\ m_1 m_4 m_5 \\ \cdots \\ m_2 m_4 m_6 \\ m_1 m_1 m_1 \\ \cdots \\ m_6 m_6 m_6 \end{pmatrix}. \tag{27}$$

The components R_k ($k = 1, \ldots, 56$) of the vector $\vec{R}$ are all possible products of the quantities m_i. Eight first components coincide with the components of the vector $\vec{r}$. The transform (26) is a linear transform of the components of the vector $\vec{R}$; it can be written as follows:

$$\tilde{\vec{R}} = \Omega \vec{R}, \qquad \tilde{R}_k = \sum_{l=1}^{56} \omega_{kl} R_l, \tag{28}$$

where ω_{kl} are elements of the matrix Ω. They can be presented through the elements γ_{ij} of the matrix Γ. The explicit form of the elements can be obtained by determining the coefficients of the quantities R_l on the right-hand side of equality (26). It should be mentioned that the first eight components of the transformed vector $\tilde{\vec{R}}$ satisfy relations (22).

The linear transform (28) is the first step in creating the transformed vector $\tilde{\vec{r}}$. The second step is the operation of projection P of the vector $\tilde{\vec{R}}$, which has 56 components on its first eight components

$$\tilde{\vec{r}} = P \tilde{\vec{R}}. \tag{29}$$

The eight components $\tilde{r}_i = \tilde{R}_i, i = 1, \ldots, 8$ satisfy relations (22). Therefore, they can be presented as matrix elements of the tensor product

$$\tilde{\tilde{\rho}} = \tilde{\rho^1} \otimes \tilde{\rho^2} \otimes \tilde{\rho^3} = \begin{pmatrix} \tilde{m_1} & 0 \\ 0 & \tilde{m_2} \end{pmatrix} \otimes \begin{pmatrix} \tilde{m_3} & 0 \\ 0 & \tilde{m_4} \end{pmatrix} \otimes \begin{pmatrix} \tilde{m_5} & 0 \\ 0 & \tilde{m_6} \end{pmatrix}. \tag{30}$$

We obtained the transform of the vector $\vec{r}$, but it is nonlinear, since the operation of projection P is a nonlinear one.

4. Block Structure of Matrix T

The physical meaning of the discussed construction of the density matrix of three two-level atoms is related to the following protocol of information processing.

One can consider a state of three two-level atoms, which is mixed, and the information under consideration is coded by three positive numbers $0 \leq a, b, c \leq 1$. The three numbers can be treated as populations of the upper levels of the three atoms. The density matrix of the three-level atoms is prepared in a totally decoherent state, i.e., the off-diagonal elements of the density matrix are equal to zero. If the evolution of the two-level atom states is determined by a Hamiltonian matrix, which has off-diagonal terms, the structure of the density matrix of the atoms changes, since in the evolution process the off-diagonal terms of the density matrix appear. Thus, in order to preserve the diagonal structure of the density matrix, the Hamiltonian matrix itself has to be diagonal, but, in this case, the initial density matrix is invariant. Information coded by the numbers a, b, and c is not processed, it is only preserved. In order to preserve the diagonal structure of the density matrix, but to change somehow the density-matrix diagonal elements, the evolution of the three-level atoms has to be determined by a master equation appropriate for describing the open quantum systems. In this case, the three positive numbers a, b, and c and their probability partners $1 - a$, $1 - b$, and $1 - c$ can be transformed by stochastic matrices associated to the semigroup of positive maps of the density operators.

We discovered that it is impossible to construct a linear transform of vector $\vec{r}$ in the general case, but one can do it in some special cases.

Let us consider the case where the matrix T from relation (11) has a block structure. The blocks are stochastic 2×2 matrices and are used to transform the probability vectors into the other probability vectors. The properties of stochastic matrices have been investigated, for example, in [19]. The block structure is organized in such a way that quantities, which form the pairs (m_1, m_2), (m_3, m_4), and (m_5, m_6) are transformed only between themselves, namely,

$$T = \begin{pmatrix} t_{11} & t_{12} & 0 & 0 & 0 & 0 \\ t_{21} & t_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & t_{33} & t_{34} & 0 & 0 \\ 0 & 0 & t_{43} & t_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & t_{55} & t_{56} \\ 0 & 0 & 0 & 0 & t_{65} & t_{66} \end{pmatrix}. \tag{31}$$

In this case, the components m_i of vector $\vec{M}$ are transformed as follows:

$$\begin{aligned} m'_1 &= t_{11}m_1 + t_{12}m_2, \qquad m'_3 = t_{33}m_3 + t_{34}m_4, \qquad m'_5 = t_{55}m_5 + t_{56}m_6, \\ m'_2 &= t_{21}m_1 + t_{22}m_2, \qquad m'_4 = t_{43}m_3 + t_{44}m_4, \qquad m'_6 = t_{65}m_5 + t_{66}m_6. \end{aligned} \tag{32}$$

In view of (32), one can obtain the transform of the components r_j of vector $\vec{r}$ as follows:

$$\begin{aligned}
r'_1 &= m'_1m'_3m'_5 = (t_{11}m_1 + t_{12}m_2)(t_{33}m_3 + t_{34}m_4)(t_{55}m_5 + t_{56}m_6) \\
&= (t_{11}t_{33}t_{55})r_1 + (t_{11}t_{33}t_{56})r_2 + (t_{11}t_{34}t_{55})r_3 + (t_{11}t_{34}t_{56})r_4 \\
&\quad +(t_{12}t_{33}t_{55})r_5 + (t_{12}t_{33}t_{56})r_6 + (t_{12}t_{34}t_{55})r_7 + (t_{12}t_{34}t_{56})r_8. \\
r'_2 &= m'_1m'_3m'_6 = (t_{11}m_1 + t_{12}m_2)(t_{33}m_3 + t_{34}m_4)(t_{65}m_5 + t_{66}m_6) \\
&= (t_{11}t_{33}t_{65})r_1 + (t_{11}t_{33}t_{66})r_2 + (t_{11}t_{34}t_{65})r_3 + (t_{11}t_{34}t_{66})r_4 \\
&\quad +(t_{12}t_{33}t_{65})r_5 + (t_{12}t_{33}t_{66})r_6 + (t_{12}t_{34}t_{65})r_7 + (t_{12}t_{34}t_{66})r_8. \\
r'_3 &= m'_1m'_4m'_5 = (t_{11}m_1 + t_{12}m_2)(t_{43}m_3 + t_{44}m_4)(t_{55}m_5 + t_{56}m_6) \\
&= (t_{11}t_{43}t_{55})r_1 + (t_{11}t_{43}t_{56})r_2 + (t_{11}t_{44}t_{55})r_3 + (t_{11}t_{44}t_{56})r_4 \\
&\quad +(t_{12}t_{43}t_{55})r_5 + (t_{12}t_{43}t_{56})r_6 + (t_{12}t_{44}t_{55})r_7 + (t_{12}t_{44}t_{56})r_8. \\
r'_4 &= m'_1m'_4m'_6 = (t_{11}m_1 + t_{12}m_2)(t_{43}m_3 + t_{44}m_4)(t_{65}m_5 + t_{66}m_6) \\
&= (t_{11}t_{43}t_{65})r_1 + (t_{11}t_{43}t_{66})r_2 + (t_{11}t_{44}t_{65})r_3 + (t_{11}t_{44}t_{66})r_4 \\
&\quad +(t_{12}t_{43}t_{65})r_5 + (t_{12}t_{43}t_{66})r_6 + (t_{12}t_{44}t_{65})r_7 + (t_{12}t_{44}t_{66})r_8. \\
r'_5 &= m'_2m'_3m'_5 = (t_{21}m_1 + t_{22}m_2)(t_{33}m_3 + t_{34}m_4)(t_{55}m_5 + t_{56}m_6) \\
&= (t_{21}t_{33}t_{55})r_1 + (t_{21}t_{33}t_{56})r_2 + (t_{21}t_{34}t_{55})r_3 + (t_{21}t_{34}t_{56})r_4 \\
&\quad +(t_{22}t_{33}t_{55})r_5 + (t_{22}t_{33}t_{56})r_6 + (t_{22}t_{34}t_{55})r_7 + (t_{22}t_{34}t_{56})r_8. \\
r'_6 &= m'_2m'_3m'_6 = (t_{21}m_1 + t_{22}m_2)(t_{33}m_3 + t_{34}m_4)(t_{65}m_5 + t_{66}m_6) \\
&= (t_{21}t_{33}t_{65})r_1 + (t_{21}t_{33}t_{66})r_2 + (t_{21}t_{34}t_{65})r_3 + (t_{21}t_{34}t_{66})r_4 \\
&\quad +(t_{22}t_{33}t_{65})r_5 + (t_{22}t_{33}t_{66})r_6 + (t_{22}t_{34}t_{65})r_7 + (t_{22}t_{34}t_{66})r_8. \\
r'_7 &= m'_2m'_4m'_5 = (t_{21}m_1 + t_{22}m_2)(t_{43}m_3 + t_{44}m_4)(t_{55}m_5 + t_{56}m_6) \\
&= (t_{21}t_{43}t_{55})r_1 + (t_{21}t_{43}t_{56})r_2 + (t_{21}t_{44}t_{55})r_3 + (t_{21}t_{44}t_{56})r_4 \\
&\quad +(t_{22}t_{43}t_{55})r_5 + (t_{22}t_{43}t_{56})r_6 + (t_{22}t_{44}t_{55})r_7 + (t_{22}t_{44}t_{56})r_8. \\
r'_8 &= m'_2m'_4m'_6 = (t_{21}m_1 + t_{22}m_2)(t_{43}m_3 + t_{44}m_4)(t_{65}m_5 + t_{66}m_6) \\
&= (t_{21}t_{43}t_{65})r_1 + (t_{21}t_{43}t_{66})r_2 + (t_{21}t_{44}t_{65})r_3 + (t_{21}t_{44}t_{66})r_4 \\
&\quad +(t_{22}t_{43}t_{65})r_5 + (t_{22}t_{43}t_{66})r_6 + (t_{22}t_{44}t_{65})r_7 + (t_{22}t_{44}t_{66})r_8.
\end{aligned} \tag{33}$$

It is the form of the linear transform of vector $\vec{r}$ corresponding to linear transform (31) of vector $\vec{M}$. With the help of (33), it can be presented in the matrix form.

Transform (31) preserves the property (6) of the state (20). It is defined by three independent parameters. It is exactly the number of parameters that is necessary to construct an arbitrary transform of the one-particle state (1). Therefore, any linear transform of qubit (1) can be presented as a linear transform of tensor product (20). Below we will study this construction in detail.

In the next section, we consider the example of transform (31).

5. Example

We consider a one-particle spin state ρ_φ described by the density matrix

$$\rho = \begin{pmatrix} a & ce^{i\varphi} \\ ce^{-i\varphi} & b \end{pmatrix}, \qquad a + b = 1. \tag{34}$$

The density matrix (34) has both nonzero diagonal and off-diagonal matrix elements. In the case of the density-matrix elements satisfying the condition $\rho^2 = \rho$, the density matrix corresponds to the pure qubit state.

Vectors $\vec{L}$ and $\vec{M}$ and matrices A and B for the state (34) have the forms

$$\vec{L} = \begin{pmatrix} a \\ ce^{i\varphi} \\ ce^{-i\varphi} \\ b \end{pmatrix}, \qquad \vec{M} = \frac{1}{2} \begin{pmatrix} 2a \\ 2b \\ 1 + c\cos\varphi \\ 1 - c\cos\varphi \\ 1 + c\sin\varphi \\ 1 - c\sin\varphi \end{pmatrix} \tag{35}$$

and

$$A = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 1 & 1/2 & 1/2 & 1 \\ 1 & -1/2 & -1/2 & 1 \\ 1 & -i/2 & i/2 & 1 \\ 1 & i/2 & -i/2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & i & -i \\ 0 & 0 & 1 & -1 & -i & i \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{36}$$

Thus, we have mapped the qubit state onto the six-dimensional vector, employing rectangular matrices (36).

We consider the transforms U_ϕ of the state (34) of the form

$$\begin{aligned} U_\phi \begin{pmatrix} a & ce^{i\varphi} \\ ce^{-i\varphi} & b \end{pmatrix} &= \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} a & ce^{i\varphi} \\ ce^{-i\varphi} & b \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} \\ &= \begin{pmatrix} a & ce^{i(\varphi+\phi)} \\ ce^{-i(\varphi+\phi)} & b \end{pmatrix}. \end{aligned} \tag{37}$$

Here we applied the unitary transform that belongs to the unitary group $U(1)$.

First, let us consider the state with $\varphi = 0$.

In this case, we have the transform of the six-vector describing the qubit state

$$\tilde{\vec{M}} = \frac{1}{2}\begin{pmatrix} 2a \\ 2b \\ 1+c\cos\phi \\ 1-c\cos\phi \\ 1+c\sin\phi \\ 1-c\sin\phi \end{pmatrix} = T_{0,\phi} \times \frac{1}{2}\begin{pmatrix} 2a \\ 2b \\ 1+c \\ 1-c \\ 1+c \\ 1-c \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & \frac{1}{2}(1+\cos\phi) & \frac{1}{2}(1-\cos\phi) & 0 & \\ 0 & 0 & \frac{1}{2}(1-\cos\phi) & \frac{1}{2}(1+\cos\phi) & 0 & \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1+\sin\phi) & \frac{1}{2}(1-\sin\phi) \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-\sin\phi) & \frac{1}{2}(1+\sin\phi) \end{pmatrix} \times \frac{1}{2}\begin{pmatrix} 2a \\ 2b \\ 1+c \\ 1-c \\ 1+c \\ 1-c \end{pmatrix}. \tag{38}$$

In view of the matrix elements of matrix $T_{0,\phi}$, one can construct the matrix (24), which transforms the vector $\vec{r}$ into the vector $\tilde{\vec{r}}$. With the help of formulas (33), the elements γ_{mn} of the matrix Γ are expressed in terms of the elements of the matrix T_ϕ.

Now we consider the case where the phase φ of the state (34) is not equal to zero.

In this case, it is necessary first to produce the transform $U_{-\varphi}$, which converts the state ρ_φ into the state ρ_0. The matrix $T_{\varphi,0}$ that acts on the vector $\vec{M}_\varphi$ has the form

$$T_{\varphi,0} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\cos\varphi+1}{2\cos\varphi} & \frac{\cos\varphi-1}{2\cos\varphi} & 0 & \\ 0 & 0 & \frac{\cos\varphi-1}{2\cos\varphi} & \frac{\cos\varphi+1}{2\cos\varphi} & 0 & \\ 0 & 0 & 0 & 0 & \frac{\sin\varphi+1}{2\cos\varphi} & \frac{\sin\varphi-1}{2\cos\varphi} \\ 0 & 0 & 0 & 0 & \frac{\sin\varphi-1}{2\cos\varphi} & \frac{\sin\varphi+1}{2\cos\varphi} \end{pmatrix}. \tag{39}$$

The transform

$$T_{\varphi,\phi} = T_{0,\phi}T_{\varphi,0}$$

converts the vector $\vec{M}_\varphi$ into the vector $\vec{M}_\phi$

$$\vec{M}_\phi = T_{\varphi,\phi}\vec{M}_\varphi. \tag{40}$$

Now one can evaluate the product of matrices $T_{0,\phi}$ and $T_{\varphi,0}$ and, using formulas (33), find the matrix $\Gamma_{\varphi,\phi}$ which transforms the vector $\vec{r}_\varphi$ into the vector $\vec{r}_\phi$,

$$\vec{r}_\phi = \Gamma_{\varphi,\phi}\vec{r}_\varphi. \tag{41}$$

This construction provides a complete description of linear transforms that act in the space of tensor products (20) and correspond to unitary transforms (37) of qubit (34).

6. Conclusions

We have shown that in quantum computation the system of three completely decoherent two-level systems can be used instead of an arbitrary one-particle spin state. Each decoherent state has the diagonal density matrix, and the whole system is described by the tensor product of the diagonal density matrices. Any linear operator, defined in the space of one-particle spin states, can be presented as a linear operator acting in the space of tensor products of the diagonal matrices.

Summarizing, we can conclude that the basic information carrier in a quantum computer is qubit, which can be realized as a two-level system. The density matrix of this system (34) depends on three parameters a, c, and φ that represent information contained in the system.

In the first step, we go from the state (34) to the three-particle decoherent state (20). This state is described by three parameters m_1, m_3, and m_5 as well, which contain now information of the original qubit. Information is processed by the linear operator (31), which has a block structure. It transforms the tensor product into another tensor product and, after performing the operation, one can go back to the qubit presentation of information with transformed parameters a', c', and φ'. The protocol of information processing presented in our paper can be considered as an analog of the classical information processing, where one uses three classical probability distributions transformed by means of stochastic matrices.

This construction can be generalized for the case of multiparticle spin states as well.

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