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SPIN TOMOGRAPHY AND STAR-PRODUCT KERNEL FOR QUBITS AND QUTRITS

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Abstract

Using the irreducible tensor-operator technique, we establish the relation between different forms of spin tomograms. Quantizer and dequantizer operators are presented in simple explicit forms and are specified for the low-spin states. The kernel of the star-product is evaluated for qubits and qutrits, and its connection with a generic formula is found.

Keywords: spin tomography, quantizer, dequantizer, star-product, kernel, qubit, qutrit.

1. Introduction

According to the conventional treatment of quantum mechanics, states of a system are associated either with the wave functions (vectors in a Hilbert space) or with the density operators. Apart from this, a new formulation of quantum states has been elaborated [1] (see also [2]) in the last few decades. This representation associates the states with the standard probability distributions. In fact, these probability distributions can be measured directly in the experiment. All the physical ingredients of quantum mechanics such as means of observables, their dispersions, etc. can be expressed in terms of the probability distributions of the corresponding quantum states. As far as continuous variables are concerned, the experiments to reconstruct the Wigner function of photon states were performed, for example in [3–7]. We point out that in the approach [1,2] the primary object in quantum mechanics associated to the quantum states is namely the probability distribution. Once the distribution is measured, it is not necessary at all to take any intermediate steps (like reconstruction of the Wigner function) in order to extract experimental information on the physical properties of a system. This implies that the quantum properties such as means of observables, variances, and other statistical characteristics can be directly obtained in view of the probability distributions. This aspect of the probability-representation approach takes place also for the states with discrete variables like spins, qubits, qudits, etc.

We concentrate here on the problem of probability representation for spin states. The quasidistribution functions for discrete spin-variable states were discussed, for example, in [8,9]. The quasidistributions such as analogs of the Wigner function [10] or the Husimi function [11] for the Lie groups, including the

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$SU(2)$ group, determine the corresponding states. In the same spirit, there exists the possibility to use fair probability distributions for spin degrees of freedom and, in fact, for other Lie groups (see, e.g., [12]).

Any spin state can be equivalently described by the density operator $\hat{\rho}$ or by the fair probability-distribution function called spin tomogram [13–18] (for states with continuous variables, see, e.g., [1, 2, 19–23]). The probability-distribution function is usually considered as an intermediate procedure for the density operator reconstruction. Apart from being a useful experimental tool, quantum tomograms themselves are a primary notion of quantum states. Using tomograms, one deals with functions instead of density operators. Various properties of these functions are discussed in [24–27]. Similarly to the density operator, any other operator can be identified with a function called tomographic symbol of the operator. Unlike the tomogram, this function is not nonnegative in the general case. To describe the standard product of operators on a Hilbert space, one can introduce the star-product of the tomographic symbols [28, 29]. The star-product is associative but noncommutative in general.

Operators and, in particular, observables can also be associated with functions called dual tomographic symbols [30, 31] (the first step toward dual symbols is taken in [22]; dual symbols are applied to study the quantumness of qubits in [32]). Ordinary and dual tomographic symbols linked together enable one to calculate the expectation values of observables. Hence it is possible to treat states, operators, and related quantities within the framework of the unified tomographic representation.

The aim of this paper is to reconsider quantizer and dequantizer operators for spin tomograms of qudit states. These operators relate tomograms with density operators, and observables with ordinary and dual tomographic symbols as well. Moreover, quantizer and dequantizer operators are constituent parts of the kernel of the star-product, which is widely used in dealing with maps of spin operators onto functions. The general procedure to use quantizer and dequantizer operators was discussed in the context of star-product quantization schemes in [29,30]. Although the explicit formulas for quantizer and dequantizer operators were obtained earlier, here we introduce another relatively simple form of these operators, show simple relations between them, check the equivalency of approaches applied in different works, and consider the cases of qubits and qutrits in detail. We also concentrate on the star-product kernel for ordinary and dual tomographic symbols.

This paper is organized as follows.

In Sec. 2, we use the irreducible tensor-operator technique to get the simple form of quantizer and dequantizer operators. In Sec. 3, the exponential representation of quantizer and dequantizer operators is reconsidered in order to illustrate its equivalency to other approaches. In Sec. 4, we derive the kernel of the unity operator on the set of qubit tomograms and that on the set of qutrit tomograms. In Sec. 5, the explicit forms of the star-product kernel for qubits and qutrits are obtained. In Sec. 6, dual tomographic symbols are briefly discussed. In Sec. 7, conclusions are presented.

2. Irreducible Tensor-Operator Representation of Quantizer and Dequantizer Operators

Unless specifically stated, qudit states with spin j are considered. We start with state vectors $|jm\rangle$ and the standard basis of the angular momentum operators $\hat{J}_x$, $\hat{J}_y$, and $\hat{J}_z$ defined through

$$\hat{\mathbf{J}}^2|jm\rangle = j(j+1)|jm\rangle, \qquad \hat{J}_z|jm\rangle = m|jm\rangle, \tag{1}$$

where m is the spin projection ($m = -j, -j+1, \ldots, j$).

The spin tomogram of a qudit state given by its density operator $\hat{\rho}$ reads

$$w(\mathbf{x}) \equiv w(m,u) = \langle jm|u\hat{\rho}u^\dagger|jm\rangle = \mathrm{Tr}\Big(\hat{\rho}u^\dagger|jm\rangle\langle jm|u\Big) = \mathrm{Tr}\Big(\hat{\rho}\hat{U}(\mathbf{x})\Big), \tag{2}$$

where u is a $(2j+1)\times(2j+1)$ unitary matrix of irreducible representation of the rotation group $SU(2)$ and $\mathbf{x}$ denotes the set of parameters $(m,u)\equiv(m,\alpha,\beta,\gamma)$, with the Euler angles α, β, and γ defining the matrix u.

The tomogram satisfies the following normalization conditions:

$$\sum_{m=-j}^{j} w(m,u) = 1, \qquad \frac{2j+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi}\sin\beta\, d\beta \int_0^{2\pi} d\gamma\; w(m,\alpha,\beta,\gamma) = 1. \tag{3}$$

We introduced the dequantizer operator in (2) as

$$\hat{U}(\mathbf{x}) = u^\dagger|jm\rangle\langle jm|u, \tag{4}$$

which is nothing else but the spin-j projector operator onto the m component along the z axis rotated by an element u of the $SU(2)$.

Given the tomogram $w(\mathbf{x})$, one can reconstruct the density operator $\hat{\rho}$ using the quantizer operator $\hat{D}(\mathbf{x})$ as follows:

$$\hat{\rho} = \int w(\mathbf{x})\hat{D}(\mathbf{x})\, d\mathbf{x}, \tag{5}$$

where

$$\int d\mathbf{x} = \sum_{m=-j}^{j} \frac{1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{\pi}\sin\beta\, d\beta \int_0^{2\pi} d\gamma. \tag{6}$$

Following [33] we write the explicit formulas for both dequantizer and quantizer operators in terms of the Clebsch–Gordan coefficients $\langle j_1m_1;j_2m_2|j_3m_3\rangle$

$$\hat{U}(\mathbf{x}) = \sum_{L=0}^{2j}\sum_{M=-L}^{L}(-1)^{j-m+M}\langle jm;j-m|L0\rangle D^{(L)}_{0-M}(\alpha,\beta,\gamma)\hat{T}^{(j)}_{LM}, \tag{7}$$

$$\hat{D}(\mathbf{x}) = \sum_{L=0}^{2j}(2L+1)\sum_{M=-L}^{L}(-1)^{j-m+M}\langle jm;j-m|L0\rangle D^{(L)}_{0-M}(\alpha,\beta,\gamma)\hat{T}^{(j)}_{LM}, \tag{8}$$

where $D^{(j)}_{m_1m_2}(\alpha,\beta,\gamma)$ is the Wigner D-function of the form

$$\begin{aligned} D^{(j)}_{m_1m_2}(\alpha,\beta,\gamma) \;&=\; e^{-im_2\alpha}e^{-im_1\gamma}\sum_s \frac{(-1)^s\sqrt{(j+m_2)!(j-m_2)!(j+m_1)!(j-m_1)!}}{s!(j-m_1-s)!(j+m_2-s)!(m_1-m_2+s)!} \\ &\times\left(\cos\frac{\beta}{2}\right)^{2j+m_2-m_1-2s}\left(-\sin\frac{\beta}{2}\right)^{m_1-m_2+2s}, \end{aligned} \tag{9}$$

and $\hat{T}^{(j)}_{LM}$ is the irreducible tensor operator for the $SU(2)$ group (also known as the polarization operator [34,35])

$$\hat{T}^{(j)}_{LM} = \sum_{m_1,m_2=-j}^{j}(-1)^{j-m_1}\langle jm_2;j-m_1|LM\rangle|jm_2\rangle\langle jm_1|. \tag{10}$$

It is worth noting that the Clebsch–Gordan coefficients can always be chosen real. Consequently, the operator $\hat{T}^{(j)}_{LM}$ is real in the basis of states $|jm\rangle$.

From formula (4) it follows that, if the operator $|jm\rangle\langle jm| = \hat{U}(m,0,0,0)$ is known, the dequantizer can easily be calculated. So we focus on finding a simple formula of this operator.

Since $D^{(L)}_{0-M}(0,0,0) = \delta_{0M}$, from (7) it follows that

$$|jm\rangle\langle jm| = \hat{U}(m,0,0,0) = \sum_{L=0}^{2j}(-1)^{j-m}\langle jm; j-m|L0\rangle\hat{T}^{(j)}_{L0} = \sum_{L=0}^{2j} f^{(j)}_L(m)\hat{S}^{(j)}_L, \tag{11}$$

where $f^{(j)}_L(m)$ is a function of the spin projection m and the operator $\hat{S}^{(j)}_L$ is proportional to the operator $\hat{T}^{(j)}_{L0}$. Consequently, $S^{(j)}_L$ is real and diagonal (and hence Hermitian) because of the peculiar form of the operator $\hat{T}^{(j)}_{L0}$

$$\hat{T}^{(j)}_{L0} = \sum_{m_1=-j}^{j}(-1)^{j-m_1}\langle jm_1; j-m_1|L0\rangle|jm_1\rangle\langle jm_1|. \tag{12}$$

Moreover, the operators $S^{(j)}_L$ and $S^{(j)}_{L'}$ are orthogonal in the sense of trace operation

$$\mathrm{Tr}\left(\hat{S}^{(j)}_L\hat{S}^{(j)}_{L'}\right) \sim \mathrm{Tr}\left(\hat{T}^{(j)}_{L0}\hat{T}^{(j)}_{L'0}\right) = \sum_{m=-j}^{j}\langle jm; j-m|L0\rangle\langle jm; j-m|L'0\rangle = \delta_{LL'}. \tag{13}$$

This implies that any Hermitian operator, being diagonal in the basis of states $|jm\rangle$, can be resolved to the linear sum of operators $\hat{S}^{(j)}_L$, $L = 0, 1, \ldots, 2j$. In other words, the matrices $S^{(j)}_L$ form a basis in the space of diagonal Hermitian matrices. On the other hand, the operators $\hat{J}^k_z$, $k = 0, 1, \ldots, 2j$ are also suitable to form the basis in the same space of operators. The transition from one basis to the other can be clarified by applying the operator $\hat{P}$ which swaps states $|jm\rangle$ and $|j-m\rangle$. Combining (12) with such a rule, one obtains $\hat{P}\hat{T}^{(j)}_{L0}\hat{P} = (-1)^L\hat{T}^{(j)}_{L0}$. It is also obvious that $\hat{P}\hat{J}^k_z\hat{P} = (-1)^k\hat{J}^k_z$. Hence, if the number L is odd, the operator $\hat{S}^{(j)}_L$ resolves to the sum of $\hat{J}_z$ to odd powers, and similarly, if the number L is even, the operator $\hat{S}^{(j)}_L$ resolves to the sum of $\hat{J}_z$ to even powers. Since $\hat{S}^{(j)}_0 \sim \hat{J}^0_z$ and $\hat{S}^{(j)}_1 \sim \hat{J}^1_z$, we may assume that the power of operators $\hat{J}_z$ in the expansion of $\hat{S}^{(j)}_L$ is not greater than L. These results can be summarized as follows:

$$\hat{S}^{(j)}_L = \sum_{k=0}^{n} a^{(j,L)}_{2k}\hat{J}^{2k}_z, \qquad \text{if} \qquad L = 2n, \tag{14}$$

$$\hat{S}^{(j)}_L = \sum_{k=0}^{n} b^{(j,L)}_{2k+1}\hat{J}^{2k+1}_z, \qquad \text{if} \qquad L = 2n+1, \tag{15}$$

or in matrix form

$$\begin{pmatrix} \hat{S}^{(j)}_0 \\ \hat{S}^{(j)}_1 \\ \hat{S}^{(j)}_2 \\ \hat{S}^{(j)}_3 \\ \hat{S}^{(j)}_4 \\ \cdots \end{pmatrix} = \begin{pmatrix} a^{(j,0)}_0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & b^{(j,1)}_1 & 0 & 0 & 0 & \cdots \\ a^{(j,2)}_0 & 0 & a^{(j,2)}_2 & 0 & 0 & \cdots \\ 0 & b^{(j,3)}_1 & 0 & b^{(j,3)}_3 & 0 & \cdots \\ a^{(j,4)}_0 & 0 & a^{(j,4)}_2 & 0 & a^{(j,4)}_4 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \begin{pmatrix} \hat{J}^0_z \\ \hat{J}^1_z \\ \hat{J}^2_z \\ \hat{J}^3_z \\ \hat{J}^4_z \\ \cdots \end{pmatrix}. \tag{16}$$

From (16) it follows that $\mathrm{Tr}(\hat{S}_{2n}^{(j)}\hat{S}_{2n+1}^{(j)})=0$, by construction. The explicit form of the coefficients $a_{2k}^{(j,L)}$ and $b_{2k+1}^{(j,L)}$ can be found readily by employing the orthogonality property (13). In fact, since the number of expansion terms in (14) increases step by step with increase in L, any operator $\hat{S}_{2n}^{(j)}$ must be orthogonal in the sense of trace operation to each $\hat{J}_z^{2k}$, $k=0,1,\ldots,n-1$. If we combine this requirement with expansion (14), we get the following system of equations:

$$\begin{pmatrix} \mathrm{Tr}\hat{J}_z^0 & \mathrm{Tr}\hat{J}_z^2 & \ldots & \mathrm{Tr}\hat{J}_z^{2n-2} \\ \mathrm{Tr}\hat{J}_z^2 & \mathrm{Tr}\hat{J}_z^4 & \ldots & \mathrm{Tr}\hat{J}_z^{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Tr}\hat{J}_z^{2n-2} & \mathrm{Tr}\hat{J}_z^{2n} & \ldots & \mathrm{Tr}\hat{J}_z^{4n-4} \end{pmatrix} \begin{pmatrix} a_0^{(j,2n)} \\ a_2^{(j,2n)} \\ \ldots \\ a_{2n-2}^{(j,2n)} \end{pmatrix} = -a_{2n}^{(j,2n)} \begin{pmatrix} \mathrm{Tr}\hat{J}_z^{2n} \\ \mathrm{Tr}\hat{J}_z^{2n+2} \\ \ldots \\ \mathrm{Tr}\hat{J}_z^{4n-2} \end{pmatrix}. \tag{17}$$

It can be proved that the determinant Δ_{2n} of the square matrix on the left-hand side of (17) is never equal to zero. This implies that one can calculate the coefficients involved using the Cramer's rule [36]. Indeed, let $a_{2n}^{(j,2n)}$ be equal to $-\Delta_{2n}$; then the coefficients read

$$a_{2k}^{(j,2n)}=\Delta_{2n}^{(k+1)}, \qquad \text{if} \qquad k=0,1,\ldots,n-1, \qquad\qquad a_{2n}^{(j,2n)}=-\Delta_{2n}, \tag{18}$$

where $\Delta_{2n}^{(i)}$ is the determinant of the matrix formed by replacing the ith column of matrix (17) by the column vector $\left(\mathrm{Tr}\hat{J}_z^{2n}\ \mathrm{Tr}\hat{J}_z^{2n+2}\ \ldots\ \mathrm{Tr}\hat{J}_z^{4n-2}\right)^{\mathrm{tr}}$.

Arguing as above, we obtain

$$b_{2k+1}^{(j,2n+1)}=\Delta_{2n+1}^{(k+1)}, \qquad \text{if} \qquad k=0,1,\ldots,n-1, \qquad\qquad b_{2n+1}^{(j,2n+1)}=-\Delta_{2n+1}, \tag{19}$$

where

$$\Delta_{2n+1}=\det\begin{pmatrix} \mathrm{Tr}\hat{J}_z^2 & \mathrm{Tr}\hat{J}_z^4 & \ldots & \mathrm{Tr}\hat{J}_z^{2n} \\ \mathrm{Tr}\hat{J}_z^4 & \mathrm{Tr}\hat{J}_z^6 & \ldots & \mathrm{Tr}\hat{J}_z^{2n+2} \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Tr}\hat{J}_z^{2n} & \mathrm{Tr}\hat{J}_z^{2n+2} & \ldots & \mathrm{Tr}\hat{J}_z^{4n-2} \end{pmatrix}, \tag{20}$$

and $\Delta_{2n+1}^{(i)}$ is the determinant of the matrix formed by replacing the ith column of matrix (20) by the column vector $\left(\mathrm{Tr}\hat{J}_z^{2n+2}\ \mathrm{Tr}\hat{J}_z^{2n+4}\ \ldots\ \mathrm{Tr}\hat{J}_z^{4n}\right)^{tr}$.

Though the explicit expressions for the coefficients $a_{2k}^{(j,L)}$ and $b_{2k+1}^{(j,L)}$ seem rather complicated, they can be readily computed by recalling that the spin projection m can take discrete values only. This results in the value of $\mathrm{Tr}\hat{J}_z^k$ being expressed by means of the corresponding Bernoulli numbers [37].

Using the formulas obtained, one can easily write the explicit form of operators $\hat{S}_L^{(j)}$ in the case of small numbers L (within a constant factor)

$$\hat{S}_0^{(j)}=\hat{J}_z^0=\hat{I}, \quad \hat{S}_1^{(j)}=\hat{J}_z, \quad \hat{S}_2^{(j)}=3\hat{J}_z^2-j(j+1)\hat{I}, \quad \hat{S}_3^{(j)}=5\hat{J}_z^3-(3j^2+3j-1)\hat{J}_z. \tag{21}$$

Now we show how to calculate functions $f_L^{(j)}(m)$ which are coefficients of expansion (11). Using the orthogonality property (13), we obtain

$$\mathrm{Tr}\left(\hat{S}_L^{(j)}|jm\rangle\langle jm|\right)=\sum_{L'=0}^{2j} f_{L'}^{(j)}(m)\mathrm{Tr}\left(\hat{S}_L^{(j)}\hat{S}_{L'}^{(j)}\right)=f_L^{(j)}(m)\mathrm{Tr}\left({\hat{S}_L^{(j)}}^{\,2}\right). \tag{22}$$

On the other hand,

$$\mathrm{Tr}\left(\hat{S}_L^{(j)}|jm\rangle\langle jm|\right) = \mathrm{Tr}\left(\sum_{k=0}^{L} c_k^{(j,L)}\hat{J}_z^k|jm\rangle\langle jm|\right) = \sum_{k=0}^{L} c_k^{(j,L)} m^k \mathrm{Tr}\left(|jm\rangle\langle jm|\right) = \sum_{k=0}^{L} c_k^{(j,L)} m^k. \tag{23}$$

Combining (14), (15), (22), and (23), we obtain

$$f_L^{(j)}(m) = \left[\mathrm{Tr}\left({\hat{S}_L^{(j)}}^2\right)\right]^{-1}\sum_{k=0}^{n} a_{2k}^{(j,L)} m^{2k}, \qquad \text{if} \qquad L = 2n, \tag{24}$$

$$f_L^{(j)}(m) = \left[\mathrm{Tr}\left({\hat{S}_L^{(j)}}^2\right)\right]^{-1}\sum_{k=0}^{n} b_{2k+1}^{(j,L)} m^{2k+1}, \qquad \text{if} \qquad L = 2n+1, \tag{25}$$

i.e., $f_L^{(j)}(m)$ has the same structure as the operator $\hat{S}_L^{(j)}$. To be more precise, one should simply replace the operator $\hat{J}_z$ by the variable m and divide the result by the normalization coefficient.

Using (12) it is not hard to prove that the functions $f_L^{(j)}(m)$ are expressed by means of the Clebsch–Gordan coefficients as follows:

$$f_L^{(j)}(m) = \left[\mathrm{Tr}\left({\hat{S}_L^{(j)}}^2\right)\right]^{-1/2}(-1)^{j-m}\langle jm; j-m|L0\rangle. \tag{26}$$

Employing the known properties of the Clebsch–Gordan coefficients [34] leads to a recurrence relation of the form

$$f_L^{(j)}(m) = \left[\frac{4(2L-1)(2L+1)}{L^2(2j-L+1)(2j+L+1)\mathrm{Tr}\left({\hat{S}_L^{(j)}}^2\right)}\right]^{1/2}$$
$$\times\left\{\left[\mathrm{Tr}\left({\hat{S}_{L-1}^{(j)}}^2\right)\right]^{1/2} m f_{L-1}^{(j)}(m) - \left[\frac{(L-1)^2(2j-L+2)(2j+L)\mathrm{Tr}\left({\hat{S}_{L-2}^{(j)}}^2\right)}{4(2L-3)(2L-1)}\right]^{1/2} f_{L-2}^{(j)}(m)\right\}. \tag{27}$$

Let us illustrate the results obtained by examples.

Qubit

$$|1/2,m\rangle\langle 1/2,m| = \frac{1}{2}\hat{I} + 2m\hat{J}_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{28}$$

Qutrit

$$\begin{aligned} |1,m\rangle\langle 1,m| &= \frac{1}{3}\hat{I} + \frac{m}{2}\hat{J}_z + \frac{3m^2-2}{6}\left(3\hat{J}_z^2 - 2\hat{I}\right) \\ &= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} + \frac{3m^2-2}{6}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{29}$$

Qudit with spin $\mathbf{j = 3/2}$

$$
\begin{aligned}
|3/2,m\rangle\langle 3/2,m| &= \frac{1}{4}\hat{I} + \frac{m}{5}\hat{J}_z + \frac{4m^2-5}{64}\left(4\hat{J}_z^2 - 5\hat{I}\right) + \frac{20m^3-41m}{720}\left(20\hat{J}_z^3 - 41\hat{J}_z\right) \\
&= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{m}{10}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix} \\
&\quad + \frac{4m^2-5}{16}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \frac{20m^3-41m}{120}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{aligned}
\tag{30}
$$

Now, in view of the explicit form of expansion (11), recalling (4), we obtain the following formula for the dequantizer operator:

$$
\hat{U}(\mathbf{x}) = \sum_{L=0}^{2j} f_L^{(j)}(m)\ u^\dagger \hat{S}_L^{(j)} u. \tag{31}
$$

A comparison of (7) with (8) leads to a simple form of the quantizer operator

$$
\hat{D}(\mathbf{x}) = \sum_{L=0}^{2j} (2L+1) f_L^{(j)}(m)\ u^\dagger \hat{S}_L^{(j)} u. \tag{32}
$$

Using these formulas along with the examples considered above, we write the dequantizer operator for qubits

$$
\hat{U}(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + m\begin{pmatrix} \cos\beta & -e^{i\alpha}\sin\beta \\ -e^{-i\alpha}\sin\beta & -\cos\beta \end{pmatrix} \tag{33}
$$

and for qutrits

$$
\begin{aligned}
\hat{U}(\mathbf{x}) &= \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m}{2}\begin{pmatrix} \cos\beta & -\frac{\sin\beta}{\sqrt{2}}e^{i\alpha} & 0 \\ -\frac{\sin\beta}{\sqrt{2}}e^{-i\alpha} & 0 & -\frac{\sin\beta}{\sqrt{2}}e^{i\alpha} \\ 0 & -\frac{\sin\beta}{\sqrt{2}}e^{-i\alpha} & -\cos\beta \end{pmatrix} \\
&\quad + \frac{3m^2-2}{6}\begin{pmatrix} \frac{3\cos^2\beta-1}{2} & -\frac{3\cos\beta\sin\beta}{\sqrt{2}}e^{i\alpha} & \frac{3\sin^2\beta}{2}e^{i2\alpha} \\ -\frac{3\cos\beta\sin\beta}{\sqrt{2}}e^{-i\alpha} & -\left(3\cos^2\beta-1\right) & \frac{3\cos\beta\sin\beta}{\sqrt{2}}e^{i\alpha} \\ \frac{3\sin^2\beta}{2}e^{-i2\alpha} & \frac{3\cos\beta\sin\beta}{\sqrt{2}}e^{-i\alpha} & \frac{3\cos^2\beta-1}{2} \end{pmatrix}.
\end{aligned}
\tag{34}
$$

The quantizer operator for qubits is simply obtained from (33) by multiplying the second term by 3. The quantizer operator for qutrits is obtained from (34) by multiplying the second and third terms by 3 and 5, respectively. It is worth noting that the quantizer and dequantizer operators of spin states are Hermitian [see (31) and (32)]. In addition, the dequantizer is positive as well. The other remarkable fact for both dequantizer and quantizer operators is that the matrix elements of $u^\dagger S_L^{(j)} u$ are in a close relation to associated Legendre functions of degree L and different orders proportional to the distance to the leading diagonal. This is also the argument for the spin tomogram $w(m,\alpha,\beta)$ (independent of γ) to be a finite sum of spherical functions $Y_l^m(\beta,\alpha)$, $l=0,1,\dots,2j$; this fact has been emphasized earlier in [24].

Low-spin tomograms are of particular interest here because any qudit tomogram and the photon-number tomogram with infinite outputs can be mapped onto qubit or qutrit tomogram [38,39].

The quasiprobability-distribution functions of continuous variables are usually illustrated by plotting on the corresponding phase space. Here, we give an illustration of the spin tomogram $w_{j\mu}(\mathbf{x})$ of the pure state $|j\mu\rangle$. The tomogram reads

$$w_{j\mu}(\mathbf{x}) = \mathrm{Tr}\Big(|j\mu\rangle\langle j\mu|\hat{U}(\mathbf{x})\Big) = |\langle jm|u|j\mu\rangle|^2 = \Big|D_{m\mu}^{(j)}(\alpha,\beta,\gamma)\Big|^2, \tag{35}$$

where the function $D_{m\mu}^{(j)}(\alpha,\beta,\gamma)$ is given by (9). From this follows that the tomogram depends only on the Euler angle β, i.e., $w_{j\mu}(\mathbf{x}) = w_{j\mu}(m,\beta)$. Different examples of this tomogram are depicted in Fig. 1. It is worth noting that the tomogram (35) tends to the following asymptotic function if $j\to\infty$ [33]:

$$\tilde{w}_{j\mu}(m,\beta) = \left(\pi j \sin^2\beta\right)^{-1/2}\left[2^{j-\mu}(j-\mu)!\right]^{-1}\exp\left(-2j\sin^2\beta\right) H_{j-\mu}^2\left(\frac{m-j\cos\beta}{\sqrt{j}\sin\beta}\right), \tag{36}$$

where $H_n(x)$ is a Hermite polynomial of degree n. The strong dependence of the asymptotic function (36) on the value of β occurs due to a pointwise but nonuniform convergence of (35) to (36).

3. Exponential Representation of Quantizer and Dequantizer Operators

The dequantizer operator can be alternatively expressed in terms of the Kronecker delta-symbol [40], i.e., in the form of the following exponential operator:

$$\hat{U}(\mathbf{x}) = \delta\left(m - u^\dagger \hat{J}_z u\right) = \frac{1}{2\pi}\int_0^{2\pi} \exp\left[i\left(m - u^\dagger \hat{J}_z u\right)\varphi\right] d\varphi. \tag{37}$$

Let us check that such a representation of the dequantizer operator completely coincides with that discussed in the previous section. To start, we notice that the operator $u\hat{U}(\mathbf{x})u^\dagger$ is diagonal in the basis of states $|jm\rangle$. Consequently, it can be resolved to the linear sum of operators $\hat{S}_L^{(j)}$. Indeed,

$$u\hat{U}(\mathbf{x})u^\dagger = \frac{1}{2\pi}\int_0^{2\pi}\exp\left[i\left(m-\hat{J}_z\right)\varphi\right] d\varphi = \sum_{L=0}^{2j} g_L^{(j)}(m)\hat{S}_L^{(j)}, \tag{38}$$

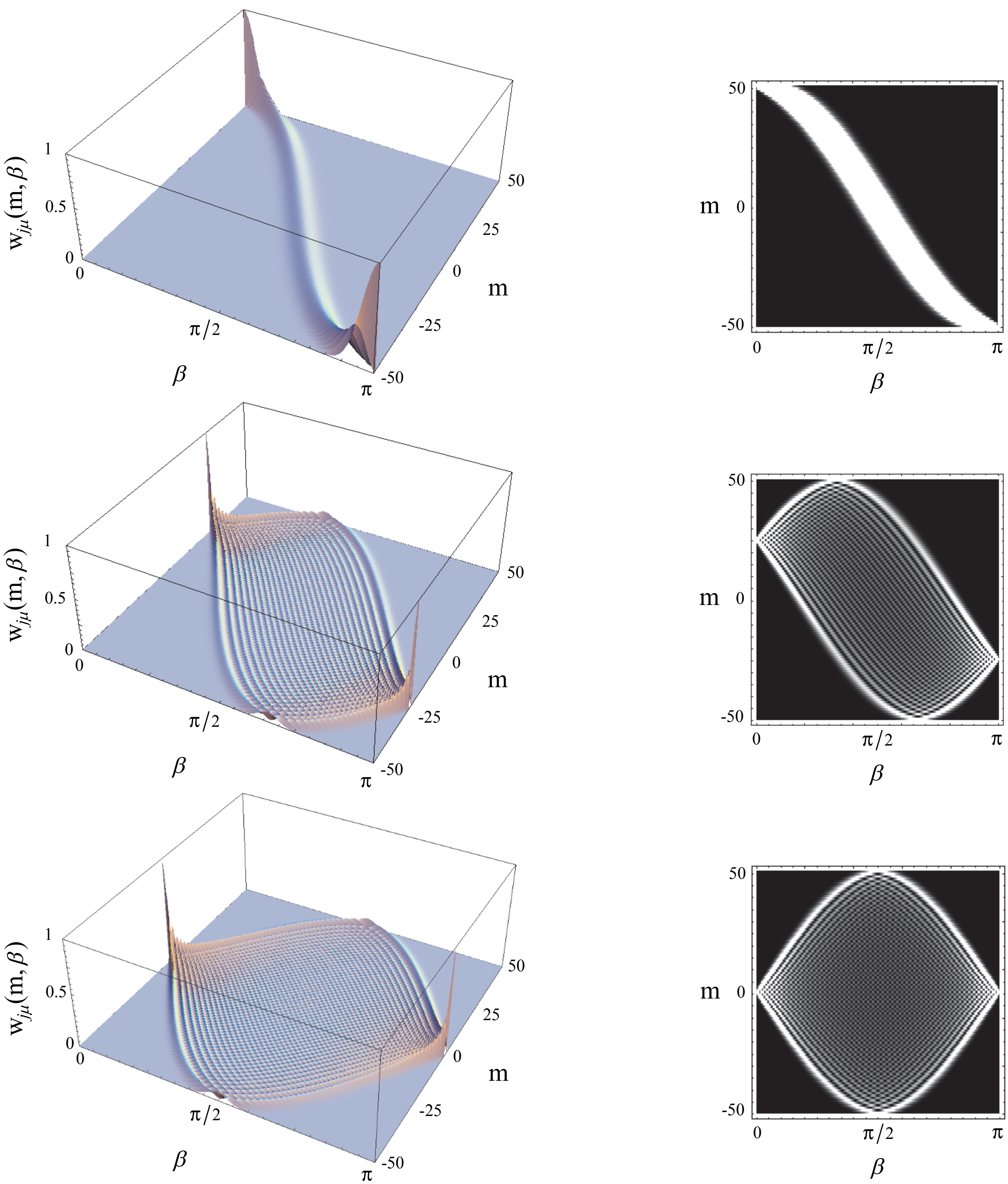

Fig. 1. Spin tomograms (on the left) and density plots (on the right) of the state $|j\mu\rangle$ with $j = 50$ and $\mu = 50$ (top), $j = 50$ and $\mu = 25$ (middle), and $j = 50$ and $\mu = 0$ (bottom).

where the expansion coefficients read

$$g_L^{(j)}(m)=\left[\mathrm{Tr}\left(\hat{S}_L^{(j)\,2}\right)\right]^{-1}\frac{1}{2\pi}\int\limits_0^{2\pi}\mathrm{Tr}\left\{\exp\left[i\left(m-\hat{J}_z\right)\varphi\right]\hat{S}_L^{(j)}\right\}d\varphi. \tag{39}$$

Now, in view of (14) and (15) or, in general, $\hat{S}_L^{(j)}=\sum_{k=0}^{L}c_k^{(j,L)}\hat{J}_z^k$, we obtain

$$\mathrm{Tr}\left\{\exp\left[i\left(m-\hat{J}_z\right)\varphi\right]\hat{S}_L^{(j)}\right\}=\sum_{m'=-j}^{j}e^{i(m-m')\varphi}\sum_{k=0}^{L}c_k^{(j,L)}m'^k \tag{40}$$

and (39) takes the form

$$g_L^{(j)}(m)=\left[\mathrm{Tr}\left(\hat{S}_L^{(j)\,2}\right)\right]^{-1}\sum_{m'=-j}^{j}\sum_{k=0}^{L}c_k^{(j,L)}m'^k\frac{1}{2\pi}\int\limits_0^{2\pi}e^{i(m-m')\varphi}d\varphi=\left[\mathrm{Tr}\left(\hat{S}_L^{(j)\,2}\right)\right]^{-1}\sum_{k=0}^{L}c_k^{(j,L)}m^k. \tag{41}$$

Comparing (41) with (24) and (25), we conclude that $g_L^{(j)}(m)\equiv f_L^{(j)}(m)$. Therefore, we have proved that as far as the dequantizer operator is concerned, exponential expression (37) is equivalent to expansion (31) through orthogonal operators $\hat{S}_L^{(j)}$ and, consequently, to formula (7) expressed in terms of irreducible tensor operators.

Similarly to the case of dequantizer, the quantizer operator $\hat{D}(\mathbf{x})$ can be represented in the exponential form [41]

$$\begin{aligned}\hat{D}(\mathbf{x}) &= \frac{2j+1}{\pi}\int\limits_0^{2\pi}\sin^2\frac{\varphi}{2}\exp\left[i\left(m-u^\dagger\hat{J}_zu\right)\varphi\right]d\varphi\\ &= u^\dagger\left\{\frac{2j+1}{\pi}\int\limits_0^{2\pi}\sin^2\frac{\varphi}{2}\exp[i(m-\hat{J}_z)\varphi]d\varphi\right\}u.\end{aligned} \tag{42}$$

Let us consider the operator $u\hat{D}(\mathbf{x})u^\dagger$ in detail. In fact, it follows easily that

$$u\hat{D}(\mathbf{x})u^\dagger=\frac{2j+1}{2\pi}\int\limits_0^{2\pi}\left\{e^{i(m-\hat{J}_z)\varphi}-\frac{1}{2}e^{i(m+1-\hat{J}_z)\varphi}-\frac{1}{2}e^{i(m-1-\hat{J}_z)\varphi}\right\}d\varphi. \tag{43}$$

Since the spin projection to an arbitrary axis can take only values from $-j$ to j, we obtain

$$\frac{1}{2\pi}\int\limits_0^{2\pi} e^{i(m-\hat{J}_z)\varphi}d\varphi = |jm\rangle\langle jm| = u\hat{U}(\mathbf{x})u^\dagger, \tag{44}$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi} e^{i(m+1-\hat{J}_z)\varphi}d\varphi = \hat{R}_+|jm\rangle\langle jm|\hat{R}_- = \begin{cases} |j,m+1\rangle\langle j,m+1| \ \text{ if } \ m=-j,\ldots,j-1, \\ 0 \ \text{ if } \ m=j, \end{cases}, \tag{45}$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi} e^{i(m-1-\hat{J}_z)\varphi}d\varphi = \hat{R}_-|jm\rangle\langle jm|\hat{R}_+ = \begin{cases} |j,m-1\rangle\langle j,m-1| \ \text{ if } \ m=-j+1,\ldots,j, \\ 0 \ \text{ if } \ m=-j, \end{cases}, \tag{46}$$

where we introduced the operators $\hat{R}_+$ and $\hat{R}_-$ specified by their matrices in the basis of states $|jm\rangle$

$$R_+ = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \qquad R_- = R_+^\dagger = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}. \tag{47}$$

Combining (43)–(46), we obtain the explicit relation between quantizer and dequantizer operators

$$\hat{D}(\mathbf{x}) \equiv \hat{D}(m,u) = (2j+1)\left[\hat{U}(\mathbf{x}) - \frac{1}{2}\hat{R}_+(u)\hat{U}(\mathbf{x})\hat{R}_-(u) - \frac{1}{2}\hat{R}_-(u)\hat{U}(\mathbf{x})\hat{R}_+(u)\right], \tag{48}$$

where $\hat{R}_+(u) = u^\dagger\hat{R}_+u$ and $\hat{R}_-(u) = u^\dagger\hat{R}_-u$. It is easy to prove that the inverse formula reads

$$\hat{U}(\mathbf{x}) = \frac{1}{2j+1}\sum_{k=0}^{\infty}\hat{D}^{(k)}, \tag{49}$$

with $\hat{D}^{(k)}$ being defined by the recurrence relations

$$\hat{D}^{(k)} = \frac{1}{2}\left[\hat{R}_+(u)\hat{D}^{(k-1)}\hat{R}_-(u) + \hat{R}_-(u)\hat{D}^{(k-1)}\hat{R}_+(u)\right], \qquad \hat{D}^{(0)} = \hat{D}(\mathbf{x}). \tag{50}$$

Now we show that the exponential form of quantizer operator (42) is in complete agreement with formula (32). Using the explicit expression (48) of the quantizer operator through the dequantizer, and employing expansion (31), proved to be identical to the exponential form, we arrive at

$$u\hat{D}(\mathbf{x})u^\dagger = (2j+1)\sum_{L=0}^{2j} f_L^{(j)}(m)\left(\hat{S}_L^{(j)} - \frac{1}{2}\hat{R}_+\hat{S}_L^{(j)}\hat{R}_- - \frac{1}{2}\hat{R}_-\hat{S}_L^{(j)}\hat{R}_+\right). \tag{51}$$

On the other hand, the diagonal operator $\left(\hat{S}_L^{(j)} - \frac{1}{2}\hat{R}_+\hat{S}_L^{(j)}\hat{R}_- - \frac{1}{2}\hat{R}_-\hat{S}_L^{(j)}\hat{R}_+\right)$ can also be resolved to the sum

$$\hat{S}_L^{(j)} - \frac{1}{2}\hat{R}_+\hat{S}_L^{(j)}\hat{R}_- - \frac{1}{2}\hat{R}_-\hat{S}_L^{(j)}\hat{R}_+ = \sum_{L'=0}^{2j} h_{L'}\hat{S}_{L'}^{(j)}, \tag{52}$$

with

$$\begin{aligned} h_L &= \left[\mathrm{Tr}\left(\hat{S}_L^{(j)\,2}\right)\right]^{-1}\left[\mathrm{Tr}\left(\hat{S}_L^{(j)}\hat{S}_L^{(j)}\right) - \frac{1}{2}\mathrm{Tr}\left(\hat{R}_+\hat{S}_L^{(j)}\hat{R}_-\hat{S}_L^{(j)}\right) - \frac{1}{2}\mathrm{Tr}\left(\hat{R}_+\hat{S}_L^{(j)}\hat{R}_-\hat{S}_L^{(j)}\right)\right] \\ &= 1 - \mathrm{Tr}\left(\hat{R}_+\hat{T}_{L0}^{(j)}\hat{R}_-\hat{T}_{L0}^{(j)}\right). \end{aligned} \tag{53}$$

In (52) only the term h_L yields nonzero contribution. Now, in view of (12) and (47), we obtain

$$\mathrm{Tr}\left(\hat{R}_+\hat{T}_{L0}^{(j)}\hat{R}_-\hat{T}_{L0}^{(j)}\right) = -\sum_{m=-j}^{j-1}\langle jm; j-m|L0\rangle\langle j, m+1; j, -m-1|L0\rangle = \frac{2(j-L)}{2j+1} \tag{54}$$

and

$$h_L = \frac{2L+1}{2j+1}. \tag{55}$$

Substituting (55) for h_L in (52) and combining the result obtained with (51), we get formula (32). This completes the proof that, for the quantizer operator, exponential expression (42) is equivalent to expansion (32) through the orthogonal operators $\hat{S}_L^{(j)}$ and, consequently, to formula (8) expressed in terms of irreducible tensor operators.

4. Delta-Function on the Tomogram Set

Using definitions (2) and (5), one can easily write

$$w(\mathbf{x}_1) = \int w(\mathbf{x}_2)\mathrm{Tr}\left(\hat{D}(\mathbf{x}_2)\hat{U}(\mathbf{x}_1)\right) d\mathbf{x}_2. \tag{56}$$

This implies that the function $\mathrm{Tr}\left(\hat{D}(\mathbf{x}_2)\hat{U}(\mathbf{x}_1)\right)$ can be treated as the kernel of the unity operator on the set of spin tomograms. As far as qubits are considered, we employ the exact formulas for quantizer and dequantizer operators (33). The result is

$$\mathrm{Tr}\left(\hat{D}(\mathbf{x}_2)\hat{U}(\mathbf{x}_1)\right) = \frac{1}{2} + 6m_1m_2\left(\cos\beta_1\cos\beta_2 + \sin\beta_1\sin\beta_2\cos(\alpha_1-\alpha_2)\right) = \frac{1}{2} + 6m_1m_2\left(\mathbf{n}_1\cdot\mathbf{n}_2\right). \tag{57}$$

In the case of qutrits, analogues calculations, with account of (34), yield

$$\mathrm{Tr}\left(\hat{D}(\mathbf{x}_2)\hat{U}(\mathbf{x}_1)\right) = \frac{1}{3} + \frac{3}{2}m_1m_2\left(\mathbf{n}_1\cdot\mathbf{n}_2\right) + \frac{5}{12}(3m_1^2-2)(3m_2^2-2)\left(3(\mathbf{n}_1\cdot\mathbf{n}_2)^2-1\right). \tag{58}$$

Here we introduced vectors $\mathbf{n}_i$, which correspond to matrices $u(\alpha_i,\beta_i,\gamma_i)$ according to the rule

$$\mathbf{n}_i = (\cos\alpha_i\sin\beta_i, \sin\alpha_i\sin\beta_i, \cos\beta_i), \tag{59}$$

and determine the axis of quantization of the spin projection for the operator $u^\dagger\hat{J}_z u$.

5. Star-Product for Qubit and Qutrit Tomograms

By construction, the tomographic symbol $f_{\hat{A}}(\mathbf{x})$ is related to the operator $\hat{A}$ as follows:

$$f_{\hat{A}}(\mathbf{x}) = \mathrm{Tr}\left(\hat{A}\hat{U}(\mathbf{x})\right), \qquad \hat{A} = \int f_{\hat{A}}(\mathbf{x})\hat{D}(\mathbf{x})d\mathbf{x}. \tag{60}$$

The symbol of the operator $\hat{A}\hat{B}$ is called the star-product of symbols $f_{\hat{A}}(\mathbf{x})$ and $f_{\hat{B}}(\mathbf{x})$. In other words,

$$f_{\hat{A}\hat{B}}(\mathbf{x}_1) = f_{\hat{A}}(\mathbf{x}_1) * f_{\hat{B}}(\mathbf{x}_1) = \mathrm{Tr}\left(\hat{A}\hat{B}\hat{U}(\mathbf{x}_1)\right) = \iint f_{\hat{A}}(\mathbf{x}_3)f_{\hat{B}}(\mathbf{x}_2)K(\mathbf{x}_3,\mathbf{x}_2,\mathbf{x}_1)\,d\mathbf{x}_2\,d\mathbf{x}_3, \tag{61}$$

where the function

$$K(\mathbf{x}_3,\mathbf{x}_2,\mathbf{x}_1) = \mathrm{Tr}\left(\hat{D}(\mathbf{x}_3)\hat{D}(\mathbf{x}_2)\hat{U}(\mathbf{x}_1)\right) \tag{62}$$

is called the kernel of the star-product scheme.

Direct calculations of kernel (62) for the qubit case yield

$$\begin{aligned} K(\mathbf{x}_3,\mathbf{x}_2,\mathbf{x}_1) \;=\; & \frac{1}{4} + 3\,m_1m_2\,(\mathbf{n}_1\cdot\mathbf{n}_2) + 9\,m_2m_3\,(\mathbf{n}_2\cdot\mathbf{n}_3) + 3\,m_1m_3\,(\mathbf{n}_3\cdot\mathbf{n}_1) \\ & + i\,18\,m_1m_2m_3\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3])\,. \end{aligned} \tag{63}$$

Here, $(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3])$ denotes the scalar triple product of the vectors $\mathbf{n}_1$, $\mathbf{n}_2$, and $\mathbf{n}_3$.

As far as qutrits are concerned, kernel (62) takes the form

$$\begin{aligned} & K(\mathbf{x}_3,\mathbf{x}_2,\mathbf{x}_1) = \frac{1}{9} + \frac{1}{2}m_1m_2\,(\mathbf{n}_1\cdot\mathbf{n}_2) + \frac{3}{2}m_2m_3\,(\mathbf{n}_2\cdot\mathbf{n}_3) + \frac{1}{2}m_1m_3\,(\mathbf{n}_3\cdot\mathbf{n}_1) \\ & + i\frac{9}{8}m_1m_2m_3\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3]) + \frac{5}{36}(3m_1^2-2)(3m_2^2-2)\left(3(\mathbf{n}_1\cdot\mathbf{n}_2)^2-1\right) \\ & + \frac{25}{36}(3m_2^2-2)(3m_3^2-2)\left(3(\mathbf{n}_2\cdot\mathbf{n}_3)^2-1\right) + \frac{5}{36}(3m_1^2-2)(3m_3^2-2)\left(3(\mathbf{n}_3\cdot\mathbf{n}_1)^2-1\right) \\ & + \frac{3}{8}(3m_1^2-2)m_2m_3\Big(3(\mathbf{n}_1\cdot\mathbf{n}_2)(\mathbf{n}_1\cdot\mathbf{n}_3) - (\mathbf{n}_2\cdot\mathbf{n}_3)\Big) \\ & + \frac{5}{8}m_1(3m_2^2-2)m_3\Big(3(\mathbf{n}_2\cdot\mathbf{n}_3)(\mathbf{n}_2\cdot\mathbf{n}_1) - (\mathbf{n}_3\cdot\mathbf{n}_1)\Big) \\ & + \frac{5}{8}m_1m_2(3m_3^2-2)\Big(3(\mathbf{n}_3\cdot\mathbf{n}_1)(\mathbf{n}_3\cdot\mathbf{n}_2) - (\mathbf{n}_1\cdot\mathbf{n}_2)\Big) \\ & + i\frac{25}{8}m_1(3m_2^2-2)(3m_3^2-2)\,(\mathbf{n}_2\cdot\mathbf{n}_3)\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3]) \\ & + i\frac{15}{8}(3m_1^2-2)m_2(3m_3^2-2)\,(\mathbf{n}_3\cdot\mathbf{n}_1)\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3]) \\ & + i\frac{15}{8}(3m_1^2-2)(3m_2^2-2)m_3\,(\mathbf{n}_1\cdot\mathbf{n}_2)\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3]) \\ & + \frac{25}{72}(3m_1^2-2)(3m_2^2-2)(3m_3^2-2)\Big\{3\,(\mathbf{n}_1\cdot\mathbf{n}_2)\,\Big(\,[\mathbf{n}_1\times\mathbf{n}_3]\cdot[\mathbf{n}_2\times\mathbf{n}_3]\,\Big) \\ & + 3\,(\mathbf{n}_2\cdot\mathbf{n}_3)\,\Big(\,[\mathbf{n}_2\times\mathbf{n}_1]\cdot[\mathbf{n}_3\times\mathbf{n}_1]\,\Big) + 3\,(\mathbf{n}_3\cdot\mathbf{n}_1)\,\Big(\,[\mathbf{n}_3\times\mathbf{n}_2]\cdot[\mathbf{n}_1\times\mathbf{n}_2]\,\Big) - 2\Big\}, \end{aligned} \tag{64}$$

where $[\mathbf{n}_i \times \mathbf{n}_j]$ is the cross product of vectors $\mathbf{n}_i$ and $\mathbf{n}_j$.

From (61) it follows that, if $\hat{A} = \hat{1}$, then

$$f_{\hat{B}}(\mathbf{x}_1) = \int f_{\hat{B}}(\mathbf{x}_2) \left(\int K(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) d\mathbf{x}_3 \right) d\mathbf{x}_2. \tag{65}$$

This implies that

$$\mathrm{Tr}\left(\hat{D}(\mathbf{x}_2) \hat{U}(\mathbf{x}_1) \right) = \int K(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) d\mathbf{x}_3. \tag{66}$$

Employing explicit formulas (57), (58), (63), and (64), one can easily check that requirement (66) is satisfied for qubits and qutrits.

6. Dual Tomographic Symbols

Dual tomographic symbols are especially convenient for calculating the expectation values of observables, i.e., the quantity $\mathrm{Tr}(\hat{\rho}\hat{A})$. Indeed, the trace of the product of two operators $\hat{A}$ and $\hat{B}$ reads

$$\mathrm{Tr}\left(\hat{A}\hat{B} \right) = \int f_{\hat{A}}(\mathbf{x}) \mathrm{Tr}\left(\hat{B}\hat{D}(\mathbf{x}) \right) d\mathbf{x} = \int f_{\hat{A}}(\mathbf{x}) f^d_{\hat{B}}(\mathbf{x})\, d\mathbf{x}, \tag{67}$$

where we introduced the dual tomographic symbol of the operator $\hat{B}$ as follows:

$$f^d_{\hat{B}}(\mathbf{x}) = \mathrm{Tr}\left(\hat{B}\hat{D}(\mathbf{x}) \right), \qquad \hat{B} = \int f^d_{\hat{B}}(\mathbf{x}) \hat{U}(\mathbf{x}) d\mathbf{x}. \tag{68}$$

It is easy to prove that the star-product kernel for dual tomographic symbols takes the form

$$K^d(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) = \mathrm{Tr}\left(\hat{U}(\mathbf{x}_3) \hat{U}(\mathbf{x}_2) \hat{D}(\mathbf{x}_1) \right). \tag{69}$$

Due to the similar structure of quantizer and dequantizer operators [see Eqs. (31) and (32)], the kernel $K^d(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1)$ differs from the kernel $K(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1)$ by numerical factors of the corresponding terms.

For qubits, one has

$$\begin{aligned} K^d(\mathbf{x}_3, \mathbf{x}_2, \mathbf{x}_1) \quad = \quad & \frac{1}{4} + 3m_1 m_2 \left(\mathbf{n}_1 \cdot \mathbf{n}_2 \right) + m_2 m_3 \left(\mathbf{n}_2 \cdot \mathbf{n}_3 \right) \\ & + 3m_1 m_3 \left(\mathbf{n}_3 \cdot \mathbf{n}_1 \right) + i6 m_1 m_2 m_3 \left(\mathbf{n}_1 \cdot [\mathbf{n}_2 \times \mathbf{n}_3] \right). \end{aligned} \tag{70}$$

In the case of qutrits, we obtain

$$
\begin{aligned}
&K^d(\mathbf{x}_3,\mathbf{x}_2,\mathbf{x}_1)=\frac{1}{9}+\frac{1}{2}m_1m_2\,(\mathbf{n}_1\cdot\mathbf{n}_2)+\frac{1}{6}m_2m_3\,(\mathbf{n}_2\cdot\mathbf{n}_3)+\frac{1}{2}m_1m_3\,(\mathbf{n}_3\cdot\mathbf{n}_1)\\
&+i\frac{3}{8}m_1m_2m_3\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3])+\frac{5}{36}(3m_1^2-2)(3m_2^2-2)\,\big(3(\mathbf{n}_1\cdot\mathbf{n}_2)^2-1\big)\\
&+\frac{1}{36}(3m_2^2-2)(3m_3^2-2)\,\big(3(\mathbf{n}_2\cdot\mathbf{n}_3)^2-1\big)+\frac{5}{36}(3m_1^2-2)(3m_3^2-2)\,\big(3(\mathbf{n}_3\cdot\mathbf{n}_1)^2-1\big)\\
&+\frac{5}{24}(3m_1^2-2)m_2m_3\Big(3(\mathbf{n}_1\cdot\mathbf{n}_2)(\mathbf{n}_1\cdot\mathbf{n}_3)-(\mathbf{n}_2\cdot\mathbf{n}_3)\Big)\\
&+\frac{1}{8}m_1(3m_2^2-2)m_3\Big(3(\mathbf{n}_2\cdot\mathbf{n}_3)(\mathbf{n}_2\cdot\mathbf{n}_1)-(\mathbf{n}_3\cdot\mathbf{n}_1)\Big)\\
&+\frac{1}{8}m_1m_2(3m_3^2-2)\Big(3(\mathbf{n}_3\cdot\mathbf{n}_1)(\mathbf{n}_3\cdot\mathbf{n}_2)-(\mathbf{n}_1\cdot\mathbf{n}_2)\Big)\\
&+i\frac{3}{8}m_1(3m_2^2-2)(3m_3^2-2)\,(\mathbf{n}_2\cdot\mathbf{n}_3)\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3])\\
&+i\frac{5}{8}(3m_1^2-2)m_2(3m_3^2-2)\,(\mathbf{n}_3\cdot\mathbf{n}_1)\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3])\\
&+i\frac{5}{8}(3m_1^2-2)(3m_2^2-2)m_3\,(\mathbf{n}_1\cdot\mathbf{n}_2)\,(\mathbf{n}_1\cdot[\mathbf{n}_2\times\mathbf{n}_3])\\
&+\frac{5}{72}(3m_1^2-2)(3m_2^2-2)(3m_3^2-2)\Big\{3\,(\mathbf{n}_1\cdot\mathbf{n}_2)\Big(\,[\mathbf{n}_1\times\mathbf{n}_3]\cdot[\mathbf{n}_2\times\mathbf{n}_3]\,\Big)\\
&+3\,(\mathbf{n}_2\cdot\mathbf{n}_3)\Big(\,[\mathbf{n}_2\times\mathbf{n}_1]\cdot[\mathbf{n}_3\times\mathbf{n}_1]\,\Big)+3\,(\mathbf{n}_3\cdot\mathbf{n}_1)\Big(\,[\mathbf{n}_3\times\mathbf{n}_2]\cdot[\mathbf{n}_1\times\mathbf{n}_2]\,\Big)-2\Big\}.
\end{aligned}
\tag{71}
$$

7. Conclusions

The tomographic-probability representation of quantum mechanics allows one to describe states and operators by special functions (tomographic symbols). Moreover, the tomograms can be measured experimentally.

Spin tomography has undergone fast development in the last few decades and has been attacked with the help of different approaches. We managed here to demonstrate the equivalency of two methods available in the literature.

We also succeeded in developing a simple form of the dequantizer and quantizer operators needed for scanning and reconstruction procedures, respectively. The explicit form of the star-product kernel is obtained for qubits and qutrits. Utilizing these expressions is straightforward while we deal with ordinary or dual tomographic symbols of operators.

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