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WIGNER FUNCTION AND BELL'S INEQUALITIES FOR EVEN AND ODD COHERENT STATES

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Abstract

The Wigner function and the symplectic tomogram of an entangled quantum state, which is a superposition of the photon's coherent states (even and odd coherent states), is studied. Photon statistics and violation of Bell's inequality for the photon state are discussed.

Keywords: even and odd coherent states, Bell's inequalities, quantum tomography.

1. Introduction

In [1] the notion of even and odd coherent states was introduced — the states are defined as superpositions of coherent states of a quantized electromagnetic field discussed, for example, in [2–4]. Quantum states can be described by the density operator [5] in different representations, in particular, by the Wigner function [6] or the optical tomogram [7] that was introduced in [8, 9] for the Wigner function reconstruction and was realized at measurements of the quantum states, for example, of photons [10].

The quantum states can be split into two classes — entangled and separable states [11] (see also [12, 13]). Entangled states, in contrast to separable states, can violate Bell's inequalities [14–16]. There exists the separability criterion [17–19] which is a necessary but not sufficient condition for separability of the states. The problem of quantum-state entanglement, i.e., finding the criterion and measure of entanglement, is not yet finally solved. This is why criteria and properties of entangled states, in particular, the connection between entanglement and fulfillment or violation of Bell's inequalities for various quantum states realized in experiments such as multimode even and odd coherent states [20, 21], are worthy of study in detail. The aim of this paper is a review and detailed consideration of the properties of the Wigner function, symplectic tomogram [7, 22], and photon statistics [23, 24], along with the study of Bell's inequality violations for even and odd coherent states.

2. Entangled States

The quantum states of composite systems can have specific strong quantum correlations corresponding to the phenomenon of entanglement [11]. The states with classical correlations are called separable states.

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In particular, for a system of two particles with density matrices $\rho_k^{(1)}$ and $\rho_k^{(2)}$, the separable state is described by the density matrix ρ which can be represented by a convex sum of tensor products of the matrices $\rho_k^{(1)}$ and $\rho_k^{(2)}$

$$\rho = \sum_k p_k \rho_k^{(1)} \otimes \rho_k^{(2)}, \tag{1}$$

with coefficients $p_k \geq 0$ satisfying the normalization condition

$$\sum_k p_k = 1. \tag{2}$$

Vice versa, the states which cannot be presented in form (1) are called entangled states. Entanglement is one of the most interesting properties which distinguishes the classical and quantum-mechanical states. As a rule, entanglement is connected with the presence of nonlocal quantum interactions in multipartite systems.

In this paper, we study the connection between the phenomenon of entanglement and possible violation of Bell's inequalities for even and odd coherent states [20, 21] within the framework of the tomographic-probability approach [7, 22].

3. Quantum Tomography

In addition to the standard methods of describing the quantum states by the density matrix and the wave function, in quantum mechanics the Wigner function was introduced [6]

$$W(\vec{q}, \vec{p}) = \int \psi\left(\vec{q} + \frac{\vec{u}}{2}\right) \psi^*\left(\vec{q} - \frac{\vec{u}}{2}\right) e^{-i\vec{p}\vec{u}}\, d\vec{u}, \tag{3}$$

where $\vec{q}$ and $\vec{p}$ are variables corresponding to the position and momentum of a multimode system. The tomogram (for one degree of freedom) is related to the wave function as follows [25]:

$$\omega\left(X, \mu, \nu\right) = \frac{1}{2\pi|\nu|} \left| \int dy\ \psi(y) \exp\left[i\left(\frac{\mu}{2\nu}y^2 - X\frac{y}{\nu}\right)\right] \right|^2, \tag{4}$$

where μ and ν are real parameters of symplectic transformation. Functions (3) and (4) completely describe the system under consideration and enable one to reconstruct the density matrix and the wave function in the position representation. Tomogram (4) is the probability density of a random variable which is the real position X; it possesses all the properties of the classical probability density, since it is a real positive normalized function. Meanwhile, the Wigner function which has similar properties for some states can be negative; this is why it is called the quasidistribution in the literature. It is easy to obtain symplectic tomograms for the Wigner function. For the two-mode case, it reads

$$\omega\left(\vec{X}, \vec{\mu}, \vec{\nu}\right) = \frac{1}{(2\pi)^2} \int W\left(\vec{q}, \vec{p}\right) \delta\left(\vec{X} - \vec{\mu}\vec{q} - \vec{\nu}\vec{p}\right) d\vec{q}\, d\vec{p}. \tag{5}$$

Thus, the probability that in the measurement the components of the vector $\vec{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ (in the two-mode case) take positive or negative values is determined by the reduced tomograms

$$\omega_{++}(\mu_1, \nu_1, \mu_2, \nu_2) = \int_0^\infty dX_1 \int_0^\infty \omega(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)\, dX_2, \quad (6)$$

$$\omega_{--}(\mu_1, \nu_1, \mu_2, \nu_2) = \int_{-\infty}^0 dX_1 \int_{-\infty}^0 \omega(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)\, dX_2, \quad (7)$$

where the index on the left-hand side of equalities (6) and (7) corresponds to infinity in the limits of integration. Analogously, the following equalities yield the probabilities of X_1 and X_2 to have different signs, i.e.,

$$\omega_{+-}(\mu_1, \nu_1, \mu_2, \nu_2) = \int_0^\infty dX_1 \int_{-\infty}^0 \omega(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)\, dX_2, \quad (8)$$

$$\omega_{-+}(\mu_1, \nu_1, \mu_2, \nu_2) = \int_{-\infty}^0 dX_1 \int_0^\infty \omega(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2)\, dX_2. \quad (9)$$

The probabilities determined by (6)–(9) were analyzed for other quantum states in [26]. The tomographic approach presented and the probabilities (6)–(9) are employed to derive the Bell's parameter.

4. Bell's Inequalities

In 1935, Einstein, Podolsky, and Rosen paid attention to the existence of quantum correlations between two particles located far away. In view of this, they draw the conclusion that quantum mechanics is incomplete and, as a consequence of the incompleteness, there exist hidden parameters λ completely describing the quantum system [27]. Later Bell formulated an inequality [14] that can be used for detecting entangled states; the violation of Bell's inequality gives evidence of the presence of purely quantum correlations in a composite system.

There exists a number of possible representations of Bell's inequalities. Various optical experiments examining its fulfillment or violation have been performed [28]. In this paper, we consider Bell's inequality in the form given by Clauser, Horne, Shimony, and Holt (CHSH) [16].

In view of (6)–(9), we determine the correlation function as follows:

$$E(\vec{\mu}, \vec{\nu}) = \omega_{++}(\vec{\mu}, \vec{\nu}) - \omega_{+-}(\vec{\mu}, \vec{\nu}) - \omega_{-+}(\vec{\mu}, \vec{\nu}) + \omega_{--}(\vec{\mu}, \vec{\nu}), \quad (10)$$

where $\vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and $\vec{\nu} = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}$. For a system with spin 1/2, function (10) determines the correlation of spin projections of the particle on different directions [16]. This means that for four possible directions (μ_1, ν_1), (μ_2, ν_2), (μ_1', ν_1'), and (μ_2', ν_2') Bell's inequality in the CHSH form reads

$$B \equiv \left| E(\mu_1, \nu_1, \mu_2, \nu_2) + E(\mu_1, \nu_1, \mu_2', \nu_2') + E(\mu_1', \nu_1', \mu_2, \nu_2) - E(\mu_1', \nu_1', \mu_2', \nu_2') \right| \le 2. \quad (11)$$

It is worth noting that violation of inequality (11) by necessity proves the entanglement of the state under study. At the same time, if inequality (11) is fulfilled, one cannot conclude that the state is separable.

We consider possible violation of Bell's inequality on the example of even and odd coherent states of the two-mode electromagnetic field.

5. Even and Odd Coherent States

The coherent state of a harmonic oscillator $|\alpha\rangle$, being the closest to the classical state of a particle moving in a quadratic potential, was first introduced as a wave packet for mechanical oscillator in [29], and later it was generalized to the case of quantum oscillators of the electromagnetic field in [2–4]. This state arises as a result of a shift of the ground state

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{12}$$

where the shift operator $D(\alpha)$ is expressed through the creation and annihilation operators $a^\dagger$ and a as follows:

$$D(\alpha) = \exp\left[\alpha a^\dagger - \alpha^* a\right], \tag{13}$$

with α being a complex number. After some algebra, we obtain that the coherent state can be represented as a superposition of the states with photon number m [4]

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_m \frac{\alpha^m}{\sqrt{m!}} |m\rangle. \tag{14}$$

We obtain the photon distribution function for the state (14) which obeys the Poisson statistics

$$P(m) = |\langle m|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2m}}{m!}. \tag{15}$$

Even and odd coherent states [1] belong to the family of the so-called nonclassical states with non-Poissonian statistics. In reality, these states are superpositions of two coherent states (14); they are also called Schrödinger cat states. We consider below examples of one- and two-mode cases.

5.1. One-Mode Even State

In the one-mode case, we define the Schrödinger cat state using the following expression:

$$|\psi\rangle = \frac{N}{\sqrt{2}} \left(|\alpha e^{i\varphi}\rangle + |-\alpha e^{-i\varphi}\rangle\right),$$
$$N = \left[1 + \exp\left(-\frac{|\alpha|^2}{2}\right)\left[\exp\left(-\alpha^2 e^{-2i\varphi}\right) + \exp\left(-\alpha^2 e^{2i\varphi}\right)\right]\right]^{-1/2},$$

where α is the positive shift amplitude and φ is the shift phase.

The probability $P(m)$ to find m photons in the state $|\psi\rangle$ strongly depends on the phase φ (see Fig. 1 where the amplitude $\alpha = 6$). For convenience, m varies continuously.

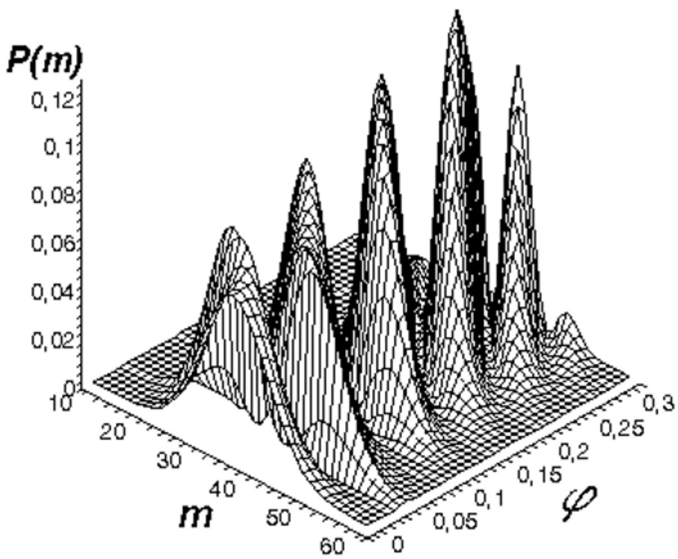

Fig. 1. Probability distribution $P(m)$ at $\alpha = 6$.

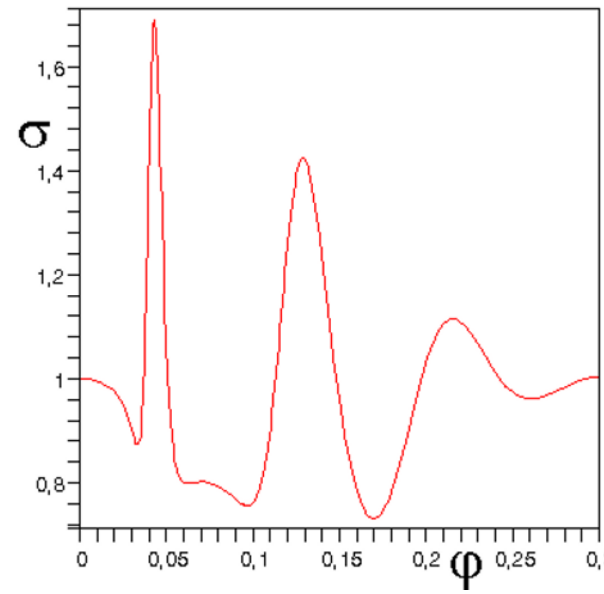

Fig. 2. Relative dispersion σ at $\alpha = 6$.

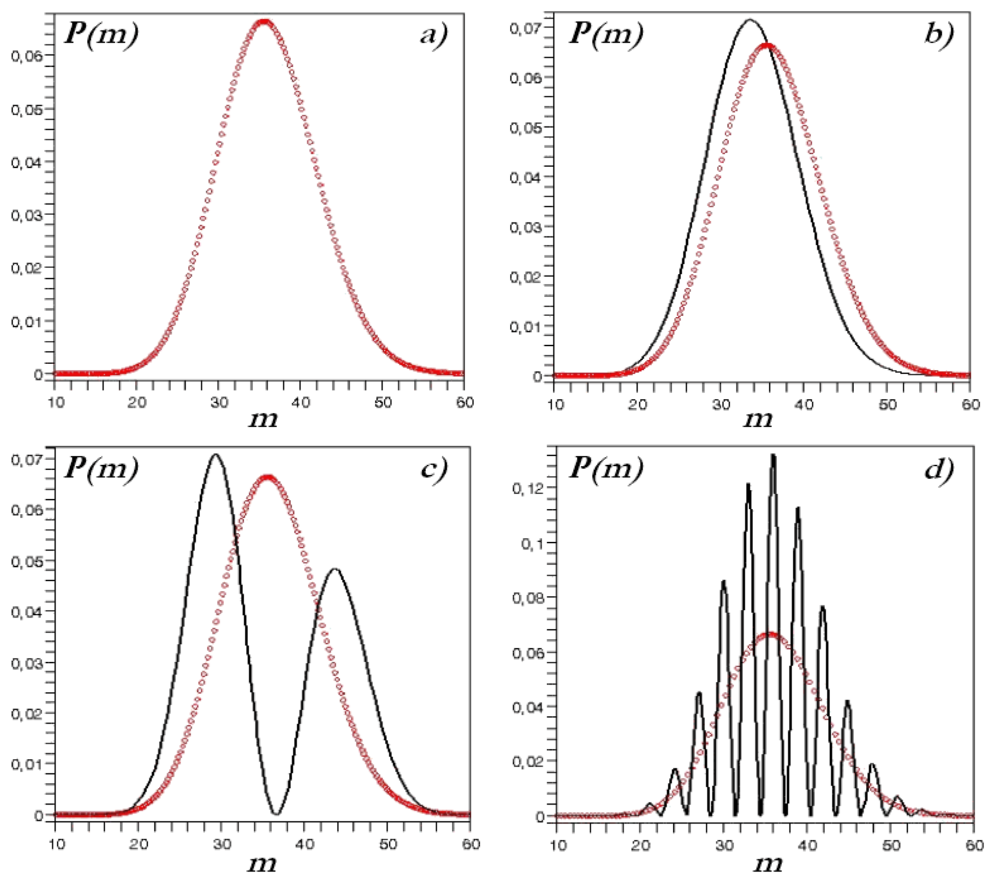

Fig. 3. Probability distribution $P(m)$ at $\alpha = 6$ and $\varphi = 0$ (a), sub-Poissonian distribution $\varphi = \pi/128$ (b), super-Poissonian distribution $\varphi = \pi/24$ (c), and strongly oscillating envelope $\varphi = \pi/3$ (d).

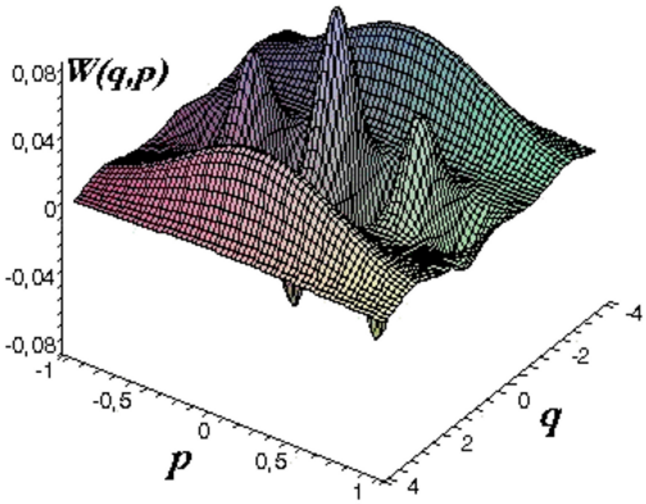

Fig. 4. Wigner function for the state $|\psi_+\rangle$ at $\alpha = \beta = 2$, $\varphi = \theta = 0$, $q_1 = q_2 = q$, and $p_1 = p_2 = p$.

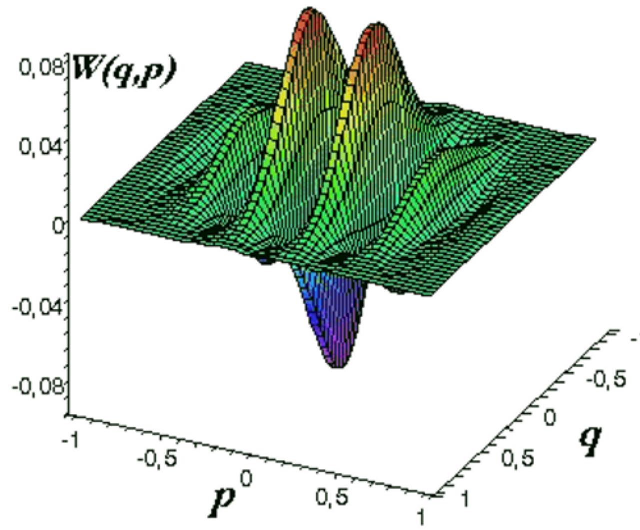

Fig. 5. Wigner function for the state $|\psi_-\rangle$ at $\alpha = \beta = 2$, $\varphi = \theta = 0$, $q_1 = q_2 = q$, and $p_1 = p_2 = p$.

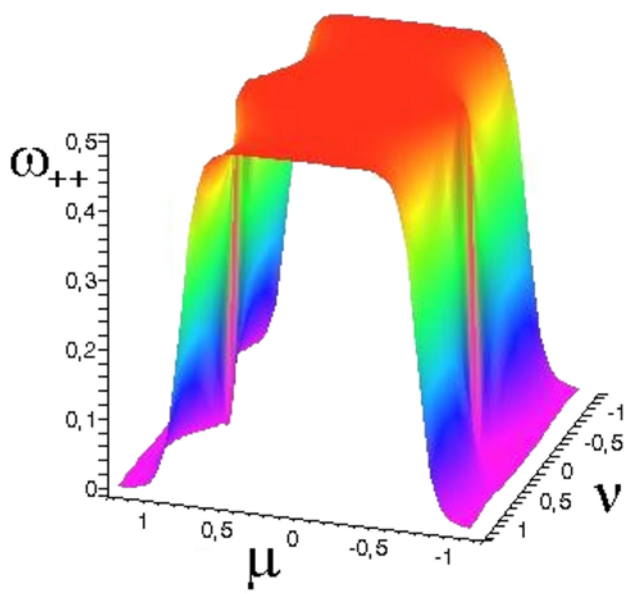

Fig. 6. Probability ω_{++} of the state $|\psi_-\rangle$ at $\alpha = \beta = 2$, $\varphi = \theta = 0$, $\mu_1 = \cos\mu$, $\mu_2 = \cos 2\mu$, $\nu_1 = \sin\nu$, and $\nu_2 = \sin 2\nu$.

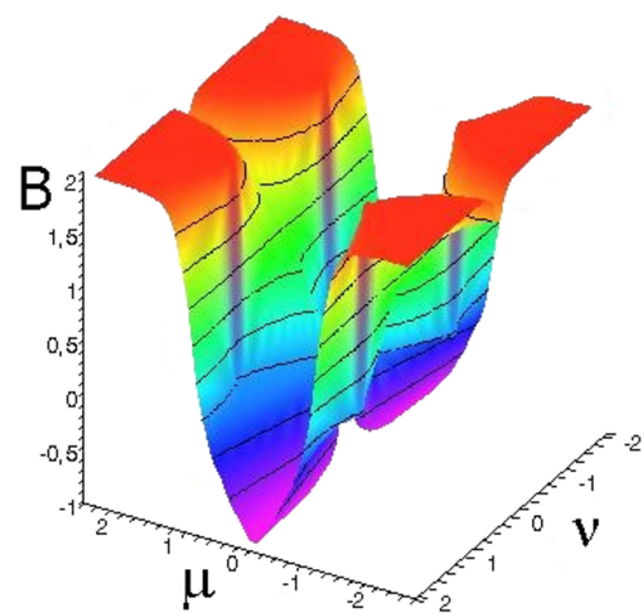

Fig. 7. Bell's parameter for the state $|\psi_-\rangle$ at $\alpha = \beta = 2$, $\varphi = \theta = 0$, $\mu_1 = |\mu|$, $\nu_1 = |\nu|$, $\mu_2 = \ln|\mu|$, $\nu_2 = \ln|\nu|$, $\mu_1' = \cos\mu$, $\nu_1' = \sin\nu$, $\mu_2' = \mu^2$, and $\nu_2' = \nu^2$.

The relative dispersion

$$\sigma^2 = \frac{\sum\limits_{m=0}^{\infty} m^2 P(m) - \left(\sum\limits_{m=0}^{\infty} m P(m)\right)^2}{\sum\limits_{m=0}^{\infty} m P(m)}$$

for the same amplitude is shown in Fig. 2. The distribution envelope following the Poisson distribution at $\varphi = 0$ (see Fig. 3a) narrows with increase in φ ($\varphi = \pi/128$) and its maximum shifts in the direction of lower m, which corresponds to the sub-Poissonian distribution at $\sigma < 1$ (see Fig. 3b). A further increase in φ ($\varphi = \pi/24$) leads to the super-Poissonian distribution at $\sigma > 1$ (see Fig. 3c), and then at ($\varphi = \pi/3$) a strong oscillating envelope arises (see Fig. 3d).

One can see that at $\varphi = 0$ the probability distribution strongly oscillates since the probability to find the odd photon state is equal to zero.

5.2. Two-Mode Even and Odd Schrödinger Cat States

In the two-mode case, the even and odd Schrödinger cat states $|\psi_+\rangle$ and $|\psi_-\rangle$ are described by the state vectors

$$|\psi_\pm\rangle = \frac{N_\pm}{\sqrt{2}} \left(\left|\alpha e^{i\varphi}\right\rangle \left|\beta e^{i\theta}\right\rangle \pm \left|-\alpha e^{-i\varphi}\right\rangle \left|-\beta e^{-i\theta}\right\rangle \right), \tag{16}$$

with the normalization constant

$$N_\pm = \left[1 \pm \exp\left(-\frac{|\alpha|^2 - |\beta|^2}{2} \right) \left[\exp\left(-\alpha^2 e^{-2i\varphi} - \beta^2 e^{-2i\theta} \right) + \exp\left(-\alpha^2 e^{2i\varphi} - \beta^2 e^{2i\theta} \right) \right] \right]^{-1/2},$$

where φ and θ are the phases and α and β are the positive amplitudes.

Being nonfactorized by construction, states (16) are entangled. The Wigner function of the state $|\psi_+\rangle$ shown in Fig. 4 consists of two peaks corresponding to two coherent states and an oscillating (interference) term located in the vicinity of the coordinate axis; namely this term is responsible for negative values of the Wigner function. Similar oscillations are observed for the state $|\psi_-\rangle$ (see Fig. 5).

For the state $|\psi_-\rangle$, the probability ω_{++} does not exceed 0.5 (see Fig. 6) and Bell's parameter calculated, in view of symplectic tomogram (5) and expressions (10) and (11), is not bigger than 2 in modulus (see Fig. 7). The maximum amplitude $B_{\max} = 1.97$. Thus, we showed that for reduced tomograms of even coherent states Bell's inequality is not violated.

A similar result was obtained for odd coherent states.

6. Conclusions

To conclude, we established that it is impossible to detect the entanglement of quantum Schrödinger cat states by checking the violation of Bell's inequalities, in view of the reduced probability distribution function obtained from tomograms of two-mode even and odd coherent states.

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