Journal of Russian Laser Research, Volume 27, Number 3, 2006

OSCILLATOR MODEL OF QUBITS AND ITS ENTANGLEMENT PROPERTIES

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Abstract

The two-mode symmetric oscillator is used to construct the qubit model as the superposition of the first excited degenerate level states of the oscillator. The entanglement properties of the oscillator states are studied using the known criterion of separability. Application to the quantum computing model based on light modes propagating in optical waveguides is briefly discussed.

Keywords: entanglement, qubits, tomographic probability, density matrix, oscillator.

1. Introduction

The quantum states of composite systems demonstrate essentially different properties in comparison with classical ones. These properties are connected with the presence of specific quantum correlations of composite systems. The quantum correlations are the feature of the entanglement phenomenon [1,2]. To date, the entanglement is the subject of intensive investigations [3–7]. States with continuous variables like the Gaussian states were studied in [8,9]. Different kinds of entanglement measure were suggested in [10–12].

The cases of discrete (spin) and continuous variables are different since in the first case one has a finite-dimensional Hilbert space of the states and in the second case one has a Hilbert space of infinite dimensionality. Nevertheless, one can combine the properties of continuous variables and spin variables by trying to model qubits (qudits) using continuous variables [13, 14]. Also there exists the scaling criterion to detect the entanglement for multimode Gaussian states. In spite of the fact that entanglement problems have been extensively studied, it is still not completely clarified both in the sense of finding an efficient entanglement criterion and the appropriate measure of entanglement that will be suitable for all situations.

The aim of this work is to construct several examples of qubits using the model of a two-mode oscillator. We study some empirical measures of entanglement and consider superposition states of independent oscillators that are obviously entangled. The coefficients of the superposition will be interpreted as qubits.

This paper is organized as follows.

In Sec. 2, we consider the phenomenon of entanglement. Section 3 is dedicated to constructing a qubit state that we will use later. In Secs. 4 and 5, we discuss notions of the density matrix and its tomogram representation and scaling transform of this matrix with reference to a qubit state. We run some of the most commonly used entanglement tests for qubit in Sec. 6. In Sec. 7, measures of entanglement for the observed state are proposed. Conclusions are presented in Sec. 8.

Manuscript submitted by the authors in English on March 5, 2006.

 1071-2836/06/2703-0204

2. Entanglement and Separability

Quite possibly, entanglement is one of the most interesting properties of quantum states, which are principally different from the classical ones. Entanglement is determined by the nonlocal correlations of quantum states of composed systems. These states play an important role in various quantum processes of data transmission and information processing. Though the concept of entanglement has been firstly introduced in 1935 by Schrödinger, it attracted attention only in the last decades (see, for example, [15] where the possibility to use it in the quantum teleportation of unknown quantum state in the two-level system from one point to the other was discussed). Also this idea was the essential ingredient of further research in quantum cryptography and quantum computing. The entanglement resource can be used in different applications in quantum technologies.

Now we give the definition of the described phenomena.

There are two different notions — separable states and entangled states. The mixed state of a bipartite system is called separable (see, for example, [16]) or nonentangled if it can be presented as a convex sum of the pure product states

$$\rho = \sum p_i \left|\psi_i\right\rangle \left\langle\psi_i\right| \otimes \left|\phi_i\right\rangle \left\langle\phi_i\right| , \tag{1}$$

where $|\psi_i\rangle$ and $|\phi_i\rangle$ are state vectors in the spaces H_A and H_B of the subsystems, respectively, and p_i are convex coefficients, i.e.,

$$p_i > 0, \qquad \sum_i p_i = 1.$$

Following [16], if a state admits such a decomposition, then it can be created using local operations; thus it cannot be an entangled state. For the general case of mixed state in the density-matrix representation, the density matrix of a composite system ρ is expressed in terms of the density matrices of subsystems $\rho_i^{(1)}$ and $\rho_i^{(2)}$

$$\rho = \sum p_i \rho_i^{(1)} \otimes \rho_i^{(2)}. \tag{2}$$

Equations (1) and (2) are quite simple, but the problem of distilling a given density matrix is a nonpolynomial-hard (NP-hard) problem. We recall that NP-hard problems are problems in the theory of complexity. The hypothesis that $P \neq NP$ is still not proved, where P is the class of all languages that can be recognized by a Turing machine working in polynomial time, and NP is the class of languages for non-polynomial Turing machines. Only P-class problems can be solved in polynomial time and, because we think that $P \neq NP$ is a strict inequality, we are unable to construct a common algorithm for determining the entanglement of mixed states as was the case for the pure states. Nevertheless, there are some cases where one can distinguish whether the state is separable or not.

When one constructs a criterion of separability, it is based on a simple property, which can be shown to hold for every separable state. This criterion provides a necessary but not sufficient condition for separability, but if the state does not satisfy the property, then it has to be an entangled one. But if the condition is fulfilled, this does not mean that the state under consideration is separable. Thus it is interesting to construct a number of tests to detect the separability.

For instance, there is a well-known criterion of separability — the positive partial transpose (PPT) criterion or Peres–Horodecki criterion [4, 5].

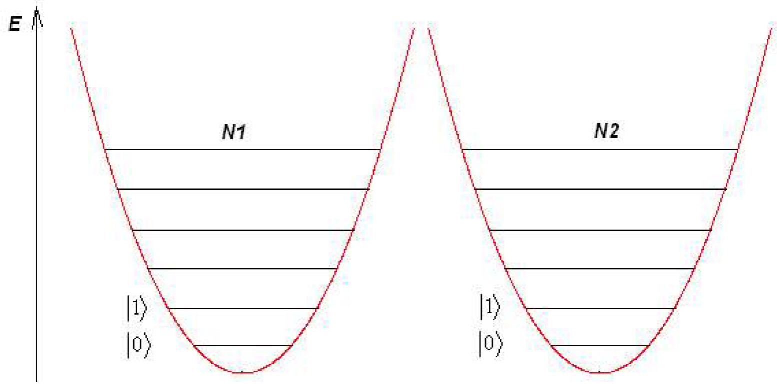

Fig. 1. Diagram of two potential wells for two independent quantum oscillators. The horizontal lines are equally spaced energy levels of the oscillators. The vertical axis corresponds to the energy.

3. Constructing the Qubit State

In this paper we consider a qubit constructed in the following way.

We take two first levels ($|0\rangle$ and $|1\rangle$) of two independent quantum oscillators (Fig. 1). The energy-conservation law in this case reads

$$N = N_1 + N_2 = 1,$$

where N_1 and N_2 are the energy levels of the first and second oscillators $N_1,\ N_2 = 0, 1, 2, \ldots$

It is common knowledge that the wave functions (we chose units with $\hbar = 1$, $m = 1$, and $\omega = 1$) of the harmonic oscillator are [17]

$$\langle q\,|n\rangle = \pi^{-1/4}\frac{1}{\sqrt{2^n n!}}\exp\left(-\frac{q^2}{2}\right)H_n\left(q\right), \tag{3}$$

where H_n(q) are Hermitian polynomials

$$H_n(\xi) = (-1)^n\, e^{\xi^2}\,\frac{d^n e^{-\xi^2}}{d\xi^n}\,. \tag{4}$$

The wave function of the ground state is a simple Gaussian

$$\langle q\,|0\rangle = \pi^{-1/4}e^{-q^2/2}. \tag{5}$$

The first excited level can be also constructed with the creation operator

$$\hat{a}^\dagger = \frac{\hat{q} - i\hat{p}}{\sqrt{2}} \tag{6}$$

and, in view of (3),

$$\langle q\,|1\rangle = \langle q|\hat{a}^\dagger\,|0\rangle = 2^{-1/2}\pi^{-1/4}qe^{-q^2/2}. \tag{7}$$

The entangled state is generated as follows:

$$|\psi\rangle = c_1\,|\varphi_1\rangle + c_2\,|\varphi_2\rangle\,, \tag{8}$$

where c_1 and c_2 are constants, in the general case, complex, and they satisfy the normalization condition

$$|c_1|^2 + |c_2|^2 = 1,$$

with $|\varphi_1\rangle = |0\rangle\,|1\rangle$ and $|\varphi_2\rangle = |1\rangle\,|0\rangle$ being the wave functions of the state. The first function in the product corresponds to the first oscillator, and the second function, to the second oscillator. If one of these constants is equal to zero and the other one is equal to one, we obtain the separable state. If $|c_1| = |c_2| = 2^{-1/2}$, then we have the Einstein–Podolsky–Rosen state which is totally entangled.

4. Density Matrix and Its Tomographic Representation

Consider our system with two canonical degrees of freedom with two pairs of canonical variables denoted by ξ_α ($\alpha = 1,2,3,4$), and the operators $\xi_\alpha = \{q_1, p_1, q_2, p_2\}$. The discussion is simplified since in our case (8)

$$\langle q_\alpha\rangle = 0, \quad \langle p_\alpha\rangle = 0, \quad \alpha = 1,2.$$

Nevertheless, if this is not so, we can always make a coordinate transformation to make the mean of the momentum and position equal to zero.

The next step is to construct a dispersion matrix

$$\sigma = \begin{pmatrix} \sigma_{q_1q_1} & \sigma_{q_1p_1} & \sigma_{q_1q_2} & \sigma_{q_1p_2} \\ \sigma_{p_1q_1} & \sigma_{p_1p_1} & \sigma_{p_1q_2} & \sigma_{p_1p_2} \\ \sigma_{q_2q_1} & \sigma_{q_2p_1} & \sigma_{q_2q_2} & \sigma_{q_2p_2} \\ \sigma_{p_2q_1} & \sigma_{p_2p_1} & \sigma_{p_2q_2} & \sigma_{p_2p_2} \end{pmatrix}$$

$$= \begin{pmatrix} \langle q_1^2\rangle & \langle q_1p_1 + p_1q_1\rangle/2 & \langle q_1q_2\rangle & \langle q_1p_2\rangle \\ \langle q_1p_1 + p_1q_1\rangle/2 & \langle p_1^2\rangle & \langle q_2p_1\rangle & \langle p_1p_2\rangle \\ \langle q_1q_2\rangle & \langle q_2p_1\rangle & \langle q_2^2\rangle & (\langle q_2p_2 + p_2q_2\rangle)/2 \\ \langle q_1p_2\rangle & \langle p_1p_2\rangle & \langle q_2p_2 + p_2q_2\rangle/2 & \langle p_2^2\rangle \end{pmatrix}. \tag{9}$$

The calculation of the matrix elements yields [17]:

$$\langle\xi_i\xi_j\rangle = \langle\psi|\,\xi_i\xi_j\,|\psi\rangle = \iint \psi^*(\xi)\xi_i\xi_j\psi(\xi)\,d\xi_i\,d\xi_j \tag{10}$$

and thus

$$\sigma = \begin{pmatrix} (|c_1|^2 + 3\,|c_2|^2)/2 & 0 & (c_1c_2^* + c_2c_1^*)/2 & -i(c_1c_2^* - c_2c_1^*)/2 \\ 0 & (|c_1|^2 + 3\,|c_2|^2)/2 & i(c_1c_2^* - c_2c_1^*)/2 & (c_1c_2^* + c_2c_1^*)/2 \\ (c_1c_2^* + c_2c_1^*)/2 & i(c_1c_2^* - c_2c_1^*)/2 & (3\,|c_1|^2 + |c_2|^2)/2 & 0 \\ -i(c_1c_2^* - c_2c_1^*)/2 & (c_1c_2^* + c_2c_1^*)/2 & 0 & (3\,|c_1|^2 + |c_2|^2)/2 \end{pmatrix}. \tag{11}$$

We can see some kind of symmetry in σ, which can be used to simplify further calculations.

A normalized matrix ρ can also be introduced; it is equal to normalized σ

$$\rho = \frac{\sigma}{N}, \tag{12}$$

where

$$N = \mathrm{Sp}\,(\sigma) = 4\left(|c_1|^2 + |c_1|^2\right). \tag{13}$$

Here we must prove that the obtained matrix ρ satisfies all properties of the density matrix, namely,

1. $\mathrm{Sp}\,(\rho) = 1$ — it is evident that after the normalization (12) this property is fulfilled;

2. $\rho^\dagger = \rho$ — simple calculations prove that this is true;

3. $\rho \geq 0$ — nonnegativity can be demonstrated with the nonnegativity of eigenvalues of ρ : $\forall c_1, c_2$, the eigenvalues are $\varepsilon = \left\{\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\right\}$.

Thus ρ is a density matrix indeed and all the entanglement criteria can be applied to it.

At this point, while setting aside for the time being the nature of ρ as a density matrix of continuous variables, let us interpret it as the density matrix of spin systems. Formally we are able to do this. The next step is to show that all our further calculations can also be done in the tomographic presentation of the density state ρ. In [18, 19] the spin tomogram of a qubit state was introduced. The unitary spin tomogram is defined as the joint probability distribution w of spin projections m_1 and m_2 depending on the unitary matrix elements $U \in U(4)$.

The rotation matrix in $3D$-space is described by the $SU(2)$-matrix

$$u_i = \begin{pmatrix} \cos(\theta_i/2)e^{i(\varphi_i+\psi_i)/2} & \sin(\theta_i/2)e^{-i((\varphi_i-\psi_i)/2} \\ -\sin(\theta_i/2)e^{i(\varphi_i-\psi_i)/2} & \cos(\theta_i/2)e^{-i(\varphi_i+\psi_i)/2} \end{pmatrix}. \tag{14}$$

For the two states, we take the tensor products

$$U(4) = u_1 \otimes u_2, \qquad U^\dagger(4) = u_1^\dagger \otimes u_2^\dagger, \tag{15}$$

where $\otimes$ denotes a tensor product.

The unitary spin tomogram of a state of two qubits with the density matrix ρ is determined by the diagonal elements of the matrix

$$\rho_u \equiv U^\dagger(4)\rho U(4). \tag{16}$$

It can be written as follows:

$$w_\rho(m_1, m_2, U) = \langle m_1, m_2|\, U^\dagger \rho U\, |m_1, m_2\rangle\,. \tag{17}$$

This tomogram completely determines ρ, and it is obvoius that ρ can be obtained from ρ_u by putting $\theta_i = \varphi_i = \psi_i = 0$.

5. Scaling Transform

We continue by presenting a nomenclature for further observing the entanglement properties of previously constructed state (8). If we define a rescaling of momentum in a system with continuous variables, then

$$\begin{aligned} x &\to x, \\ p &\to \lambda p. \end{aligned} \tag{18}$$

For $\lambda \in [-1, 1]$ it defines a semigroup of maps, which are not canonical almost everywhere. It is almost evident that the map (18) is physically equivalent to a rescaling of time $t \to \lambda t$, or $\hbar \to \lambda\hbar$, or $i \to \lambda i$. In the case $\lambda = 1$, it is the identical map, and if $\lambda = -1$ it reduces to time reversal [20]. The point is that (18) is not a canonical map, in general, and thus it plays a fundamental role in the detection of entanglement. Now, first let us look at the consequences of partial scaling on the dispersion matrix of a two-mode quantum state. We change the scale of the momentum variable of the second subsystem by a factor of λ ($|\lambda| \leq 1$), i.e.,

$$p_2 \to \lambda p_2. \tag{19}$$

We will call λ the scaling parameter.

The following notation will be used further [see (11)]:

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{43} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} & \sigma_{14} \\ 0 & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 \\ \sigma_{41} & \sigma_{42} & 0 & \sigma_{43} \end{pmatrix}. \tag{20}$$

In view of (19), the dispersion matrix (20) transforms into the following one:

$$\sigma^S = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \lambda\sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \lambda\sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \lambda\sigma_{34} \\ \lambda\sigma_{41} & \lambda\sigma_{42} & \lambda\sigma_{43} & \lambda^2\sigma_{43} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & 0 & \sigma_{13} & \lambda\sigma_{14} \\ 0 & \sigma_{22} & \sigma_{23} & \lambda\sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 \\ \lambda\sigma_{41} & \lambda\sigma_{42} & 0 & \lambda^2\sigma_{43} \end{pmatrix}. \tag{21}$$

Similar to (12) we construct a normalized version of σ^S

$$\rho^S = \frac{\sigma^S}{N^S}\,, \tag{22}$$

where

$$N^S = Sp(\sigma^S) = \left(\frac{5}{2} + \frac{3\lambda^2}{2}\right)|c_1|^2 + \left(\frac{7}{2} + \frac{\lambda^2}{2}\right)|c_2|^2\,. \tag{23}$$

The spin tomogram of (22) is determined by the matrix

$$\rho_u^S \equiv U^\dagger(4)\rho^S U(4). \tag{24}$$

Now we have the whole apparatus for running the entanglement tests.

6. Entanglement Tests

In this section, we will show what is obtained if some of the most commonly used entanglement tests are implemented to the qubit state constructed. While processing these tests some summary will be made.

6.1. Robertson–Schrödinger Inequality

The first idea of detecting nonseparability in our work is to check whether the criterion of separability based on the Robertson–Schrödinger [21] uncertainty relation works in our case. This criterion is applied to the matrix obtained by considering continuous variables.

For one degree of freedom, applying a canonical transformation leads to the Schrödinger uncertainty relation in a simple form [21] (as previously, $\hbar \equiv 1$)

$$\left\langle (q - \langle q\rangle)^2 \right\rangle \left\langle (p - \langle p\rangle)^2 \right\rangle - \left\langle \frac{qp + pq}{2} - \langle q\rangle \langle p\rangle \right\rangle^2 \geq \frac{1}{4}. \tag{25}$$

If there are no correlations between $\mathbf{q}$ and $\mathbf{p}$, we have the usual Heisenberg uncertainty relation

$$\left\langle (q - \langle q\rangle)^2 \right\rangle \left\langle (p - \langle p\rangle)^2 \right\rangle \geq \frac{1}{4}. \tag{26}$$

For simplicity of discussion we write (25) in the form

$$\det C = \det \left(V + \frac{i}{2}\Omega \right), \tag{27}$$

where

$$V = \begin{pmatrix} \langle q^2\rangle & \langle qp + pq\rangle/2 \\ \langle qp + pq\rangle/2 & \langle p^2\rangle \end{pmatrix} = \begin{pmatrix} \sigma_{qq} & \sigma_{qp} \\ \sigma_{pq} & \sigma_{pp} \end{pmatrix} \tag{28}$$

and

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

In the general case of N degrees of freedom,

$$V_{\alpha\beta} = \frac{1}{2} \langle \{\xi_\alpha, \xi_\beta\}\rangle, \qquad \alpha, \beta = 1, 2, \ldots 2N. \tag{30}$$

Thus the condition can be written as follows:

$$C_{\alpha\beta} = V_{\alpha\beta} + \frac{i}{2}\Sigma_{\alpha\beta} \geq 0, \tag{31}$$

where $\Sigma = \text{diag}\,(\Omega, \Omega, \ldots, \Omega)$.

For the pure and mixed states, it is easy to show that

$$\det V \geq \frac{1}{4^N}. \tag{32}$$

Inequality (31) is always true for V representing a physical state, so it can be used to detect the entanglement. While $C_{\alpha\beta}$ (and, of course, $V_{\alpha\beta}$) are invariants of symplectic transformations, they are not invariant under scaling of ξ_α.

While we use scaled values, we will use index S. For example, scaled V is V^S. A scaled representation of (31) has the consequence

$$\det \left(V^S + \frac{i}{2}\Sigma \right) \geq 0, \tag{33}$$

and Eq. (33) reduces to the following:

$$A\lambda^2 + B\lambda + C \geq 0. \tag{34}$$

It was shown in [21] that in Eq. (34)

$$A = \det V - \frac{1}{4}(\sigma_{33}\sigma_{44} - \sigma_{34}^2), \quad B = \frac{1}{2}\left(\sigma_{41}\sigma_{23} - \sigma_{13}\sigma_{24}\right), \quad C = \frac{1}{16} - \frac{1}{4}\left(\sigma_{11}\sigma_{22} - \sigma_{12}^2\right). \tag{35}$$

Equation (34) is always true if

$$B^2 - 4AC \geq 0. \tag{36}$$

Written in block matrix form

$$V^S + \frac{i}{2}\Sigma = \begin{pmatrix} V_1 & V_{12} \\ V_{12}^T & V_2 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} \Omega & 0 \\ 0 & \Omega \end{pmatrix}. \tag{37}$$

So the condition (36) can be expressed as follows:

$$|\det V_{12}|^2 - (4\det V - \det V_2)\left(\frac{1}{4} - \det V_1\right) \leq 0. \tag{38}$$

Inequality (38) can be used to detect entangled states of continuous variables.

Let us see what will we get if (33) [or (38)] is applied to our prepared state. Our mixed configuration (8) describes a physical state and thus we must have (33). The most interesting conditions are the following ones:

(i) Separable states $\{|c_1|=1,\ c_2=0\}$ and $\{c_1=0,\ |c_2|=1\}$;

(ii) Fully entangled state $\{|c_1| = 1/\sqrt{2}, |c_2| = 1/\sqrt{2}\}$.

So one has

$\det\left(\sigma + \frac{i}{2}\Sigma\right)$:

$\lvert c_1\rvert = 1,\ c_2 = 0$	$\lvert c_1\rvert = 0,\ c_2 = 1$	$\lvert c_1\rvert = 1/\sqrt{2}$ $\lvert c_2\rvert = 1/\sqrt{2}$
0	0	0

This determinant is equal to zero for all $c_1, c_2 \in$ C, which was predicted.

Changing the scale of the momentum variable of the second subsystem (19) we get a new matrix, which is also tested by (33). For example, for $\lambda = -1$,

$\det\left(\sigma^S + \frac{i}{2}\Sigma\right)$:

$\lvert c_1\rvert = 1,\ c_2 = 0$	$c_1 = 0,\ \lvert c_2\rvert = 1$	$\lvert c_1\rvert = 1/\sqrt{2}$ $\lvert c_2\rvert = 1/\sqrt{2}$
0	0	$0.25 > 0$

Quite the same result $\det\left(\sigma^S + \frac{i}{2}\Sigma\right) \geq 0$ is obtained for $\forall c_1, c_2 \in$ C and $\forall\,|\lambda| \leq 1$.

As an example, we can show that the same result is obviously obtained for the tomographic representation (where, for instance, $\theta_1 = \pi/6$, $\theta_2 = \pi/8$, $\varphi_1 = \pi/4$, $\varphi_2 = \pi/6$, $\psi_1 = \pi/8$, and $\psi_2 = \pi/9$), namely,

$\det\left(\rho_u^S + \frac{i}{2}\Sigma\right)$:

$\lvert c_1\rvert = 1,\ c_2 = 0$ $\lambda = 1$	$c_1 = 0,\ \lvert c_2\rvert = 1$ $\lambda = 1$	$\lvert c_1\rvert = 1/\sqrt{2},\ \lvert c_2\rvert = 1/\sqrt{2}$ $\lambda = 1$
$\approx 0.0256348 > 0$	$\approx 0.0256348 > 0$	$\approx 0.0256348 > 0$

$\det\left(\rho_u^S + \frac{i}{2}\Sigma\right)$:

$\lvert c_1\rvert = 1,\ c_2 = 0$ $\lambda = -1$	$c_1 = 0,\ \lvert c_2\rvert = 1$ $\lambda = -1$	$\lvert c_1\rvert = 1/\sqrt{2},\ \lvert c_2\rvert = 1/\sqrt{2}$ $\lambda = -1$
$\approx 0.0256348 > 0$	$\approx 0.0256348 > 0$	$\approx 0.0406877 > 0$

We have also shown that $\det\left(\rho_u^S + \frac{i}{2}\Sigma\right) \geq 0$ for any angles. But if entanglement is detected, this determinant should be less than zero. A possible explanation is that our state is not completely described by second moments only, and the Robertson–Schrödinger criterion for higher moments should be tried. Such results mean that the chosen criterion (in the form used) cannot be implemented for the detection of entanglement in our particular case, and thus we must consider other criteria.

6.2. Partial Transpose Criterion

We concentrate on the density matrix ρ obtained using the superposition state of two oscillators, which will be considered as the density matrix of the spin system. Taking the transpose of a matrix is a positive, but not completely positive map [20, 22]:

$$T : \rho \to \rho^T, \qquad \rho \geq 0 \Rightarrow \rho^T \geq 0. \tag{39}$$

According to the Peres–Horodecki criterion [4], relation (39) provides a necessary condition for a state of a bipartite system to detect the entanglement. As noted previously, this map is equivalent to time reversal.

Suppose that we have a bipartite spin system

$$S = S_1 \times S_2,$$

which lets us distill our whole system in Hilbert space as a tensor product

$$H = H_1 \times H_2,$$

where H_1 and H_2 are Hilbert subspaces, and the spin system is described as the density matrix, in view of (11),

$$\zeta = \rho \geq 0. \tag{40}$$

The distilling of Hilbert space as a tensor product induces the representation

$$\zeta \equiv \zeta_{a\alpha,b\beta},$$

where Latin indices refer to the first system and Greek indices to the second one. In the next step, the operation of partial transpose is introduced, which consists in performing the transposition operation of the second system only

$$T_2 = I \otimes T : \zeta \equiv \zeta_{a\alpha,b\beta} \to \zeta^{T_2} \equiv \zeta_{a\beta,b\alpha}. \tag{41}$$

So (41) is constructed as a partial time reversal of just the second subsystem; thus, if the system is separable, then (41) is transformed into a positive matrix; otherwise, into a nonpositive one.

In the case $\forall c_1, c_2 \in C$, we have that $\zeta^{T_2} > 0$ (the eigenvalues of ζ^{T_2} are quite the same as for ζ, i.e., $\varepsilon = \{\frac{1}{8}, \frac{1}{8}, \frac{3}{8}, \frac{3}{8}\}$). This mean that the constructed matrix is separable for $\forall c_1, c_2 \in C$ and does not preserve information on the entanglement in the ground state of the oscillator function (8).

It might be interesting to analyze the entanglement properties of other matrices based on (11) whose origin is empirical. The results of a trial-and-error method will be demonstrated.

First of all, we consider the matrix

$$\zeta = \frac{\sigma + \frac{\imath}{2}\Sigma}{\mathrm{Sp}\,\sigma}. \tag{42}$$

The eigenvalues of this matrix are

$$\begin{aligned} \varepsilon_1 &= \frac{1}{8}(1 - \frac{1}{|c_1|^2 + |c_2|^2}) = 0 \geq 0, \\ \varepsilon_2 &= \frac{1}{8}(1 + \frac{1}{|c_1|^2 + |c_2|^2}) = \frac{1}{4} \geq 0, \\ \varepsilon_3 &= \frac{1}{8}(3 - \frac{1}{|c_1|^2 + |c_2|^2}) = \frac{1}{4} \geq 0, \\ \varepsilon_4 &= \frac{1}{8}(3 + \frac{1}{|c_1|^2 + |c_2|^2}) = \frac{1}{2} \geq 0; \end{aligned} \tag{43}$$

thus (42) satisfy the properties of the density matrix. After the partial transpose, the derived matrix ζ^{T_2} has quite the same eigenvalues (43) as matrix (42) but does not depend on c_1, c_2, and there is no information on the state (8).

The next matrix to investigate is

$$\zeta = \rho + \frac{\imath}{2}\Sigma. \tag{44}$$

But there are negative eigenvalues of (44), i.e., $\varepsilon = \left\{-\frac{3}{8}, \frac{5}{8}, -\frac{1}{8}, \frac{7}{8}\right\}$, which is why matrix (44) cannot be considered as the density matrix, because it is negatively defined. Nevertheless, further it will be shown that some information on the entanglement properties of the initial state (8) can be obtained.

6.3. Scaling transform

In [20] it was shown that time scaling is equivalent to $i \to \lambda i$. According to the results of the previous subsection, it can be predicted that for nonentanglement of (8) using spin-like matrices ρ and $(\mathrm{Sp}\,\sigma)^{-1}\left(\sigma + \frac{\imath}{2}\Sigma\right)$ will be detected but, nevertheless, the scaling transform will be observed. For the case of a qubit generic state written as

$$\chi = \frac{1}{2}\begin{pmatrix} 1+z & x-iy \\ x+iy & 1-z \end{pmatrix}, \tag{45}$$

scaling can be defined as follows:

$$\chi \to T_\lambda \chi = \frac{1}{2}\begin{pmatrix} 1+z & x-i\lambda y \\ x+i\lambda y & 1-z \end{pmatrix}. \tag{46}$$

The matrix representation of map (46) reads

$$T_\lambda \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(1+\lambda) & \frac{1}{2}(1-\lambda) & 0 \\ 0 & \frac{1}{2}(1-\lambda) & \frac{1}{2}(1+\lambda) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{47}$$

That is why T_λ transforms the vector

$$\vec{\chi} = \begin{pmatrix} \chi_{11} \\ \chi_{12} \\ \chi_{21} \\ \chi_{22} \end{pmatrix}, \tag{48}$$

which corresponds to (41)

$$\chi = \begin{pmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix}, \tag{49}$$

into a new one

$$\vec{\chi}_\lambda = \begin{pmatrix} \chi_{11} \\ \{[(1+\lambda)/2]\chi_{12}\} + \{[(1-\lambda)/2]\chi_{21}\} \\ \{[(1+\lambda)/2]\chi_{21}\} + \{[(1-\lambda)/2]\chi_{12}\} \\ \chi_{22} \end{pmatrix}; \tag{50}$$

thus

$$\chi_\lambda = \begin{pmatrix} \chi_{11} & \{[(1+\lambda)/2]\chi_{12}\} + \{[(1-\lambda)/2]\chi_{21}\} \\ \{[(1+\lambda)/2]\chi_{21}\} + \{[(1-\lambda)/2]\chi_{12}\} & \chi_{22} \end{pmatrix}. \tag{51}$$

Note that T_λ can be written as a convex sum of the identity map and the transposition [20]

$$T_\lambda = \frac{1+\lambda}{2} I + \frac{1-\lambda}{2} T, \tag{52}$$

where T_λ is a positive map operation.

Let us see what results will be obtained if this map is applied to the matrix

$$\rho_\lambda = T_\lambda \rho, \tag{53}$$

$$\left(\frac{\sigma + \frac{\imath}{2}\Sigma}{Sp\,\sigma} \right)_\lambda = T_\lambda \left(\frac{\sigma + \frac{\imath}{2}\Sigma}{Sp\,\sigma} \right). \tag{54}$$

Our calculation shows that for $\forall c_1, c_2 \in \mathrm{C}$ and $\forall\,|\lambda| \leq 1$, maps (53) and (54) are nonnegative, and this criterion cannot be used.

We employed a trial-and-error method for checking the entanglement properties of the observed state. The result of our attempts is that none of the applied criteria can detect the entanglement of the initial-state function (8) but, nevertheless, the entanglement measure can still be constructed and its value can tell us whether the state in our particular case is separable or not.

7. Measure of Entanglement

One of the most important concepts in entanglement theory is measure that is used as a unit of the entanglement. Now we will review the commonly accepted set of properties that all measures of entanglement should share. For a general density matrix ρ, which can be divided into two or more subsystems, the quantity $M_x(\rho)$ (the label x is used to denote a generic measure) qualifies as an entanglement monotone if it satisfies the following conditions [4, 5, 16]:

1. $M_x(\rho) \geq 0,\ M_x(\rho) = 0$, if ρ is separable.

2. $M_x(\rho)$ is not increased on the average by local operations and classical communication. For example, with any state ρ and partition $\{A, B\}$ local unitary transformations $\hat{U} = \hat{U}_A \otimes \hat{U}_B$ do not affect $M_x(\rho)$ (see [4]).

3. The entanglement measure should satisfy inequality $\sum_i p_i M_x(\rho_i) \geq M_x(\sum_i p_i \rho_i)$.

In this section, we review and use several entanglement measures. In addition to computing these entanglement quantities, we introduce a new measure for (8). We will not consider measures having the nature of entropy because this is not our task in this paper.

(a) A measure taken in some articles reads

$$M(\lambda) = \sum_{k=1}^{4} \left(\left| \rho^S_{u,kk} \right| - \rho^S_{u,kk} \right) \tag{55}$$

[see (24)] and because

$$\sum_k \rho^S_{u,kk} = \mathrm{Sp}\, \rho^S_u = 1,$$

measure (55) can be simplified

$$M(\lambda) = \sum_{k=1}^{4} \left| \rho^S_{u,kk} \right| - 1. \tag{56}$$

Our calculations showed that $M(\forall\lambda) \equiv 0$ and this measure cannot be used for characterizing the entanglement.

(b) We can consider another type of measure, which is very close to spin measures. Consider the matrix [see (42)]

$$\zeta^{T_2}_u = U^\dagger(4) \zeta^{T_2} U(4). \tag{57}$$

Then the measure reads

$$M = \sum_{k=1}^{4} \left| \zeta^{T_2}_{u,kk} \right| - 1. \tag{58}$$

We have made calculations for all cases of c_1, c_2 and obtained the result that $M \equiv 0$. Thus using this measure for this case failed.

(c) We failed in the previous attempts of taking measures but there is one that seems suitable for the entanglement measure and satisfies the properties for being it. Though it is empirical, it satisfies all the properties of the measure and correctly behaves in our case of the oscillator qubit model. The form of the measure is

$$M(\lambda) = \sum_{k=1}^{4} |\varepsilon_k| - 2, \tag{59}$$

where ε_k are eigenvalues of our matrix $\rho_u^S + \frac{i}{2}\Sigma$ [see (24)]. We calculated these eigenvalues in the explicit form for $\lambda = -1$

$$\begin{aligned}
\varepsilon_1 &= \frac{1}{4} - \frac{\sqrt{25|c_1|^4 + 9|c_2|^4 + 34|c_1|^2|c_2|^2}}{8(|c_1|^2 + |c_2|^2)}, \\
\varepsilon_2 &= \frac{1}{4} + \frac{\sqrt{25|c_1|^4 + 9|c_2|^4 + 34|c_1|^2|c_2|^2}}{8(|c_1|^2 + |c_2|^2)}, \\
\varepsilon_3 &= \frac{1}{4} - \frac{\sqrt{9|c_1|^4 + 25|c_2|^4 + 34|c_1|^2|c_2|^2}}{8(|c_1|^2 + |c_2|^2)}, \\
\varepsilon_4 &= \frac{1}{4} + \frac{\sqrt{9|c_1|^4 + 25|c_2|^4 + 34|c_1|^2|c_2|^2}}{8(|c_1|^2 + |c_2|^2)}.
\end{aligned} \tag{60}$$

We have shown that (where, for example, $\theta_1 = \pi/6$, $\theta_2 = \pi/8$, $\varphi_1 = \pi/4$, $\varphi_2 = \pi/6$, $\psi_1 = \pi/8$, $\psi_2 = \pi/9$, as it was considered before) for $\lambda = -1$, measure (57) is

$\|c_1\| = 1$, $c_2 = 0$	$c_1 = 0$, $\|c_2\| = 1$	$\|c_1\| = 0.6$, $\|c_2\| = 0.8$	$\|c_1\| = 1/\sqrt{2}$, $\|c_2\| = 1/\sqrt{2}$
0	0	0.0551029	0.0594582

We also calculated that, if $|c_1| = 1/\sqrt{2}$ and $|c_2| = 1/\sqrt{2}$, M has a maximum equal to 0.0594582 and a minimum equal to zero when $|c_1| = 1$ and $c_2 = 0$ or $c_1 = 0$ and $|c_2| = 1$.

In Figs. 2, 3, and 4 some plots of M(c_1,c_2) are presented in different projections for angles $\theta_1 = \pi/6$, $\theta_2 = \pi/8$, $\varphi_1 = \pi/4$, $\varphi_2 = \pi/6$, $\psi_1 = \pi/8$, and $\psi_2 = \pi/9$ for real c_1 and c_2 satisfying the normalization condition $|c_1|^2 + |c_2|^2 = 1$. It is also clear, that for any other angles taken, the common course of the plots (and also the measure behavior) will be identical.

Even in this step it is evident that this measure is a correct measure for characterizing the entanglement properties. Our calculations showed that we can obtain the same results for $\forall c_1, c_2 \in \mathrm{C}$ and $\forall |\lambda| \leq 1$. Moreover, as can be seen from the plots, we have the maximum of (59) in the case of totally entangled states and the minimum in the case of separable states. Also it is true that the more entangled state corresponds to the higher value of measure and, correspondingly, the less entangled state corresponds to the lower value of measure.

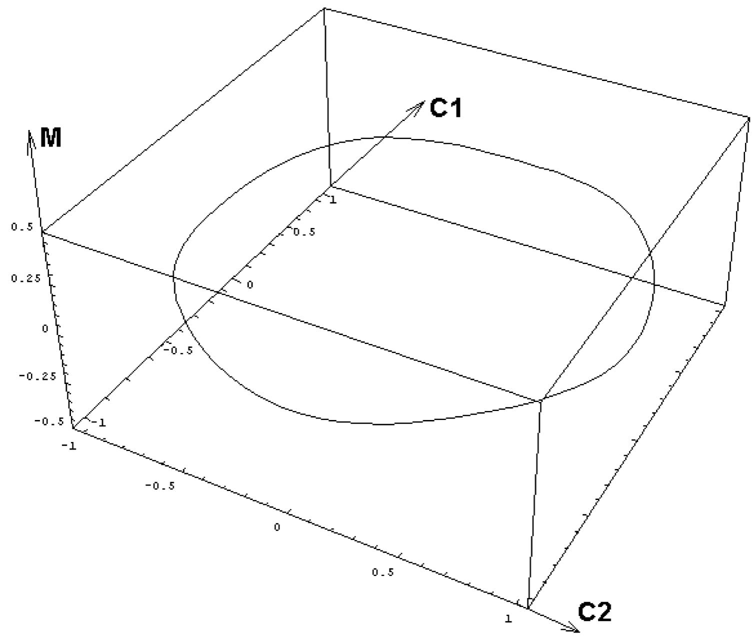

Fig. 2. $M(c_1, c_2)$-isometric projection.

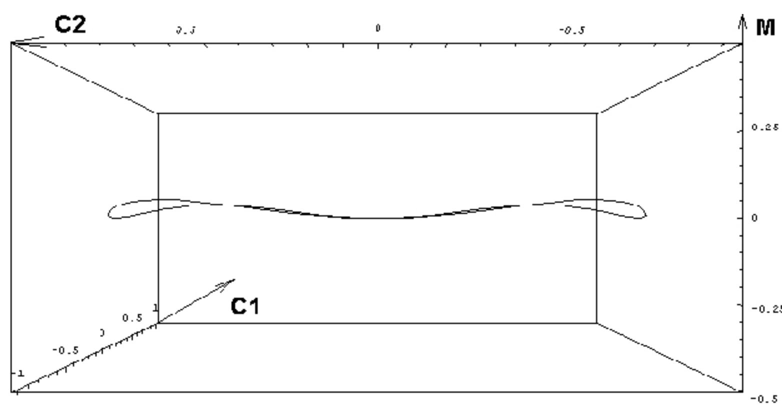

Fig. 3. $M(c_1, c_2)$-front projection.

8. Conclusions

To conclude, we summarize the main results of the paper.

We considered a very simple example of the entangled superposition state of a two-mode oscillator. We interpret the coefficients of the superposition as qubit. By construction, the state under consideration is an entangled state of two continuous variables. We checked whether the known criteria of entanglement can detect the entanglement. The result turned out to be negative. Thus both the Peres–Horodecki criterion

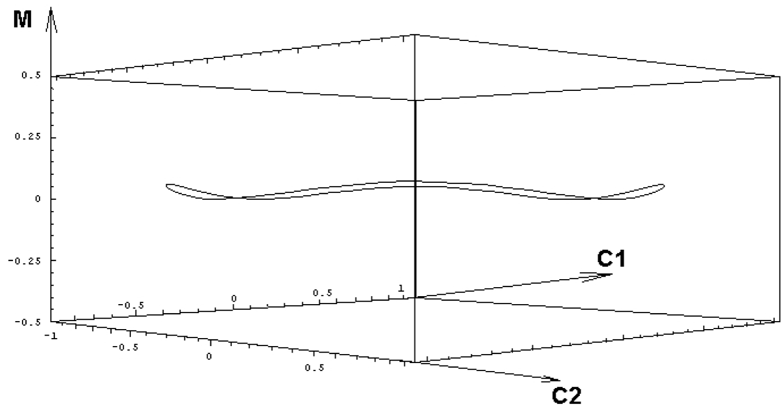

Fig. 4. $M(c_1, c_2)$-isometric projection.

[3–5] and partial scaling criterion [21, 23] did not detect the entanglement. Some ansatz is suggested to study the properties of qubits made using the two-mode oscillator states. One could consider the model of qubits constructed by a two-mode oscillator on the basis of the classical modes of electromagnetic radiation propagating along an optical waveguide [24]. In fact, electromagnetic waves in optical waveguides are described by a Schrödinger-like equation due to the Fock–Leontovich paraxial approximation [25,26]. This property was suggested [24] for use in modeling some elements of quantum computing. The entanglement properties are important for this purpose. We hope that the analysis of qubits constructed in the present work may be useful for a deeper understanding of the possibilities of using optical fibers in the context of modeling some elements of quantum computing.

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