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The Well-Posedness and Discontinuous Galerkin Approximation for the Non-Newtonian Stokes–Darcy–Forchheimer Coupling System

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Abstract

We study the non-Newtonian Stokes–Darcy–Forchheimer system modeling the free fluid coupled with the porous medium flow with shear/velocity-dependent viscosities. The unique existence is proved by using the theory of nonlinear monotone operator and a coupled inf-sup condition. Moreover, we apply the discontinuous Galerkin (DG) method with P^k/P^{k-1}-DG element for numerical discretization and obtain the well-posedness, stability, and error estimate. For both the continuous and the discrete problem, we explore the convergence of the Picard iteration (or called Kacǎnov method). The theoretical results are confirmed by the numerical examples.

Keywords Non-Newtonian flow · Stokes–Darcy–Forchheimer · Nonlinear monotone theory · Discontinuous Galerkin method · Error estimates · Picard iteration

1 Introduction

The Stokes–Darcy system modeling the interaction between the free Newtonian fluid and porous medium flow finds a wide application in engineering simulations. The mathematical and numerical analysis of this model has been extensively studied in recent years. However, many real-world fluid problems involve the non-Newtonian fluid (e.g., molten

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plastics, biological fluids, and engine oils with polymeric additives), and the Newtonian models are not suitable to apply. Several models exist for non-Newtonian free fluids, as discussed in [26]. These models include the Stokes/Navier–Stokes equations with varying types of shear-dependent viscosity (commonly referred to as the generalized Newtonian flow), the Oldroyd-B model, and the Peterlin viscoelastic model, among others. When dealing with non-Newtonian flow in a porous medium, it is generally agreed that the Darcy-Forchheimer model [13], which takes into account viscosity that varies with velocity, is a more suitable approach than the Darcy system. In this paper, we focus on the Stokes–Darcy–Forchheimer system with shear/velocity-dependent viscosities for the non-Newtonian fluid flow passing through the coupled porous medium region, which has massive applications in real-world simulation [17, 24], such as industrial filtering, plasma separation from blood, and so on.

There exist numerous works on numerical methods for coupled fluid models. We only select a few to introduce in the following, which we think are closely related to our model. For the non-Newtonian Stokes–Darcy equations with Beavers–Joseph–Saffman(BJS) interface condition, the finite element method (FEM) is proposed and analyzed [10], as well as the mortar FEM [11] and domain decomposition approach [19]. For the Newtonian (Navier–)Stokes–Darcy–Forchheimer model, the well-posedness and numerical discretization have been studied in [1, 2, 6, 7, 9], including the FEM, the fully-Mixed FEM, and a study on the residual-based a posteriori error estimates. The discontinuous Galerkin (DG) method satisfying the local conservation law is popular for coupled fluid computation. For the Stokes–Darcy system, [27, 29, 33] employs the DG method and mixed FEM for discretization, while [35, 36] adopts the DG scheme combined with the penalty and Nitsche's approaches to treating the interface condition. In the present work, we apply the DG method for the non-Newtonian Stokes–Darcy–Forchheimer model with shear/velocity-dependent viscosities. To our knowledge, the well-posedness and numerical analysis have not been investigated for this nonlinear system.

It is important to note that various non-Newtonian fluids exhibit varying types of viscosity that are dependent on shear. In this article, we will provide some models for viscosity commonly utilized in describing biological fluids, paints, and similar substances [8, 10]. We denote by $\boldsymbol{D}(\boldsymbol{u}) := (\nabla \boldsymbol{u} + \nabla^\top \boldsymbol{u})/2$ the deformation tensor of fluids, and by $g_1(|\boldsymbol{D}(\boldsymbol{u})|)$ the dynamic viscosity [31] as a function of $\boldsymbol{D}(\boldsymbol{u})$ expressed as

$$g_1(|\boldsymbol{D}(\boldsymbol{u})|) = \nu_\infty + (\nu_0 - \nu_\infty) G(\sqrt{2}|\boldsymbol{D}(\boldsymbol{u})|) \qquad (\nu_0 > \nu_\infty > 0). \tag{1.1}$$

The effective viscosity $g_2(\cdot)$ [10, 25] for the non-Newtonian porous media flow is determined by

$$g_2(|\boldsymbol{u}|) = \nu_\infty + (\nu_0 - \nu_\infty) G(\sqrt{2}|\boldsymbol{u}|) \qquad (\nu_0 > \nu_\infty > 0). \tag{1.2}$$

We listed some choices of $G(s)$ in Table 1.

The strong nonlinearity of $g_1(\cdot)$ and $g_2(\cdot)$ (see the above models), together with the Forchheimer term $C_F|\boldsymbol{u_2}|\boldsymbol{u_2}$ (see (2.2a)), lead to analytical difficulties. The motivation of the present paper is to develop the well-posedness and numerical analysis for the Stokes–Darcy–Forchheimer model with shear/velocity-dependent viscosities. For the PDE model, we utilize the monotone theory [15, 23] to show the well-posedness. And we demonstrate a coupled inf-sup condition (see Lemma 2.4) to show the existence of pressure. The Picard iteration (or called Kačanov method) is applied to solve the nonlinear coupled system, and we develop the convergence theorems under different assumptions on $g_1(\cdot)$ and $g_2(\cdot)$.

For numerical approximation, we adopt the DG method with P^k/P^{k-1}-DG element $(k = 1, 2)$ for the velocity and pressure in both the Stokes and Darcy–Forchheimer regions.

Table 1 The models for the dynamic viscosity of the non-Newtonian flow

Model	$G(s)$	
Carreau	$(1+(Ks)^2)^{(r-1)/2}$	$0 \le r < 1$
Carreau–Yasuda	$(1+(Ks)^a)^{(r-1)/a}$	$a > 0, 0 \le r < 1$
Cross	$(1+(Ks)^r)^{-1}$	$0 < r \le 1$
Modified cross	$(1+(Ks)^r)^a$	$r > 0, a < 0, ar + 1 \ge 0$
Powell–Eyring	$\sinh^{-1}(Ks)/(Ks)$	

We show the well-posedness and a-priori estimates, and also investigate the convergence of the Picard iteration for the nonlinear discrete problem. For error analysis, we introduce the Lagrange multiplier formulation (cf. [21]) using P^k-DG element for the Lagrange multiplier, and obtain the error estimate $O(h^k)$ for both velocity and pressure. Several numerical experiments are provided to confirm the theoretical results.

The rest of this article is arranged as follows. In Sect. 2, we derive the variational form of the coupled system and show the well-posedness. Moreover, we prove two convergence theorems for the Picard iteration. The DG scheme is presented in Sect. 3. We demonstrate the well-posedness, the convergence of Picard iteration, and the error estimates. Section 4 is devoted to numerical experiments.

2 The PDE Model and the Well-Posedness

2.1 The PDE Model and Notations

Let Ω be an open smooth bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ consisting of the free fluid region Ω_1 and porous medium region Ω_2 separated by the interface Γ. We denote by $\boldsymbol{n}$ the unit outward normal vector to $\partial\Omega$, and by $\boldsymbol{n_{12}}$ the unit normal vector to Γ outward Ω_2, and set $\Gamma_i := \partial\Omega_i \backslash \Gamma$ $(i = 1, 2)$ and $\boldsymbol{n_{21}} := -\boldsymbol{n_{12}}$. The velocities and pressures of the fluids in (Ω_1, Ω_2) are denoted by $\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2})$ and $p = (p_1, p_2)$. The model below represents the coupled non-Newtonian flow in Ω_1 and Ω_2.

The free flow in Ω_1 is governed by the non-Newtonian Stokes equations with shear-dependent viscosity and no-slip boundary condition on Γ_1:

$$-\nabla \cdot \big(g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1}) - p_1\mathbf{I}\big) = \boldsymbol{f_1} \qquad \text{in } \Omega_1, \tag{2.1a}$$

$$\nabla \cdot \boldsymbol{u_1} = 0 \qquad \text{in } \Omega_1, \tag{2.1b}$$

$$\boldsymbol{u_1} = \mathbf{0} \qquad \text{on } \Gamma_1. \tag{2.1c}$$

The flow in the porous medium domain Ω_2 is described by the Darcy–Forchheimer system with the velocity-dependent viscosity $g_2(|\boldsymbol{u_2}|)$:

$$\boldsymbol{K}^{-1} g_2(|\boldsymbol{u_2}|)\boldsymbol{u_2} + C_F|\boldsymbol{u_2}|\boldsymbol{u_2} + \nabla p_2 = \boldsymbol{f_2} \qquad \text{in } \Omega_2, \tag{2.2a}$$

$$\nabla \cdot \boldsymbol{u_2} = 0 \qquad \text{in } \Omega_2, \tag{2.2b}$$

$$\boldsymbol{u_2} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_2, \tag{2.2c}$$

where C_F is the Forchheimer coefficient and $\boldsymbol{K}$ is a symmetric positive definite matrix representing the permeability of the porous medium satisfying

$$k_{\max}|\boldsymbol{\xi}|^2 \geq \boldsymbol{\xi}^\top \boldsymbol{K}\boldsymbol{\xi} \geq k_{\min}|\boldsymbol{\xi}|^2 \quad \text{with } k_{\max} \geq k_{\min} > 0, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d.$$

We will make assumptions on the boundedness and continuity of $g_1(\cdot)$ and $g_2(\cdot)$ (see (2.4) and (2.5)).

On interface Γ, we enforce the conservation law, force balance, and the BJS condition, stated as follows:

$$\boldsymbol{u_1} \cdot \boldsymbol{n_{12}} + \boldsymbol{u_2} \cdot \boldsymbol{n_{21}} = 0 \qquad \text{on } \Gamma, \tag{2.3a}$$

$$p_1 - \big(g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1})\boldsymbol{n_{12}}\big) \cdot \boldsymbol{n_{12}} = p_2 \qquad \text{on } \Gamma, \tag{2.3b}$$

$$\boldsymbol{u_1} \cdot \boldsymbol{t_l} + r_l\big(g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1})\boldsymbol{n_{12}}\big) \cdot \boldsymbol{t_l} = 0 \qquad \text{on } \Gamma, \tag{2.3c}$$

where r_l denote frictional constants and $\boldsymbol{t_l}$ denote the tangent vector on Γ $(l = 1, \ldots, d-1)$.

Remark 2.1 Beavers and Joseph proposed an experimental condition (called the BJ condition) that states the connection between the slip velocity and the shear stress along the interface, i.e.,

$$(\boldsymbol{u_1} - \boldsymbol{u_2}) \cdot \boldsymbol{t_l} = -\frac{r_l}{\mu}\Big(\big(2\mu \boldsymbol{D}(\boldsymbol{u_1}) - p_1\mathbf{I}\big)\boldsymbol{n_{12}}\Big) \cdot \boldsymbol{t_l} \qquad \text{on } \Gamma, \quad l = 1, \ldots, d-1,$$

where c_l is the frictional constant, μ is the viscosity. In view of the fact that the tangent velocity of the porous medium is much smaller than the other terms, Saffman proposed simplified interface condition [20, 30], also known as the BJS condition:

$$\boldsymbol{u_1} \cdot \boldsymbol{t_l} = -2\mu r_l \boldsymbol{D}(\boldsymbol{u_1})\boldsymbol{n_{12}} \cdot \boldsymbol{t_l} \qquad \text{on } \Gamma, \quad l = 1, \ldots, d-1.$$

In this work, we use the same BJS condition as [10, 21] with dynamic viscosity and demonstrate the well-posedness of the coupling model. It is worth mentioning that the well-posedness of the BJ condition is unclear.

We assume that $g_1(\cdot)$ is positive, bounded and Lipschitz continuous, and $g_1(|\boldsymbol{A}|)\boldsymbol{A}$ is strongly monotone, i.e., there exist positive constants $\nu_{1\infty}$, ν_{10}, C_{g_1} and C such that, for any symmetric matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{d\times d}$:

$$\nu_{1\infty} \leq g_1(|\boldsymbol{A}|) \leq \nu_{10}, \tag{2.4a}$$

$$\big|g_1(|\boldsymbol{A}|) - g_1(|\boldsymbol{B}|)\big| \leq C_{g_1}|\boldsymbol{A} - \boldsymbol{B}|, \tag{2.4b}$$

$$\big(g_1(|\boldsymbol{A}|)\boldsymbol{A} - g_1(|\boldsymbol{B}|)\boldsymbol{B}, \boldsymbol{A} - \boldsymbol{B}\big) \geq C|\boldsymbol{A} - \boldsymbol{B}|^2. \tag{2.4c}$$

Likewise, we suppose $g_2(\cdot)$ satisfies the positivity, boundedness and Lipschitz continuous, and $g_2(|\boldsymbol{a}|)\boldsymbol{a}$ is strongly monotone, saying for all $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$,

$$\nu_{2\infty} \leq g_2(|\boldsymbol{a}|) \leq \nu_{20}, \tag{2.5a}$$

$$\big|g_2(|\boldsymbol{a}|) - g_2(|\boldsymbol{b}|)\big| \leq C_{g_2}|\boldsymbol{a} - \boldsymbol{b}|, \tag{2.5b}$$

$$\big(g_2(|\boldsymbol{a}|)\boldsymbol{a} - g_2(|\boldsymbol{b}|)\boldsymbol{b}, \boldsymbol{a} - \boldsymbol{b}\big) \geq C|\boldsymbol{a} - \boldsymbol{b}|^2, \tag{2.5c}$$

where $\nu_{2\infty}$, ν_{20}, C_{g_2}, and C are all positive constants.

Remark 2.2 Suppose $g_1(s), g_2(s) \in C^1(\mathbb{R})$. We see that

$$\begin{aligned} g_1(|\boldsymbol{A}|)\boldsymbol{A} - g_1(|\boldsymbol{B}|)\boldsymbol{B} &= \int_0^1 \frac{d}{d\xi}\Big[g_1(|\boldsymbol{B}+\xi(\boldsymbol{A}-\boldsymbol{B})|)ig(\boldsymbol{B}+\xi(\boldsymbol{A}-\boldsymbol{B})\big)\Big] \\ &= \int_0^1 \Big[g_1'\big(|\boldsymbol{B}+\xi(\boldsymbol{A}-\boldsymbol{B})|\big)\frac{\big(\boldsymbol{B}+\xi(\boldsymbol{A}-\boldsymbol{B}), \boldsymbol{A}-\boldsymbol{B}\big)}{|\boldsymbol{B}+\xi(\boldsymbol{A}-\boldsymbol{B})|}\big(\boldsymbol{B} \\ &\quad + \xi(\boldsymbol{A}-\boldsymbol{B})\big) + g_1(|\boldsymbol{B}+\xi(\boldsymbol{A}-\boldsymbol{B})|)(\boldsymbol{A}-\boldsymbol{B})\Big]\, d\xi, \\ g_2(|\boldsymbol{a}|)\boldsymbol{a} - g_2(|\boldsymbol{b}|)\boldsymbol{b} &= \int_0^1 g_2\big(|\boldsymbol{b}+\xi(\boldsymbol{a}-\boldsymbol{b})|\big)(\boldsymbol{a}-\boldsymbol{b}) \\ &\quad + g_2'\big(|\boldsymbol{b}+\xi(\boldsymbol{a}-\boldsymbol{b})|\big)\frac{\big(\boldsymbol{b}+\xi(\boldsymbol{a}-\boldsymbol{b}), \boldsymbol{a}-\boldsymbol{b}\big)}{|\boldsymbol{b}+\xi(\boldsymbol{a}-\boldsymbol{b})|}\big(\boldsymbol{b}+\xi(\boldsymbol{a}-\boldsymbol{b})\big)\, d\xi. \end{aligned}$$

Since we have assumed $g_1(\cdot) \geq \nu_{1\infty} > 0$ and $g_2(\cdot) \geq \nu_{2\infty} > 0$, the monotonicity (2.4c) and (2.5c) follow if $g_1(s)$ and $g_2(s)$ are nondecreasing, i.e., $g_1'(\cdot), g_2'(\cdot) \geq 0$. Otherwise, we have to assume that $g_1'(|\boldsymbol{A}|)\boldsymbol{A}$ and $g_2'(|\boldsymbol{a}|)\boldsymbol{a}$ are bounded such that

$$\min_{\boldsymbol{A}\in\mathbb{R}^{d\times d}_{\mathrm{sym}}} \Big(g_1(|\boldsymbol{A}|) - \big|g_1'(|\boldsymbol{A}|)\big||\boldsymbol{A}|\Big) \geq C, \quad \min_{\boldsymbol{a}\in\mathbb{R}^d} \Big(g_2(|\boldsymbol{a}|) - \big|g_2'(|\boldsymbol{a}|)\big||\boldsymbol{a}|\Big) \geq C.$$

Remark 2.3 Note that the viscosity of the non-Newtonian fluid model we present in Table 1 displays shear-thinning behavior, which means that viscosity monotonically decreases as the shear rate increases, i.e., $G'(s) \leq 0$. Most fluids show the shear thinning behavior in real life, such as blood, food, and beverages modeled by the famous Carreau fluid, and the flow in the porous medium, the Cross model, is generally considered [14, 18]. It is easy to observe that that $G(s)$ presented in Table 1 satisfies

$$G'(s) \leq 0, \quad \lim_{s\downarrow 0} G(s) = 1, \quad \lim_{s\to\infty} G(s) = 0.$$

We see that $g_1(\cdot)$ (resp. $g_2(\cdot)$) given by (1.1) (resp. (1.2)) allows a wide range of shear rate values, where the zero shear-rate (resp. velocity) corresponds to viscosity ν_0, and infinite shear-rate (resp. velocity) corresponds to ν_∞, indicating the above assumptions (2.4a) (resp. (2.5a)). In addition, it is not difficult to validate that the assumptions (2.4b) and (2.4c) (resp. (2.5b) and (2.5c)) also hold for the models listed in Table 1.

To state the weak formulation, we introduce the function spaces:

$$\begin{aligned} &X_1 := \{\boldsymbol{v_1} \in H^1(\Omega_1)^d : \ \boldsymbol{v_1} = 0 \ \text{ on } \Gamma_1\}, \quad M_1 := L^2(\Omega_1), \\ &X_2 := \{\boldsymbol{v_2} \in L^3(\Omega_2)^d : \nabla\cdot\boldsymbol{v_2} \in L^3(\Omega_2), \boldsymbol{v_2}\cdot\boldsymbol{n_{21}} = 0 \ \text{ on } \Gamma_2\}, \ M_2 := L^{\frac{3}{2}}(\Omega_2), \\ &X := X_1 \times X_2 = \{\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}) : \boldsymbol{v_1} \in X_1, \ \boldsymbol{v_2} \in X_2\}, \\ &M := \{q = (q_1, q_2) \in M_1 \times M_2 : (q_1, 1)_{\Omega_1} + (q_2, 1)_{\Omega_2} = 0\}, \end{aligned}$$

where $(\cdot,\cdot)_\omega$ represents the $L^2(\omega)$ inner product. We endow the above spaces with the following norms:

$$\begin{aligned} &\|\boldsymbol{v_1}\|_{X_1} := \|\boldsymbol{v_1}\|_{H_0^1(\Omega_1)}, \ \|q\|_M := \|q_1\|_{M_1} + \|q_2\|_{M_2} = \|q_1\|_{L^2(\Omega_1)} + \|q_2\|_{L^{\frac{3}{2}}(\Omega_2)}, \\ &\|\boldsymbol{v_2}\|_{X_2} := \|\boldsymbol{v_2}\|_{L^3(\Omega_2)} + \|\nabla\cdot\boldsymbol{v_2}\|_{L^3(\Omega_2)}, \quad \|\boldsymbol{v}\|_X := \|\boldsymbol{v_1}\|_{X_1} + \|\boldsymbol{v_2}\|_{X_2}. \end{aligned}$$

Noting that $\boldsymbol{v_2} \cdot \boldsymbol{n_{12}} \in (W_{00}^{\frac{1}{3},\frac{3}{2}}(\Gamma))^*$ for all $\boldsymbol{v_2} \in X_2$, we set the space

$$V := \{\boldsymbol{v} \in X : \int_\Gamma \mu(\boldsymbol{v_1} \cdot \boldsymbol{n_{12}} + \boldsymbol{v_2} \cdot \boldsymbol{n_{21}})\, ds = 0 \ \ (\forall \mu \in W_{00}^{\frac{1}{3},\frac{3}{2}}(\Gamma))\},$$

which weakly enforces the interface condition (2.3a), i.e., the continuity of normal velocities on Γ.

For any $\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2})$, $\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}) \in X$ and $q = (q_1, q_2) \in M$, we set

$$\begin{aligned}
a_1(\boldsymbol{u_1}, \boldsymbol{v_1}) &:= \int_{\Omega_1} g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1}) : \boldsymbol{D}(\boldsymbol{v_1})\, dx \\
&\quad + \sum_{l=1}^{d-1} \int_\Gamma r_l^{-1}(\boldsymbol{u_1} \cdot \boldsymbol{t_l})(\boldsymbol{v_1} \cdot \boldsymbol{t_l})\, ds, \\
a_2(\boldsymbol{u_2}, \boldsymbol{v_2}) &:= \int_{\Omega_2} \left(\boldsymbol{K}^{-1} g_2(|\boldsymbol{u_2}|)\boldsymbol{u_2} + C_F|\boldsymbol{u_2}|\boldsymbol{u_2}\right) \cdot \boldsymbol{v_2}\, dx, \\
b_1(\boldsymbol{v_1}, q_1) &:= -\int_{\Omega_1} p_1 \nabla \cdot \boldsymbol{v_1}\, dx, \qquad b_2(\boldsymbol{v_2}, q_2) := -\int_{\Omega_2} p_2 \nabla \cdot \boldsymbol{v_2}\, dx, \\
a(\boldsymbol{u}, \boldsymbol{v}) &:= a_1(\boldsymbol{u_1}, \boldsymbol{v_1}) + a_2(\boldsymbol{u_2}, \boldsymbol{v_2}), \qquad b(\boldsymbol{v}, p) := b_1(\boldsymbol{v_1}, p_1) + b_2(\boldsymbol{v_2}, p_2), \\
(\boldsymbol{f}, \boldsymbol{v}) &:= \int_{\Omega_1} \boldsymbol{f_1} \cdot \boldsymbol{v_1}\, dx + \int_{\Omega_2} \boldsymbol{f_2} \cdot \boldsymbol{v_2}\, dx.
\end{aligned}$$

2.2 Weak Formulation

Testing (2.1a) and (2.2a) by $\boldsymbol{v_1} \in X_1$ and $\boldsymbol{v_2} \in X_2$ respectively, applying the integration by parts, and using the interface conditions (2.3), we obtain the weak formulation:

Find $(\boldsymbol{u}, p) \in V \times M$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, q) = 0 & \forall\, q \in M. \end{cases} \tag{2.6}$$

We introduce the space $\Lambda := W_{00}^{\frac{1}{3},\frac{3}{2}}(\Gamma)$ with the norm $\|\mu\|_\Lambda := \|\mu\|_{W_{00}^{\frac{1}{3},\frac{3}{2}}(\Gamma)}$, and the bilinear form

$$b_I(\boldsymbol{v}, \lambda) := \int_\Gamma \lambda(\boldsymbol{v_1} \cdot \boldsymbol{n_{12}} + \boldsymbol{v_2} \cdot \boldsymbol{n_{21}})\, ds \qquad (\forall \boldsymbol{v} \in X,\ \lambda \in \Lambda),$$

which satisfies the inf-sup condition [10, Lemma 3.2]: there is a constant $\beta_I > 0$ such that

$$\beta_I \|\lambda\|_\Lambda \le \sup_{(\boldsymbol{v_1}, \boldsymbol{v_2}) \in W^{1,3}(\Omega_1) \times X_2} \frac{b_I(\boldsymbol{v}, \lambda)}{\|\boldsymbol{v_1}\|_{W^{1,3}(\Omega_1)} + \|\boldsymbol{v_2}\|_{X_2}} \le \sup_{\boldsymbol{v} \in X} \frac{b_I(\boldsymbol{v}, \lambda)}{\|\boldsymbol{v}\|_X}. \tag{2.7}$$

It follows from (2.7) that (2.6) is equivalent to the Lagrange multiplier form:

Find $(\boldsymbol{u}, p, \lambda) \in X \times M \times \Lambda$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) + b_I(\boldsymbol{v}, \lambda) = (\boldsymbol{f}, \boldsymbol{v}) & \forall\, \boldsymbol{v} \in X, \\ b(\boldsymbol{u}, q) = 0 & \forall\, q \in M, \\ b_I(\boldsymbol{u}, \mu) = 0 & \forall\, \mu \in \Lambda. \end{cases} \tag{2.8}$$

In fact, for sufficiently smooth $(\boldsymbol{u}, p)$, one can verify that

$$\lambda = p_2 = p_1 - \big(g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1})\boldsymbol{n_{12}}\big) \cdot \boldsymbol{n_{12}} \quad \text{on } \Gamma. \tag{2.9}$$

2.3 The Unique Existence of Weak Solution

We now turn to the well-posedness of (2.6) (also (2.8)). To this end, we first prove that (2.11) admits a unique solution $\boldsymbol{u}$. Then, we derive a coupled inf-sup condition, which implies the unique existence of p such that $(\boldsymbol{u}, p)$ solves (2.6). The well-posedness of (2.8) is a direct result from (2.7) and the unique existence of (2.6).

Setting

$$\mathring{V} := \{\boldsymbol{v} \in V : \ \nabla \cdot \boldsymbol{v_1} = 0 \ \text{ a.e. on } \Omega_1, \ \nabla \cdot \boldsymbol{v_2} = 0 \ \text{ a.e. on } \Omega_2\},$$

and $\mathring{V}^*$ the dual of $\mathring{V}$, we define a nonlinear operator $A : \mathring{V} \to \mathring{V}^*$,

$$(A(\boldsymbol{u}), \boldsymbol{v}) := a(\boldsymbol{u}, \boldsymbol{v}) \quad (\forall\, \boldsymbol{u}, \boldsymbol{v} \in \mathring{V}), \tag{2.10}$$

and consider the weak formulation:

Find $\boldsymbol{u} \in \mathring{V}$ such that

$$(A(\boldsymbol{u}), \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \mathring{V}. \tag{2.11}$$

Let us consider the existence and uniqueness of the (2.11). According to [23], we need to verify the hemicontinuity, coercivity, and monotonicity of A, which are illustrated by the following lemmas.

Lemma 2.1 *The operator $A : \mathring{V} \to \mathring{V}^*$ is continuous and bounded with the estimate*

$$\|A(\boldsymbol{u})\|_{\mathring{V}^*} \le c(\|\boldsymbol{u_1}\|_{X_1} + \|\boldsymbol{u_2}\|_{X_2} + \|\boldsymbol{u_2}\|_{X_2}^2). \tag{2.12}$$

Proof For any $\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2}) \in \mathring{V}$, we have $\boldsymbol{D}(\boldsymbol{u_1}) \in L^2(\Omega_1)^{d\times d}$ and $\boldsymbol{u_2} \in L^3(\Omega_2)^d$ by the definition of space X. And the continuity condition of $g_1(\cdot)$ and $g_2(\cdot)$ (see (2.4b) and (2.5b)) implies $g_1(|\boldsymbol{D}(\boldsymbol{u_1}(x))|)$ and $g_2(|\boldsymbol{u_2}(x)|)$ are measurable on Ω_1 and Ω_2 respectively. We begin by proving the boundedness of A. Recalling the definition of operator norm and applying (2.10), we get

$$\|A(\boldsymbol{u})\|_{\mathring{V}^*} = \sup_{\boldsymbol{v}\in\mathring{V}} \frac{(A(\boldsymbol{u}), \boldsymbol{v})}{\|\boldsymbol{v}\|_{\mathring{V}}} = \sup_{\boldsymbol{v}\in\mathring{V}} \frac{a_1(\boldsymbol{u_1}, \boldsymbol{v_1}) + a_2(\boldsymbol{u_2}, \boldsymbol{v_2})}{\|\boldsymbol{v}\|_X}. \tag{2.13}$$

By using (2.4a), (2.5a) and Hölder's inequality [12],

$$\begin{aligned}
a_1(\boldsymbol{u_1}, \boldsymbol{v_1}) &\le \nu_{10}\|\boldsymbol{D}(\boldsymbol{u_1})\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{v_1})\|_{L^2(\Omega_1)} + c\|\boldsymbol{u_1}\|_{L^2(\Gamma)}\|\boldsymbol{v_1}\|_{L^2(\Gamma)} \\
&\le c\|\boldsymbol{u_1}\|_{X_1}\|\boldsymbol{v_1}\|_{X_1}, \\
a_2(\boldsymbol{u_2}, \boldsymbol{v_2}) &\le c(\|\boldsymbol{u_2}\|_{L^3(\Omega_2)} + \|\boldsymbol{u_2}\|_{L^3(\Omega_2)}^2)\|\boldsymbol{v_2}\|_{X_2} \\
&\le c(\|\boldsymbol{u_2}\|_{X_2} + \|\boldsymbol{u_2}\|_{X_2}^2)\|\boldsymbol{v_2}\|_{X_2}.
\end{aligned}$$

Combining (2.13) with these inequalities, we get (2.12).

We proceed to show the continuity of A. Let $\{\boldsymbol{u}^{(n)}\}$ be a sequence in $\mathring{V}$ satisfying $\|\boldsymbol{u}^{(n)} - \boldsymbol{u}\|_X \to 0$. Then there exists a subsequence $\{\boldsymbol{u}^{(n')}\}$ and a function $\boldsymbol{h} = (\boldsymbol{h_1}, \boldsymbol{h_2}) \in \mathring{V}$ such that

$$\boldsymbol{u}_{\boldsymbol{1}}^{(n')}(x) \to \boldsymbol{u_1}(x) \quad \text{a.e. } x \in \Omega_1, \quad \boldsymbol{u}_{\boldsymbol{2}}^{(n')}(x) \to \boldsymbol{u_2}(x) \quad \text{a.e. } x \in \Omega_2, \tag{2.14}$$

$$\left|\boldsymbol{u}_{\mathbf{1}}^{(n')}(x)\right| \le |\boldsymbol{h}_{\mathbf{1}}(x)| \ \ \forall n', \ \text{a.e. } x \in \Omega_1, \ \left|\boldsymbol{u}_{\mathbf{2}}^{(n')}(x)\right| \le |\boldsymbol{h}_{\mathbf{2}}(x)| \ \ \forall n', \ \text{a.e. } x \in \Omega_2. \tag{2.15}$$

For any $\boldsymbol{w} \in \mathring{V}$,

$$\begin{aligned}
&\left|\langle A(\boldsymbol{u}^{(n')}) - A(\boldsymbol{u}), \boldsymbol{w}\rangle\right| \\
&\le \left| \int_{\Omega_1} I_1^{(n')} \boldsymbol{D}(\boldsymbol{w}_{\mathbf{1}})\, dx \right| + \left| \sum_{l=1}^{d-1} \int_{\Gamma} r_l^{-1} I_2^{(n')} (\boldsymbol{w}_{\mathbf{1}} \cdot \boldsymbol{t}_{\boldsymbol{l}})\, ds \right| + c \left| \int_{\Omega_2} I_3^{(n')} \boldsymbol{w}_{\mathbf{2}}\, dx \right| \\
&\le \|I_1^{(n')}\|_{L^2(\Omega_1)} \|\boldsymbol{w}_{\mathbf{1}}\|_{L^2(\Omega_1)} + r_l^{-1} \|I_2^{(n')}\|_{L^2(\Gamma)} \|\boldsymbol{w}_{\mathbf{1}} \cdot \boldsymbol{t}_{\boldsymbol{l}}\|_{L^2(\Gamma)} \\
&\quad + c \|I_3^{(n')}\|_{L^{\frac{3}{2}}(\Omega_2)} \|\boldsymbol{w}_{\mathbf{2}}\|_{L^3(\Omega_2)} \\
&\le c\Big(\|I_1^{(n')}\|_{L^2(\Omega_1)} + \|I_2^{(n')}\|_{L^2(\Gamma)} + \|I_3^{(n')}\|_{L^{\frac{3}{2}}(\Omega_2)} \Big) \|\boldsymbol{w}\|_X,
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
&I_1^{(n')} = g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{(n')})|)\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{(n')}) - g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}})|)\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}), \ I_2^{(n')} = (\boldsymbol{u}_{\mathbf{1}}{}^{(n')} - \boldsymbol{u}_{\mathbf{1}}) \cdot \boldsymbol{t}_{\boldsymbol{l}} \\
&I_3^{(n')} = g_2(|\boldsymbol{u}_{\mathbf{2}}^{(n')}|)\boldsymbol{u}_{\mathbf{2}}^{(n')} - g_2(|\boldsymbol{u}_{\mathbf{2}}|)\boldsymbol{u}_{\mathbf{2}} + |\boldsymbol{u}_{\mathbf{2}}^{(n')}|\boldsymbol{u}_{\mathbf{2}}^{(n')} - |\boldsymbol{u}_{\mathbf{2}}|\boldsymbol{u}_{\mathbf{2}}.
\end{aligned}$$

According to (2.15),

$$\begin{aligned}
&\|I_1^{(n')}\|_{L^2(\Omega_1)} + \|I_2^{(n')}\|_{L^2(\Gamma)} \\
&\qquad\qquad \le c(\|\boldsymbol{D}(\boldsymbol{h}_{\mathbf{1}})\|_{L^2(\Omega_1)} + \|\boldsymbol{h}_{\mathbf{1}}\|_{X_1} + \|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}})\|_{L^2(\Omega_1)} + \|\boldsymbol{u}_{\mathbf{1}}\|_{X_1}), \\
&\|I_3^{(n')}\|_{L^{\frac{3}{2}}(\Omega_2)} \le c(\|\boldsymbol{h}_{\mathbf{2}}\|_{L^{\frac{3}{2}}(\Omega_2)} + \|\boldsymbol{h}_{\mathbf{2}}\|^2_{L^3(\Omega_2)} + \|\boldsymbol{u}_{\mathbf{2}}\|_{L^{\frac{3}{2}}(\Omega_2)} + \|\boldsymbol{u}_{\mathbf{2}}\|^2_{L^3(\Omega_2)}).
\end{aligned}$$

By (2.14) and the continuity of $g_1(\cdot)$ and $g_2(\cdot)$, we obtain

$$\begin{aligned}
&g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{(n')}(x))|) \to g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}(x))|) \quad \text{a.e. } x \in \Omega_1, \\
&g_2(|\boldsymbol{u}_{\mathbf{2}}^{(n')}(x)|) \to g_2(|\boldsymbol{u}_{\mathbf{2}}(x)|) \quad \text{a.e. } x \in \Omega_2,
\end{aligned}$$

which implies

$$I_1^{(n')}(x), \ I_2^{(n')}(x) \to 0 \quad \text{a.e. } x \in \Omega_1, \quad I_3^{(n')}(x) \to 0 \quad \text{a.e. } x \in \Omega_2.$$

Then, we conclude by the dominated convergence theorem [5] that

$$\|I_1^{(n')}\|_{L^2(\Omega_1)} \to 0, \quad \|I_2^{(n')}\|_{L^2(\Gamma)} \to 0, \quad \|I_3^{(n')}\|_{L^{\frac{3}{2}}(\Omega_2)} \to 0. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\|A(\boldsymbol{u}^{(n')}) - A(\boldsymbol{u})\|_{\mathring{V}^*} \le c\Big(\|I_1^{(n')}\|_{L^2(\Omega_1)} + \|I_2^{(n')}\|_{L^2(\Gamma)} + \|I_3^{(n')}\|_{L^{\frac{3}{2}}(\Omega_2)} \Big) \to 0.$$

The proof is completed. □

Remark 2.4 The continuity implies A is hemicontinuous, i.e., $\forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathring{V}$,

$$\lim_{\xi \to 0} \langle A(\boldsymbol{u} + \xi \boldsymbol{v}) - A(\boldsymbol{u}), w\rangle = 0.$$

Lemma 2.2 *The operator A is coercive, i.e.,*

$$\frac{(A(\boldsymbol{v}), \boldsymbol{v})}{\|\boldsymbol{v}\|_X} \to +\infty \qquad as \quad \|\boldsymbol{v}\|_X \to \infty. \tag{2.18}$$

In particular,

$$(A(\boldsymbol{v}), \boldsymbol{v}) \geq c\|\boldsymbol{v_1}\|_{X_1}^2 + \frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{v_2}\|_{L^2(\Omega_2)}^2 + C_F\|\boldsymbol{v_2}\|_{X_2}^3 \quad (\forall\, \boldsymbol{v} \in \mathring{V}). \tag{2.19}$$

Proof For all $\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}) \in \mathring{V}$, by the Korn inequality [12], Poincaré inequality [5] and $\nabla \cdot \boldsymbol{v_2} = 0$,

$$\begin{aligned}(A(\boldsymbol{v}), \boldsymbol{v}) = a(\boldsymbol{v}, \boldsymbol{v}) &\geq \nu_{1\infty}\|\boldsymbol{D}(\boldsymbol{v_1})\|_{L^2(\Omega_1)}^2 + \frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{v_2}\|_{L^2(\Omega_2)}^2 + C_F\|\boldsymbol{v_2}\|_{L^3(\Omega_2)}^3 \\ &\geq c\|\boldsymbol{v_1}\|_{X_1}^2 + \frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{v_2}\|_{L^2(\Omega_2)}^2 + C_F\|\boldsymbol{v_2}\|_{L^3(\Omega_2)}^3 \geq 0.\end{aligned}$$

Hence, we obtain (2.19), which implies (2.18). □

Lemma 2.3 *The operator A is strongly monotone, i.e., for all $\boldsymbol{u}, \boldsymbol{v} \in \mathring{V}$,*

$$(A(\boldsymbol{u}) - A(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}) \geq c(\|\boldsymbol{u_1} - \boldsymbol{v_1}\|_{X_1}^2 + \|\boldsymbol{u_2} - \boldsymbol{v_2}\|_{L^2(\Omega_2)}^2). \tag{2.20}$$

Proof Let $\boldsymbol{u} = (\boldsymbol{u_1}, \boldsymbol{u_2})$, $\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}) \in \mathring{V}$. We make the decomposition

$$\begin{aligned}(A(\boldsymbol{u}) - A(\boldsymbol{v}), \boldsymbol{u} - \boldsymbol{v}) = &\big(a_1(\boldsymbol{u_1}, \boldsymbol{u_1} - \boldsymbol{v_1}) - a_1(\boldsymbol{v_1}, \boldsymbol{u_1} - \boldsymbol{v_1})\big) \\ &+ \int_{\Omega_2} \boldsymbol{K}^{-1}\big(g_2(|\boldsymbol{u_2}|)\boldsymbol{u_2} - g_2(|\boldsymbol{v_2}|)\boldsymbol{v_2}\big) \cdot (\boldsymbol{u_2} - \boldsymbol{v_2})\, dx \\ &+ \int_{\Omega_2} C_F\big(|\boldsymbol{u_2}|\boldsymbol{u_2} - |\boldsymbol{v_2}|\boldsymbol{v_2}\big) \cdot (\boldsymbol{u_2} - \boldsymbol{v_2})\, dx =: I_1 + I_2 + I_3.\end{aligned}$$

By the monotonicity (2.4c) of $g_1(\cdot)$, Korn inequality and Poincaré inequality, we have

$$I_1 \geq C\|\boldsymbol{D}(\boldsymbol{u_1} - \boldsymbol{v_1})\|_{L^2}^2 \geq CC_k\|\nabla(\boldsymbol{u_1} - \boldsymbol{v_1})\|_{L^2}^2 \geq c\|(\boldsymbol{u_1} - \boldsymbol{v_1})\|_{X_1}^2. \tag{2.21}$$

For I_2, (2.5c) implies

$$I_2 \geq c\|\boldsymbol{u_2} - \boldsymbol{v_2}\|_{L^2(\Omega_2)}^2. \tag{2.22}$$

For I_3, we calculate as follows

$$\begin{aligned}I_3 &= \int_{\Omega_2} C_F \int_0^1 \frac{d}{d\xi}\Big[\big|\boldsymbol{v_2} + \xi(\boldsymbol{u_2} - \boldsymbol{v_2})\big|\big(\boldsymbol{v_2} + \xi(\boldsymbol{u_2} - \boldsymbol{v_2})\big)\Big] \cdot (\boldsymbol{u_2} - \boldsymbol{v_2})\, dx \\ &= \int_{\Omega_2} C_F \int_0^1 \Bigg[\frac{\big((\boldsymbol{v_2} + \xi(\boldsymbol{u_2} - \boldsymbol{v_2}))^2, (\boldsymbol{u_2} - \boldsymbol{v_2})^2\big)}{|\boldsymbol{v_2} + \xi(\boldsymbol{u_2} - \boldsymbol{v_2})|} \\ &\qquad + |\boldsymbol{v_2} + \xi(\boldsymbol{u_2} - \boldsymbol{v_2})|(\boldsymbol{u_2} - \boldsymbol{v_2})^2\Bigg]\, d\xi\, dx \geq 0.\end{aligned} \tag{2.23}$$

Hence, we conclude (2.20). □

Theorem 2.1 *There exists a unique solution $u \in \mathring{V}$ to* (2.11) satisfying

$$\|\boldsymbol{u_1}\|_{X_1}^2 + \|\boldsymbol{u_2}\|_{L^2(\Omega_2)}^2 + \|\boldsymbol{u_2}\|_{X_2}^3 \leq C(\|\boldsymbol{f_1}\|_{X_1^*}^2 + \|\boldsymbol{f_2}\|_{X_2^*}^{\frac{3}{2}}). \tag{2.24}$$

Proof Since $A : \mathring{V} \to \mathring{V}^*$ is hemicontinuous, coercive, and monotone (by Lemmas 2.1 to 2.3), the unique existence of $A\boldsymbol{u} = \boldsymbol{f}$ follows from [23, Theorem 2.1]. To obtain the a-priori estimate, we substitute $\boldsymbol{v} = \boldsymbol{u}$ into (2.11) and apply (2.19)

$$c\|\boldsymbol{u_1}\|_{X_1}^2 + \frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{u_2}\|_{L^2(\Omega_2)}^2 + C_F\|\boldsymbol{u_2}\|_{L^3(\Omega_2)}^3 \le (A(\boldsymbol{u}), \boldsymbol{u}) = a(\boldsymbol{u}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{u})$$
$$\le \|\boldsymbol{f_1}\|_{X_1^*}\|\boldsymbol{u_1}\|_{X_1} + \|\boldsymbol{f_2}\|_{X_2^*}\|\boldsymbol{u_2}\|_{X_2}$$
$$\le \frac{1}{2c}\|\boldsymbol{f_1}\|_{X_1^*}^2 + \frac{c}{2}\|\boldsymbol{u_1}\|_{X_1}^2 + (\frac{2}{3})^{\frac{3}{2}} C_F^{-\frac{1}{2}}\|\boldsymbol{f_2}\|_{X_2^*}^{\frac{3}{2}} + \frac{C_F}{2}\|\boldsymbol{u_2}\|_{L^3(\Omega_2)}^3,$$

where we have used Young's inequality and $\|\boldsymbol{u_2}\|_{X_2} = \|\boldsymbol{u_2}\|_{L^3(\Omega_2)}$ (by $\nabla \cdot \boldsymbol{u_2} = 0$) in the last inequality. Hence, we conclude (2.24). □

We have obtained the existence of $\boldsymbol{u}$. Now let us prove an inf-sup condition which implies the unique existence of $p \in M$ to (2.6).

Lemma 2.4 *There is a constant $\beta > 0$ such that*

$$\inf_{p\in M}\sup_{\boldsymbol{v}\in V}\frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_X\|p\|_M} \ge \beta. \tag{2.25}$$

Remark 2.5 A different inf-sup condition for the linear Stokes–Darcy equations is established in [21, Lemma 3.2]:

$$\inf_{p\in L^2(\Omega_1)\times L^2(\Omega_2)}\sup_{\boldsymbol{v}\in H_0^1(\Omega_1)\times H(\text{div};\Omega_2;)}\frac{b(\boldsymbol{v}, p)}{(\|\boldsymbol{v_1}\|_{W^{1,2}(\Omega_1)} + \|\boldsymbol{v_2}\|_{H(\text{div};\Omega_2)})\|p\|_{L^2(\Omega)}} \ge \beta.$$

We modify this inf-sup condition and prove (2.25) to handle the case with $\boldsymbol{v_2} \in L^3(\Omega_2)$, $\nabla \cdot \boldsymbol{v_2} \in L^3(\Omega_2)$ and $p_2 \in L^{\frac{3}{2}}(\Omega_2)$.

Proof We make decomposition $p_1 = \mathring{p}_1 + \xi_1$, $p_2 = \mathring{p}_2 + \xi_2$, where

$$(\mathring{p}_1, 1)_{\Omega_1} = 0, \qquad \xi_1 = \frac{1}{|\Omega_1|}(p_1, 1)_{\Omega_1}, \quad (\mathring{p}_2, 1)_{\Omega_2} = 0, \qquad \xi_2 = \frac{1}{|\Omega_2|}(p_2, 1)_{\Omega_2}.$$

Since $\int_{\Omega_1} q_1 dx + \int_{\Omega_2} q_2 dx = 0$, we discover

$$\xi_1|\Omega_1| + \xi_2|\Omega_2| = 0. \tag{2.26}$$

There exists a $\mathring{\boldsymbol{v}}_1 \in W_0^{1,2}(\Omega_1)$ such that

$$-\nabla \cdot \mathring{\boldsymbol{v}}_1 = \frac{\mathring{p}_1}{\|\mathring{p}_1\|_{L^2(\Omega_1)}} \quad \text{in } \Omega_1, \quad \|\mathring{\boldsymbol{v}}_1\|_{W^{1,2}(\Omega_1)} \le C\frac{\|\mathring{p}_1\|_{L^2(\Omega_1)}}{\|\mathring{p}_1\|_{L^2(\Omega_1)}} \le C. \tag{2.27}$$

We select $\hat{p}_2 = \frac{|p_2|^{-\frac{1}{2}}p_2}{\|p_2\|_{L^{\frac{3}{2}}(\Omega_2)}^{\frac{1}{2}}} \in L^{\frac{3}{2}}(\Omega_2)$ (note that $\|\hat{p}_2\|_{L^3(\Omega_2)} = 1$), and set $\hat{\xi}_2 := \frac{1}{|\Omega_2|}(\hat{p}_2, 1)_{\Omega_2}$, $\mathring{\hat{p}}_2 := \hat{p}_2 - \hat{\xi}_2$. By Hölder's inequality, we have

$$\hat{\xi}_2 \le \frac{1}{|\Omega_2|}\|1\|_{L^{\frac{3}{2}}(\Omega_2)}\|\hat{p}_2\|_{L^3(\Omega_2)} \le \frac{1}{|\Omega_2|^{\frac{1}{3}}}. \tag{2.28}$$

Note that

$$\int_{\Omega_2}\hat{p}_2 p_2 dx = \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)}. \tag{2.29}$$

For $\mathring{\hat{p}}_2$, there exists a $\mathring{\hat{\boldsymbol{v}}}_\mathbf{2} \in W_0^{1,3}(\Omega_2)$ satisfying

$$- \nabla \cdot \mathring{\hat{\boldsymbol{v}}}_\mathbf{2} = \mathring{\hat{p}}_2 \quad \text{in}\, \Omega_1, \quad \|\mathring{\hat{\boldsymbol{v}}}_\mathbf{2}\|_{W^{1,3}(\Omega_2)} \le C\|\mathring{\hat{p}}_2\|_{L^{\frac{3}{2}}(\Omega_2)}. \tag{2.30}$$

For arbitrary constant $C^* > 0$, there exist $\tilde{\boldsymbol{v}}_\mathbf{1}$, $\tilde{\boldsymbol{v}}_\mathbf{2}$ such that

$$- \nabla \cdot \tilde{\boldsymbol{v}}_\mathbf{1} = \frac{C^*}{|\Omega_1|} \quad \text{in}\, \Omega_1, \quad \tilde{\boldsymbol{v}}_\mathbf{1} = 0 \quad \text{on}\, \Gamma_1, \quad \tilde{\boldsymbol{v}}_\mathbf{1} = -\frac{C^*}{|\Gamma|}\boldsymbol{n}_\mathbf{12} \quad \text{on}\, \Gamma, \tag{2.31}$$

$$-\nabla \cdot \tilde{\boldsymbol{v}}_\mathbf{2} = -\frac{C^*}{|\Omega_2|} \quad \text{in}\, \Omega_2, \quad \tilde{\boldsymbol{v}}_\mathbf{2} = 0 \quad \text{on}\, \Gamma_2, \quad \tilde{\boldsymbol{v}}_\mathbf{2} = -\frac{C^*}{|\Gamma|}\boldsymbol{n}_\mathbf{12} \quad \text{on}\, \Gamma, \tag{2.32}$$

$$\|\tilde{\boldsymbol{v}}_\mathbf{1}\|_{W^{1,2}(\Omega_1)} \le CC^*, \quad \|\tilde{\boldsymbol{v}}_\mathbf{2}\|_{W^{0,3}(\mathrm{div},\Omega_2)} \le \|\tilde{\boldsymbol{v}}_\mathbf{2}\|_{W^{1,3}(\Omega_2)} \le CC^*. \tag{2.33}$$

Now, we set $\boldsymbol{v}_\mathbf{1} := \mathring{\boldsymbol{v}}_\mathbf{1} + \tilde{\boldsymbol{v}}_\mathbf{1}$, $\boldsymbol{v}_\mathbf{2} := \mathring{\hat{\boldsymbol{v}}}_\mathbf{2} + \tilde{\boldsymbol{v}}_\mathbf{2}$. In view of (2.27), (2.31) and the definition of p_1, we find

$$\begin{aligned} -(\nabla \cdot \boldsymbol{v}_\mathbf{1}, p_1)_{\Omega_1} &= \left(\frac{\mathring{p}_1}{\|\mathring{p}_1\|_{L^2(\Omega_1)}}, p_1\right)_{\Omega_1} + \left(\frac{C^*}{|\Omega_1|}, p_1\right)_{\Omega_1} \\ &= \frac{1}{\|\mathring{p}_1\|_{L^2(\Omega_1)}}(\mathring{p}_1, \mathring{p}_1 + \xi_1)_{\Omega_1} + C^*\xi_1 = \|\mathring{p}_1\|_{L^2(\Omega_1)} + C^*\xi_1. \end{aligned} \tag{2.34}$$

It follows from (2.30) and (2.32) that,

$$\begin{aligned} -(\nabla \cdot \boldsymbol{v}_\mathbf{2}, p_2)_{\Omega_2} &= (\mathring{\hat{p}}_2, p_2)_{\Omega_2} - \frac{C^*}{|\Omega_2|}(1, p_2)_{\Omega_2} = (\hat{p}_2 - \hat{\xi}_2, p_2)_{\Omega_2} - C^*\xi_2 \\ &= \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} - \hat{\xi}_2(1, p_2)_{\Omega_2} - C^*\xi_2 = \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} - \hat{\xi}_2\xi_2|\Omega_2| - C^*\xi_2. \end{aligned} \tag{2.35}$$

Combining (2.34) and (2.35), and applying (2.26) and (2.28), we get

$$\begin{aligned} -(\nabla \cdot \boldsymbol{v}_\mathbf{1}, p_1)_{\Omega_1} &- (\nabla \cdot \boldsymbol{v}_\mathbf{2}, p_2)_{\Omega_2} \\ &\ge \|\mathring{p}_1\|_{L^2(\Omega_1)} + \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} + C^*\xi_1 - |\Omega_2|^{\frac{2}{3}}\xi_2 - C^*\xi_2 \\ &= \|\mathring{p}_1\|_{L^2(\Omega_1)} + \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} + C^*\left(1 + \frac{|\Omega_1|}{|\Omega_2|}\right)\xi_1 + \frac{|\Omega_1|}{|\Omega_2|^{\frac{1}{3}}}\xi_1. \end{aligned}$$

Taking sufficiently large γ and $C^* = \gamma \frac{\xi_1}{|\xi_1|}$, we obtain

$$\begin{aligned} -(\nabla \cdot \boldsymbol{v}_\mathbf{1}, p_1)_{\Omega_1} &- (\nabla \cdot \boldsymbol{v}_\mathbf{2}, p_2)_{\Omega_2} \\ &\ge \|\mathring{p}_1\|_{L^2(\Omega_1)} + \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} + \gamma\left(1 + \frac{|\Omega_1|}{|\Omega_2|}\right)\frac{1}{|\xi_1|}\xi_1^2 + \frac{|\Omega_1|}{|\Omega_2|^{\frac{1}{3}}}\xi_1 \\ &\ge \|\mathring{p}_1\|_{L^2(\Omega_1)} + \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} + C_0|\xi_1| \ge \|p_1\|_{L^2(\Omega_1)} + \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)}. \end{aligned}$$

Next, by the triangle inequality, (2.27) and (2.33), we have

$$\begin{aligned} \|\boldsymbol{v}_\mathbf{1}\|_{W^{1,2}(\Omega_1)} &= \|\mathring{\boldsymbol{v}}_\mathbf{1} + \tilde{\boldsymbol{v}}_\mathbf{1}\|_{W^{1,2}(\Omega_1)} \le \|\mathring{\boldsymbol{v}}_\mathbf{1}\|_{W^{1,2}(\Omega_1)} + \|\tilde{\boldsymbol{v}}_\mathbf{1}\|_{W^{1,2}(\Omega_1)} \\ &\le C + CC^* \le C(1+\gamma). \end{aligned} \tag{2.36}$$

Similarly,

$$\begin{aligned}\|\boldsymbol{v_2}\|_{W^{0,3}(\mathrm{div},\Omega_2)} &\le \|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)} \le \|\mathring{\boldsymbol{v}}_{\boldsymbol{2}}\|_{W^{1,3}(\Omega_2)} + \|\tilde{\boldsymbol{v}}_{\boldsymbol{2}}\|_{W^{1,3}(\Omega_2)} \\ &\le C\|\mathring{\hat{p}}_2\|_{L^{\frac{3}{2}}(\Omega_2)} + CC^* \ (\text{by}(2.33) and (2.30)) \\ &\le C\|\hat{p}_2 - \hat{\xi}_2\|_{L^{\frac{3}{2}}(\Omega_2)} + CC^* \le C(\|\hat{p}_2\|_{L^{\frac{3}{2}}(\Omega_2)} + |\Omega_2|^{-\frac{1}{3}}) + CC^* \\ &\le C(1+\gamma) \quad (\text{by}(2.28)).\end{aligned} \tag{2.37}$$

Therefore,

$$\|\boldsymbol{v_1}\|_{X_1} + \|\boldsymbol{v_2}\|_{X_2} \le C(1+\gamma).$$

In summary, for any $p \in M$, we have found $(\boldsymbol{v_1}, \boldsymbol{v_2})$ satisfying

$$\sup_{\boldsymbol{v}\in V} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v_1}\|_{X_1} + \|\boldsymbol{v_2}\|_{X_2}} \ge \frac{\|p_1\|_{L^2(\Omega_1)} + \|p_2\|_{L^{\frac{3}{2}}(\Omega_2)}}{C(1+\gamma)},$$

which implies (2.25). □

Remark 2.6 In view of the first inequality of (2.37), we can replace $\|\boldsymbol{v_2}\|_{X_2}$ of the above two inequalities by $\|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)}$, which results

$$\sup_{\boldsymbol{v}\in W} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v_1}\|_{W^{1,2}(\Omega_1)} + \|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)}} \ge C\|p\|_M \quad (\forall p \in M), \tag{2.38}$$

where

$$W := \{\boldsymbol{v} = (\boldsymbol{v_1}, \boldsymbol{v_2}) : \boldsymbol{v_1} \in W^{1,2}_{0,\Gamma_1}(\Omega_1)^d,\ \boldsymbol{v_2} \in W^{1,3}_{0,\Gamma_2}(\Omega_2)^d,\ b_I(\boldsymbol{v}, \mu) = 0\,(\forall \mu \in \Lambda)\}.$$

Theorem 2.2 *There exists a unique $p \in M$ and $\lambda \in \Lambda$ such that $(\boldsymbol{u}, p)$ solves (2.6) and $(\boldsymbol{u}, p, \lambda)$ solves (2.8). Moreover, we have*

$$\|p\|_M + \|\lambda\|_\Lambda \le C(\|\boldsymbol{f_1}\|_{X_1^*} + \|\boldsymbol{f_1}\|^{\frac{4}{3}}_{X_1^*} + \|\boldsymbol{f_2}\|_{X_2^*} + \|\boldsymbol{f_2}\|^{\frac{3}{4}}_{X_2^*}). \tag{2.39}$$

Proof It follows from Theorem 2.1 and Lemma 2.4 that there exists a unique $(\boldsymbol{u}, p) \in V \times M$ of (2.6). Furthermore, by (2.25)

$$\|p\|_M \le C \sup_{\boldsymbol{v}\in V} \frac{b(\boldsymbol{v}, p)}{\|\boldsymbol{v}\|_X} = \frac{(\boldsymbol{f}, \boldsymbol{v}) - a(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{v}\|_X}.$$

In view of

$$(\boldsymbol{f}, \boldsymbol{v}) \le \|\boldsymbol{f_1}\|_{X_1^*}\|\boldsymbol{v_1}\|_{X_1} + \|\boldsymbol{f_2}\|_{X_2^*}\|\boldsymbol{v_2}\|_{X_2}, \tag{2.40a}$$

$$\begin{aligned}a(\boldsymbol{u}, \boldsymbol{v}) &\le c\|\boldsymbol{u_1}\|_{X_1}\|\boldsymbol{v_1}\|_{X_1} + \frac{\nu_{20}}{k_{\min}}\|\boldsymbol{u_2}\|_{L^2(\Omega_2)}\|\boldsymbol{v_2}\|_{L^2(\Omega_2)} \\ &\quad + C_F\|\boldsymbol{u_2}\|^2_{L^3(\Omega_2)}\|\boldsymbol{v_2}\|_{L^3(\Omega_2)},\end{aligned} \tag{2.40b}$$

together with $L^3(\Omega_2) \subset L^2(\Omega_2)$ and Theorem 2.1, we obtain

$$\begin{aligned}\|p\|_M &\le C(\|\boldsymbol{f_1}\|_{X_1^*} + \|\boldsymbol{f_2}\|_{X_2^*} + \|\boldsymbol{u_1}\|_{X_1} + \|\boldsymbol{u_2}\|_{L^2(\Omega_2)} + \|\boldsymbol{u_2}\|^2_{L^3(\Omega_2)}) \\ &\le C(\|\boldsymbol{f_1}\|_{X_1^*} + \|\boldsymbol{f_1}\|^{\frac{4}{3}}_{X_1^*} + \|\boldsymbol{f_2}\|^{\frac{3}{4}}_{X_2^*} + \|\boldsymbol{f_2}\|_{X_2^*}).\end{aligned} \tag{2.41}$$

Moreover, applying (2.7), we have the unique existence of (2.8), and the boundedness of λ:

$$\|\lambda\|_\Lambda \leq C \sup_{\boldsymbol{v}\in X} \frac{b_I(\boldsymbol{v},\lambda)}{\|\boldsymbol{v}\|_X} = C \sup_{\boldsymbol{v}\in X} \frac{(\boldsymbol{f},\boldsymbol{v}) - a(\boldsymbol{u},\boldsymbol{v}) - b(\boldsymbol{v},p)}{\|\boldsymbol{v}\|_X}.$$

By (2.40a), (2.40b) and

$$\begin{aligned} |b(\boldsymbol{v},p)| &\leq \|\nabla\cdot\boldsymbol{v_1}\|_{L^2(\Omega_1)}\|p_1\|_{L^2(\Omega_1)} + \|\nabla\cdot\boldsymbol{v_2}\|_{L^3(\Omega_2)}\|p_2\|_{L^{\frac{3}{2}}(\Omega_2)} \\ &\leq C\|\boldsymbol{v}\|_X\|p\|_M, \end{aligned}$$

we get

$$\|\lambda\|_\Lambda \leq C(\|\boldsymbol{f_1}\|_{X_1^*} + \|\boldsymbol{f_2}\|_{X_2^*} + \|\boldsymbol{u_1}\|_{X_1} + \|\boldsymbol{u_2}\|_{L^2(\Omega_2)} + \|\boldsymbol{u_2}\|^2_{L^3(\Omega_2)} + \|p\|_M).$$

Together with (2.41) we conclude (2.39). □

2.4 Picard Iteration

To solve the nonlinear coupled problem (2.11), we utilize the Picard iterative method (also called Kačanov method) [3, 34]. For any $\boldsymbol{w},\boldsymbol{u},\boldsymbol{v}\in\mathring{V}$, we define

$$\begin{aligned} \hat{a}_1(\boldsymbol{w_1};\boldsymbol{u_1},\boldsymbol{v_1}) &:= \int_{\Omega_1} g_1(|\boldsymbol{D}(\boldsymbol{w_1})|)\boldsymbol{D}(\boldsymbol{u_1})\colon \boldsymbol{D}(\boldsymbol{v_1})\,dx \\ &\quad + \sum_{l=1}^{d-1}\int_\Gamma r_l^{-1}(\boldsymbol{u_1}\cdot\boldsymbol{t}_l)(\boldsymbol{v_1}\cdot\boldsymbol{t}_l)\,ds, \\ \hat{a}_2(\boldsymbol{w_2};\boldsymbol{u_2},\boldsymbol{v_2}) &:= \int_{\Omega_2}\left(\boldsymbol{K}^{-1}g_2(|\boldsymbol{w_2}|)\boldsymbol{u_2} + C_F|\boldsymbol{w_2}|\boldsymbol{u_2}\right)\cdot\boldsymbol{v_2}\,dx, \\ \hat{a}(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) &:= \hat{a}_1(\boldsymbol{w_1};\boldsymbol{u_1},\boldsymbol{v_1}) + \hat{a}_2(\boldsymbol{w_2};\boldsymbol{u_2},\boldsymbol{v_2}). \end{aligned}$$

Note that $\hat{a}_1(\boldsymbol{u_1};\boldsymbol{u_1},\boldsymbol{v_1}) = a_1(\boldsymbol{u_1},\boldsymbol{v_1})$ and $\hat{a}_2(\boldsymbol{u_2};\boldsymbol{u_2},\boldsymbol{v_2}) = a_2(\boldsymbol{u_2},\boldsymbol{v_2})$. (2.11) is equivalently expressed as:

Find $\boldsymbol{u}\in\mathring{V}$ such that

$$\hat{a}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in\mathring{V}. \tag{2.42}$$

The Kačanov method is stated as follows.

Given $\boldsymbol{u}^{(0)} = (\boldsymbol{u_1}^{(0)},\boldsymbol{u_2}^{(0)})$, for $l = 1, 2, \ldots,$ find $\boldsymbol{u}^{(l)}\in\mathring{V}$ such that

$$\hat{a}(\boldsymbol{u}^{(l-1)};\boldsymbol{u}^{(l)},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in\mathring{V}. \tag{2.43}$$

(2.43) is a linear elliptic problem admitting a unique solution. We turn to the convergence of Kačanov method. For briefness, we set

$$\boldsymbol{\varepsilon}^{(l)} = (\boldsymbol{\varepsilon_1}^{(l)},\boldsymbol{\varepsilon_2}^{(l)}) := (\boldsymbol{u_1} - \boldsymbol{u_1}^{(l)}, \boldsymbol{u_2} - \boldsymbol{u_2}^{(l)}).$$

Theorem 2.3 *Set the constants* $C_1 := \frac{C_{g1}^2}{\nu_{1\infty}^2}$ *and* $C_2 := (\frac{k_{\max}}{\nu_{2\infty}})^2(\frac{C_{g2}}{k_{\min}} + C_F)^2$. *Under the assumption that*

$$\max\left(C_1\|\boldsymbol{D}(\boldsymbol{u_1})\|^2_{L^\infty(\Omega_1)}, C_2\|\boldsymbol{u_2}\|^2_{L^\infty(\Omega_2)}\right) =: C_I < 1, \tag{2.44}$$

The unique solution $\boldsymbol{u}^{(l)}$ *of Picard iteration* (2.43) *converges to* $\boldsymbol{u}$. *In particular,*

$$\begin{aligned} &\frac{\nu_{1\infty}}{2}\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l)})\|^2_{L^2(\Omega_1)} + \frac{\nu_{2\infty}}{2k_{\max}}\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)}\|^2_{L^2(\Omega_2)} \\ &\quad \le C_I\Big(\frac{\nu_{1\infty}}{2}\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l-1)})\|^2_{L^2(\Omega_1)} + \frac{\nu_{2\infty}}{2k_{\max}}\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l-1)}\|^2_{L^2(\Omega_2)}\Big). \end{aligned} \tag{2.45}$$

Proof Subtracting (2.43) from (2.42) and taking $\boldsymbol{v} = \boldsymbol{\varepsilon}^{(l)}$, we have

$$\hat{a}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{\varepsilon}^{(l)}) - \hat{a}(\boldsymbol{u}^{(l-1)}; \boldsymbol{u}^{(l)}, \boldsymbol{\varepsilon}^{(l)}) = 0,$$

which yields

$$\begin{aligned} &\nu_{1\infty}\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l)})\|^2_{L^2(\Omega_1)} + \sum_{l=1}^{d-1} r_l^{-1}\|\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l)}\cdot \boldsymbol{t}_l\|^2_{L^2(\Gamma)} + \frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)}\|^2_{L^2(\Omega_2)} \\ &\quad + C_F\int_{\Omega_2} |\boldsymbol{u}_{\mathbf{2}}^{(l-1)}|(\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)})^2 dx \le \hat{a}(\boldsymbol{u}^{(l-1)}; \boldsymbol{u}, \boldsymbol{\varepsilon}^{(l)}) - \hat{a}(\boldsymbol{u}; \boldsymbol{u}, \boldsymbol{\varepsilon}^{(l)}) \\ &= \int_{\Omega_1}\Big(g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{(l-1)})|) - g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}})|)\Big)\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}) : \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l)})\, dx \\ &\quad + \int_{\Omega_2} \boldsymbol{K}^{-1}\big(g_2(\boldsymbol{u}_{\mathbf{2}}^{(l-1)}) - g_2(\boldsymbol{u}_{\mathbf{2}})\big)\boldsymbol{u}_{\mathbf{2}}\cdot \boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)}\, dx \\ &\quad + \int_{\Omega_2} C_F\big(|\boldsymbol{u}_{\mathbf{2}}^{(l-1)}| - |\boldsymbol{u}_{\mathbf{2}}|\big)\boldsymbol{u}_{\mathbf{2}}\cdot \boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)}\, dx \\ &\le C_{g_1}\|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}})\|_{L^\infty(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l-1)})\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l)})\|_{L^2(\Omega_1)} \\ &\quad + (\frac{C_{g2}}{k_{\min}} + C_F)\|\boldsymbol{u}_{\mathbf{2}}\|_{L^\infty(\Omega_2)}\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l-1)}\|_{L^2(\Omega_2)}\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)}\|_{L^2(\Omega_2)}. \end{aligned}$$

Applying the Schwarz inequality to the above inequality, we achieve (2.45), which implies that $\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{(l)})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{\varepsilon}_{\mathbf{2}}^{(l)}\|_{L^2(\Omega_2)} \le CC_I^l \downarrow 0$ as $l \to \infty$ if $C_I < 1$. □

Remark 2.7 If $\|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}})\|^2_{L^\infty(\Omega_1)}$ and $\|\boldsymbol{u}_{\mathbf{2}}\|_{L^\infty(\Omega_2)}$ are sufficiently small, then the assumption (2.44) is satisfied and the convergence is guaranteed. However, since the boundedness of $\|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}})\|^2_{L^\infty(\Omega_1)}$ and $\|\boldsymbol{u}_{\mathbf{2}}\|_{L^\infty(\Omega_2)}$ is unprescribed, the assumption (2.44) is nontrivial to verify. Nevertheless, our numerical examples show the Picard iteration is quite applicable (see Sect. 4).

Next we consider the case that $C_F = 0$ (i.e., the absence of the Forchheimer term $C_F|\boldsymbol{u}_{\mathbf{2}}|\boldsymbol{u}_{\mathbf{2}}$) and $g_1(\cdot)$, $g_2(\cdot)$ are non-increasing functions. We show that the Picard iteration converges without the assumption (2.44). In this case, the model becomes the non-Newtonian Stokes–Darcy system, and we introduce the function spaces:

$$\begin{aligned} \tilde{X} &:= \{\boldsymbol{v} = (\boldsymbol{v}_{\mathbf{1}}, \boldsymbol{v}_{\mathbf{2}}) \in H^1(\Omega_1)\times H(\text{div}; \Omega_2) : \boldsymbol{v}_{\mathbf{1}}|_{\Gamma_1} = 0,\ \boldsymbol{v}_{\mathbf{2}}\cdot \boldsymbol{n}|_{\Gamma_2} = 0\}, \\ \tilde{V} &:= \{\boldsymbol{v} \in \tilde{X} : \int_\Gamma \mu(\boldsymbol{v}_{\mathbf{1}}\cdot \boldsymbol{n}_{12} + \boldsymbol{v}_{\mathbf{2}}\cdot \boldsymbol{n}_{21})\, ds = 0\ \forall \mu \in H_{00}^{\frac{1}{2}}(\Gamma)\}, \\ \mathring{\tilde{V}} &:= \{\boldsymbol{v} \in \tilde{V} : \nabla\cdot \boldsymbol{v}_{\mathbf{1}} = 0,\ \nabla\cdot \boldsymbol{v}_{\mathbf{2}} = 0\}. \end{aligned}$$

Given $\boldsymbol{w} \in \mathring{\tilde{V}}$, we define the bilinear form: for any $\boldsymbol{u}, \boldsymbol{v} \in \mathring{\tilde{V}}$,

$$B(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v}) := \int_{\Omega_1} g_1(|\boldsymbol{D}(\boldsymbol{w_1})|)\boldsymbol{D}(\boldsymbol{u_1}) : \boldsymbol{D}(\boldsymbol{v_1})\,dx \\ + \sum_{l=1}^{d-1}\int_\Gamma r_l^{-1}(\boldsymbol{u_1}\cdot\boldsymbol{t}_l)(\boldsymbol{v_1}\cdot\boldsymbol{t}_l)\,ds + \int_{\Omega_2}\boldsymbol{K}^{-1}g_2(|\boldsymbol{w_2}|)\boldsymbol{u_2}\cdot\boldsymbol{v_2}\,dx.$$

The variational form and the Picard iteration are presented as follows.

Find $\boldsymbol{u}\in\mathring{\tilde{V}}$ such that

$$B(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in\mathring{\tilde{V}}. \tag{2.46}$$

Given $\boldsymbol{u}^{(0)} = (\boldsymbol{u}_1^{(0)},\boldsymbol{u}_2^{(0)})$, for $l=1,2,\ldots,$ find $\boldsymbol{u}^{(l)}\in\mathring{\tilde{V}}$ such that

$$B(\boldsymbol{u}^{(l-1)};\boldsymbol{u}^{(l)},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\,\boldsymbol{v}\in\mathring{\tilde{V}}. \tag{2.47}$$

It is apparent that, under (2.4) and (2.5), (2.47) admits a unique solution.

Proposition 2.1 *If $g_1(\cdot)'\le 0$ and $g_2(\cdot)'\le 0$, then the Picard iteration (2.47) converges, i.e.,*

$$\boldsymbol{u}^{(l)}\to\boldsymbol{u} \quad in\ \mathring{\tilde{V}} \qquad (l\to\infty).$$

Proof We define a functional $F:\mathring{\tilde{V}}\to\mathbb{R}$:

$$F(\boldsymbol{u}) := 2^{-1}\Big(\int_{\Omega_1}\int_0^{|\boldsymbol{D}(\boldsymbol{u_1})|^2} g_1(\tau)\,d\tau\,dx \\ + \sum_{l=1}^{d-1}\int_\Gamma r_l^{-1}\int_0^{|\boldsymbol{u_1}\cdot\boldsymbol{t}_l|^2} 1\,d\tau\,ds + \int_{\Omega_2}\boldsymbol{K}^{-1}\int_0^{|\boldsymbol{u}|^2} g_2(\tau)\,d\tau\,dx\Big).$$

It is easy to check that

$$\langle F'(\boldsymbol{u}),\boldsymbol{v}-\boldsymbol{w}\rangle = B(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v}-\boldsymbol{w}) \qquad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\mathring{\tilde{V}}.$$

Since $g_1(\cdot)'\le 0$ and $g_2(\cdot)'\le 0$,

$$g_i(\beta)(\beta-\alpha)\le\int_\alpha^\beta g_i(\tau)\,d\tau\le g_i(\alpha)(\beta-\alpha) \quad (0\le\alpha\le\beta<\infty,\ i=1,2),$$

which implies

$$F(\boldsymbol{v})-F(\boldsymbol{u})\le 2^{-1}\big(B(\boldsymbol{u};\boldsymbol{v},\boldsymbol{v})-B(\boldsymbol{u};\boldsymbol{u},\boldsymbol{u})\big). \tag{2.48}$$

Moreover, $B(\boldsymbol{w};\cdot,\cdot)$ satisfies the continuity and coercivity (by (2.4a), (2.5a), Korn inequality and $\nabla\cdot\boldsymbol{u}=\nabla\cdot\boldsymbol{v}=0$): for all $\boldsymbol{u},\boldsymbol{v}\in\mathring{\tilde{V}}$,

$$|B(\boldsymbol{w};\boldsymbol{u},\boldsymbol{v})| \le \nu_{10}\|\boldsymbol{D}(\boldsymbol{u_1})\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{v_1})\|_{L^2(\Omega_1)} + \frac{\nu_{20}}{k_{\min}}\|\boldsymbol{u_2}\|_{L^2(\Omega_2)}\|\boldsymbol{v_2}\|_{L^2(\Omega_2)} \\ \le C\|\boldsymbol{u}\|_{H^1(\Omega_1)\times H(\mathrm{div},\Omega_2)}\|\boldsymbol{v}\|_{H^1(\Omega_1)\times H(\mathrm{div},\Omega_2)}, \tag{2.49a}$$

$$B(\boldsymbol{w};\boldsymbol{u}-\boldsymbol{v},\boldsymbol{u}-\boldsymbol{v}) \ge \nu_{1\infty}\|\boldsymbol{D}(\boldsymbol{u_1}-\boldsymbol{v_1})\|^2_{L^2(\Omega_1)} + \frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{u_2}-\boldsymbol{v_2}\|^2_{L^2(\Omega_2)} \\ \ge C\|\boldsymbol{u}-\boldsymbol{v}\|^2_{H^1(\Omega_1)\times H(\mathrm{div},\Omega_2)}, \tag{2.49b}$$

It follows from (2.48), (2.49) and [34, Theorem 25.L] that $\boldsymbol{u}^{(l)}\to\boldsymbol{u}$ in $\mathring{\tilde{V}}$. □

3 Discretization Scheme

We adopt the discontinuous Galerkin method to discretize the coupled system (2.1)–(2.3). In this section, we first prove the unique existence of the discrete solution and investigate the convergence of the Picard iteration, and then show the error estimates.

3.1 The Discontinuous Galerkin Approximation

We denote by $\mathcal{T}_i^h$ a regular and quasi-uniform triangulation of Ω_i, by Γ_i^h the set of interior facets, by $h_E := \mathrm{diam}(E)$ the diameter of the element E, and by $h := \max_{E\in\mathcal{T}_1^h\cap\mathcal{T}_2^h} h_E$ the mesh size. We introduce the discontinuous finite element spaces:

$$
\begin{aligned}
&X_i^h := \{\boldsymbol{v_i^h} \in L^2(\Omega_i)^d : \boldsymbol{v_i^h}|_E \in P^k(E)^d \ (\forall E \in \mathcal{T}_i^h)\}, \quad X^h := X_1^h \times X_2^h,\\
&M_i^h := \{q_i^h \in M_i : q_i^h|_E \in P^{k-1}(E) \ (\forall E \in \mathcal{T}_i^h)\},\\
&M^h := \{q^h = (q_1^h, q_2^h) \in M_1^h \times M_2^h : (q_1^h, 1)_{\Omega_1} + (q_2^h, 1)_{\Omega_2} = 0\},
\end{aligned}
$$

where the $P^k(E)$ is the space of polynomials of degree k on the element E ($k = 1$ or 2).

Let ξ be a scalar or vector-valued function. We now introduce the average $\{\!\{\xi\}\!\}$ and jump $[\![\xi]\!]$ on each interior facet $e \in \partial E_1 \cup \partial E_2$,

$$
\{\!\{\xi\}\!\} = \frac{1}{2}(\xi|_{E_1} + \xi|_{E_2}), \quad [\![\xi]\!] = \xi|_{E_1} - \xi|_{E_2}.
$$

If $e \in \partial\Omega$ and $e \in E_1$, then $\{\!\{\xi\}\!\} = \xi|_{E_1}$, $[\![\xi]\!] = \xi|_{E_1}$. The norms of the above spaces are given as follows

$$
\begin{aligned}
&\|\boldsymbol{v_1^h}\|_{X_1^h} := |||\nabla \boldsymbol{v_1^h}|||_{L^2(\Omega_1)} + \Big(\sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_{1e}}{|e|}\|[\![\boldsymbol{v_1^h}]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}},\\
&\|\boldsymbol{v_2^h}\|_{X_2^h} := \|\boldsymbol{v_2^h}\|_{L^2(\Omega_2)} + \Big(\sum_{e\in\Gamma_2^h\cup\Gamma_2} \frac{\sigma_{2e}}{|e|}\|[\![\boldsymbol{v_2^h}\cdot\boldsymbol{n_e}]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}},\\
&\|\boldsymbol{v_2^h}\|_{X_2^h, W^{0,r}(\mathrm{div})} := \|\boldsymbol{v_2^h}\|_{L^r(\Omega_2)} + |||\nabla\cdot\boldsymbol{v_2^h}|||_{L^r(\Omega_2)} + \|\boldsymbol{v_2^h}\|_{X_2^h},\\
&\|\boldsymbol{v_2^h}\|_{X_2^h, W^{1,2}} := \|\boldsymbol{v_2^h}\|_{L^2(\Omega_2)} + |||\nabla\boldsymbol{v_2^h}|||_{L^2(\Omega_2)} + \Big(\sum_{e\in\Gamma_2^h\cup\Gamma_2} \frac{\sigma_{2e}}{|e|}\|[\![\boldsymbol{v_2^h}]\!]\|_{L^2(e)}^2\Big)^{\frac{1}{2}},\\
&\|\boldsymbol{v^h}\|_{X^h} := \|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h},\\
&\|q^h\|_{M^h} := \|q_1^h\|_{M_1^h} + \|q_2^h\|_{M_2^h} = \|q_1^h\|_{L^2(\Omega_1)} + \|q_2^h\|_{L^{\frac{3}{2}}(\Omega_2)},
\end{aligned}
$$

where the parameters σ_{1e} and σ_{2e} are positive constants, $|e|$ is the length of edge e, and for $i = 1, 2$,

$$
|||\nabla\boldsymbol{v_i^h}|||_{L^2(\Omega_i)}^2 := \sum_{E\in\mathcal{T}_i^h} \|\nabla\boldsymbol{v_i^h}\|_{L^2(E)}^2, \ |||\nabla\cdot\boldsymbol{v_2^h}|||_{L^2(\Omega_2)}^2 := \sum_{E\in\mathcal{T}_2^h} \|\nabla\cdot\boldsymbol{v_2^h}\|_{L^2(E)}^2.
$$

The following trace inequalities (cf. [28]) will be used for analysis.

$$
\|\nabla\boldsymbol{v}\cdot\boldsymbol{n_e}\|_{L^2(e)}^2 \le C_t(h_E^{-1}\|\boldsymbol{v}\|_{H^1(E)}^2 + h_E\|\boldsymbol{v}\|_{H^2(E)}^2) \quad (\forall\boldsymbol{v}\in H^2(E),\ \forall e\subset\partial E), \tag{3.1a}
$$

$$
\|\nabla\boldsymbol{v}\cdot\boldsymbol{n_e}\|_{L^2(e)} \le C_t h_E^{-\frac{1}{2}}\|\nabla\boldsymbol{v}\|_{L^2(E)} \quad (\forall\boldsymbol{v}\in P^k(E),\ \forall e\subset\partial E). \tag{3.1b}
$$

Assume that $\mathcal{T}_1^h$ and $\mathcal{T}_2^h$ match each other at interface Γ. We denote by $\mathcal{E}^h$ the partition of the interface Γ inherited from $\mathcal{T}_1^h$ (also $\mathcal{T}_1^h$), and set

$$\Lambda^h := \{\mu^h \in L^2(\Gamma) : \mu^h|_e \in P^k(e)\ (\forall e \in \mathcal{E}^h)\} \quad (k = 1, 2),$$
$$V^h := \{\boldsymbol{v^h} \in X^h : \sum_{e\in\Gamma} \int_e \mu^h(\boldsymbol{v_1^h} \cdot \boldsymbol{n_{12}} + \boldsymbol{v_2^h} \cdot \boldsymbol{n_{21}})\, ds = 0\ \ (\forall \mu^h \in \Lambda^h)\}.$$

For any $\boldsymbol{u^h} = (\boldsymbol{u_1^h}, \boldsymbol{u_2^h})$, $\boldsymbol{v^h} = (\boldsymbol{v_1^h}, \boldsymbol{v_2^h}) \in X^h$ and $q^h = (q_1^h, q_2^h) \in M^h$, we set

$$\begin{aligned}
a_1^h(\boldsymbol{u_1^h}, \boldsymbol{v_1^h}) :=& \sum_{E\in\mathcal{T}_1^h} \int_E g_1(|\boldsymbol{D}(\boldsymbol{u_1^h})|)\boldsymbol{D}(\boldsymbol{u_1^h}) : \boldsymbol{D}(\boldsymbol{v_1^h})\, dx \\
& - \sum_{e\in\Gamma_1^h\cup\Gamma_1} \int_e \{\!\{ g_1(|\boldsymbol{D}(\boldsymbol{u_1^h})|)\boldsymbol{D}(\boldsymbol{u_1^h})\boldsymbol{n_e} \}\!\} \cdot [\![\boldsymbol{v_1^h}]\!]\, ds \\
& + \sum_{e\in\Gamma_1^h\cup\Gamma_1} \int_e \frac{\sigma_{1e}}{|e|} [\![\boldsymbol{u_1^h}]\!] \cdot [\![\boldsymbol{v_1^h}]\!]\, ds + \sum_{l=1}^{d-1}\sum_{e\in\Gamma} \int_e r_l^{-1}(\boldsymbol{u_1^h} \cdot \boldsymbol{t_l})(\boldsymbol{v_1^h} \cdot \boldsymbol{t_l})\, ds, \\
a_2^h(\boldsymbol{u_2^h}, \boldsymbol{v_2^h}) :=& \sum_{E\in\mathcal{T}_2^h} \int_E \left(\boldsymbol{K}^{-1} g_2(|\boldsymbol{u_2^h}|)\boldsymbol{u_2^h} + C_F|\boldsymbol{u_2^h}|\boldsymbol{u_2^h}\right) \cdot \boldsymbol{v_2^h}\, dx \\
& + \sum_{e\in\Gamma_2^h\cup\Gamma_2} \int_e \frac{\sigma_{2e}}{|e|} [\![\boldsymbol{u_2^h} \cdot \boldsymbol{n_e}]\!][\![\boldsymbol{v_2^h} \cdot \boldsymbol{n_e}]\!]\, ds, \\
b_i^h(\boldsymbol{v_i^h}, q_i^h) :=& -\sum_{E\in\mathcal{T}_i^h} \int_E q_i^h \nabla \cdot \boldsymbol{v_i^h}\, dx + \sum_{e\in\Gamma_i^h\cup\Gamma_i} \int_e \{\!\{q_i^h\}\!\}[\![\boldsymbol{v_i^h} \cdot \boldsymbol{n_e}]\!]\, ds\ (i = 1, 2), \\
a^h(\boldsymbol{u^h}, \boldsymbol{v^h}) :=& a_1^h(\boldsymbol{u_1^h}, \boldsymbol{v_1^h}) + a_2^h(\boldsymbol{u_2^h}, \boldsymbol{v_2^h}),\ \ b^h(\boldsymbol{v^h}, p^h) := b_1^h(\boldsymbol{v_1^h}, p_1^h) + b_2(\boldsymbol{v_2^h}, p_2^h).
\end{aligned}$$

Multiplying the equations with the test function $\boldsymbol{v_1^h} \in X_1^h$ and $\boldsymbol{v_2^h} \in X_2^h$, integrating by parts over element E and summing over all element, we obtain the discrete scheme:

Find $(\boldsymbol{u^h}, p^h) \in V^h \times M^h$ such that

$$\begin{cases} a^h(\boldsymbol{u^h}, \boldsymbol{v^h}) + b^h(\boldsymbol{v^h}, p^h) = (\boldsymbol{f}, \boldsymbol{v^h}) & \forall\ \boldsymbol{v^h} \in V^h, \\ b^h(\boldsymbol{u^h}, q^h) = 0 & \forall\ q^h \in M^h. \end{cases} \tag{3.2}$$

We introduce the quasi-local interpolation [29, 33] $\Pi := (\Pi_1, \Pi_2) : X \to X_h$ satisfying

$$b_i^h(\Pi_i \boldsymbol{v}_i - \boldsymbol{v}_i, p_i^h) = 0 \quad (\forall p^h \in M_i^h,\ i = 1, 2), \tag{3.3}$$

and the stability and error estimates

$$\|\Pi_1 \boldsymbol{v_1}\|_{X_1} \le c\|\boldsymbol{v_1}\|_{X_1}, \quad \|\Pi_1 \boldsymbol{v_1} - \boldsymbol{v_1}\|_{X_1} \le ch^{s-1}|\boldsymbol{v_1}|_{H^s(\Omega_1)} \quad (1 \le s \le k+1), \tag{3.4}$$

and, for any $E \in \mathcal{T}_2^h$,

$$\|\Pi_2 \boldsymbol{v_2} - \boldsymbol{v_2}\|_{H^m(E)} \le ch_E^{s-m}|\boldsymbol{v_2}|_{H^s(E)} \qquad (1 \le s \le k+1,\ m = 0, 1), \tag{3.5a}$$
$$\|\nabla \cdot (\Pi_2 \boldsymbol{v_2} - \boldsymbol{v_2})\|_{L^2(E)} \le ch_E^s|\nabla \cdot \boldsymbol{v_2}|_{H^s(E)} \qquad (0 \le s \le k). \tag{3.5b}$$

By the definition of Π_2 ([29, 33]), we have $b_2(\Pi_2 \boldsymbol{v_2} - \boldsymbol{v_2}, q_2^h) = 0$ $(\forall q_2^h \in M_2^h)$ and $\int_e (\Pi_2 \boldsymbol{v_2} - \boldsymbol{v_2}) \cdot \boldsymbol{n_e} \boldsymbol{w_2^h} \cdot \boldsymbol{n_e}\, ds = 0$ $(\forall \boldsymbol{w_2^h} \in X_2^h)$, which also implies that

$$\|[\![\Pi_2 \boldsymbol{v_2} \cdot \boldsymbol{n_e}]\!]\|_{L^2(e)} = 0 \quad (\forall e \in \Gamma_2^h \cup \Gamma_2),$$

$$||| \nabla \cdot \Pi_2 \boldsymbol{v_2} |||_{L^r(\Omega_2)} \le C \, ||| \nabla \cdot \boldsymbol{v_2} \, |||_{L^r(\Omega_2)} \quad (\forall r \in (1, \infty)).$$

Together with

$$\begin{aligned} \|\Pi_2 \boldsymbol{v_2}\|_{L^3(\Omega_2)} &\le \|\boldsymbol{v_2}\|_{L^3(\Omega_2)} + \|\boldsymbol{v_2} - \Pi_2 \boldsymbol{v_2}\|_{L^3(\Omega_2)} \\ &\le \|\boldsymbol{v_2}\|_{L^3(\Omega_2)} + ch\|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)} \le c\|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)}, \end{aligned}$$

we see that

$$\|\Pi_2 \boldsymbol{v_2}\|_{X_2^h, W^{0,3}(\mathrm{div})} \le c\|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)}$$

is valid. Noting that $\Pi \boldsymbol{v} \in V^h$ $(\forall \boldsymbol{v} \in V)$, we have (by (2.38)): for all $p^h \in M^h$,

$$\begin{aligned} &\sup_{\boldsymbol{v^h} \in V^h} \frac{b^h(\boldsymbol{v^h}, p^h)}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h, W^{0,3}(\mathrm{div})}} \ge \sup_{\boldsymbol{v} \in V} \frac{b^h(\boldsymbol{v}, p^h)}{\|\Pi_1 \boldsymbol{v_1}\|_{X_1^h} + \|\Pi_2 \boldsymbol{v_2}\|_{X_2^h, W^{0,3}(\mathrm{div})}} \\ &= \sup_{\boldsymbol{v} \in V} \frac{b(\boldsymbol{v}, p^h)}{\|\Pi_1 \boldsymbol{v_1}\|_{X_1^h} + \|\Pi_2 \boldsymbol{v_2}\|_{X_2^h, W^{0,3}(\mathrm{div})}} \ge c \sup_{\boldsymbol{v} \in W} \frac{b(\boldsymbol{v}, p^h)}{\|\boldsymbol{v_1}\|_{W^{1,2}(\Omega_1)} + \|\boldsymbol{v_2}\|_{W^{1,3}(\Omega_2)}} \\ &\ge c\beta \|p^h\|_{M^h}. \end{aligned} \tag{3.6}$$

The following inf-sup condition can be proved analogously:

$$\sup_{\boldsymbol{v^h} \in V^h} \frac{b^h(\boldsymbol{v^h}, p^h)}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h, W^{1,2}}} \ge c\beta (\|p_1^h\|_{L^2(\Omega_1)} + \|p_2^h\|_{L^2(\Omega_2)}) \quad (\forall p^h \in M^h). \tag{3.7}$$

Moreover, we set the bilinear form

$$b_I^h(\boldsymbol{v^h}, \lambda^h) := \sum_{e \in \Gamma} \int_e \lambda^h (\boldsymbol{v_1^h} \cdot \boldsymbol{n_{12}} + \boldsymbol{v_2^h} \cdot \boldsymbol{n_{21}}) \, ds \qquad (\forall \boldsymbol{v^h} \in X^h, \ \lambda^h \in \Lambda^h),$$

which satisfies (see [36, Lemma 4.6] for the definition of the discrete $H^{-\frac{1}{2}}$ norm $\|\cdot\|_{-\frac{1}{2}, \Lambda^h}$):

$$\sup_{\boldsymbol{v^h} \in X^h} \frac{b_I^h(\boldsymbol{v^h}, \lambda^h)}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h, W^{1,2}}} \ge C \|\lambda^h\|_{-\frac{1}{2}, \Lambda^h}. \tag{3.8}$$

It follows from (3.8) that (3.2) is equivalent to the discrete Lagrange multiplier problem: Find $(\boldsymbol{u^h}, p^h, \lambda^h) \in V^h \times M^h \times \Lambda^h$ such that

$$\begin{cases} a^h(\boldsymbol{u^h}, \boldsymbol{v^h}) + b^h(\boldsymbol{v^h}, p^h) + b_I^h(\boldsymbol{v^h}, \lambda^h) = (\boldsymbol{f}, \boldsymbol{v^h}) & \forall \ \boldsymbol{v^h} \in X^h, \\ b^h(\boldsymbol{u^h}, q^h) = 0 & \forall \ q^h \in M^h, \\ b_I^h(\boldsymbol{u^h}, \mu^h) = 0 & \forall \ \mu^h \in \Lambda^h. \end{cases} \tag{3.9}$$

For pressure p and Lagrange multiplier λ, we introduce the L^2-projection. Define $P_M : L^2(\Omega_i) \to M_i^h$, $P_\Lambda : L^2(\Gamma) \to \Lambda_h$,

$$\int_{\Omega_i} q_i^h (p - P_M p) \, dx = 0 \qquad \forall q_i^h \in M_i^h \quad (i = 1, 2), \tag{3.10a}$$

$$\int_\Gamma \mu^h (P_\Lambda \lambda - \lambda) \, ds = 0 \qquad \forall \mu^h \in \Lambda^h. \tag{3.10b}$$

The following error estimates are valid:

$$\|P_M p - p\|_{H^m(E)} \le c h_E^{k-m} |p|_{H^k(E)} \qquad (\forall E \in \mathcal{T}_i^h, \ m = 0, 1), \tag{3.11a}$$

$$\|P_\Lambda \lambda - \lambda\|_{H^m(e)} \le c h_e^{k-m} |\lambda|_{H^k(e)} \qquad (h_e = \mathrm{diam}(e), \ \forall e \in \mathcal{E}^h, \ m = 0, 1). \tag{3.11b}$$

3.2 Existence and Uniqueness of the Discrete Problem

We consider the well-posedness of (3.9) (also (3.13)) in this subsection. Set

$$\mathring{V}^h := \{\boldsymbol{v^h} \in V^h : b^h(\boldsymbol{v^h}, q^h) = 0, \quad (\forall q^h \in M^h)\},$$

and denote by $(\mathring{V}^h)^*$ the dual of $\mathring{V}^h$. Define the nonlinear operator $A^h : \mathring{V}^h \to (\mathring{V}^h)^*$,

$$(A^h(\boldsymbol{u^h}), \boldsymbol{v^h}) := a^h(\boldsymbol{u^h}, \boldsymbol{v^h}) \quad (\forall\, \boldsymbol{u^h}, \boldsymbol{v^h} \in \mathring{V}^h). \tag{3.12}$$

We consider the discrete problem:
Find $\boldsymbol{u^h} \in \mathring{V}^h$ such that

$$(A^h(\boldsymbol{u^h}), \boldsymbol{v^h}) = (\boldsymbol{f}, \boldsymbol{v^h}) \quad \forall\, \boldsymbol{v^h} \in \mathring{V}^h. \tag{3.13}$$

As with Sect. 2.3, we will verify the hemicontinuity, coercivity, and monotonicity of A^h, which are presented by the following lemmas.

Lemma 3.1 *The operator A^h is continuous and bounded, satisfying: for any $\boldsymbol{u^h}, \boldsymbol{v^h} \in V^h$,*

$$\begin{aligned} |\langle A^h(\boldsymbol{u^h}), \boldsymbol{v^h}\rangle| \le c\big(&\|\boldsymbol{u_1^h}\|_{X_1^h}\|\boldsymbol{v_1^h}\|_{X_1^h} \\ &+ \|\boldsymbol{u_2^h}\|_{X_2^h}\|\boldsymbol{v_2^h}\|_{X_2^h} + \|\boldsymbol{u_2^h}\|^2_{L^3(\Omega_2)}\|\boldsymbol{v_2^h}\|_{L^3(\Omega_2)}\big). \end{aligned} \tag{3.14}$$

Proof In the following, we only prove the boundedness (3.14). The continuity can be proved in the same way as Lemma 2.1. For any $\boldsymbol{u^h} = (\boldsymbol{u_1^h},\ \boldsymbol{u_2^h})$, $\boldsymbol{v^h} = (\boldsymbol{v_1^h}, \boldsymbol{v_2^h}) \in \mathring{V}^h$, using (3.1b) and Schwarz inequality, we calculate as

$$\begin{aligned} &- \sum_{e\in\Gamma_1^h\cup\Gamma_1} \int_e \{\!\!\{ g_1(|\boldsymbol{D}(\boldsymbol{u_1^h})|)\boldsymbol{D}(\boldsymbol{u_1^h})\boldsymbol{n_e} \}\!\!\} \cdot [\![\boldsymbol{v_1^h}]\!]\, ds \\ &\le c\Big(\sum_{E\in\mathcal{T}_1^h} \|\nabla \boldsymbol{u_1^h}\|_{L^2(E^+)} + \|\nabla \boldsymbol{u_1^h}\|_{L^2(E^-)}\Big)\|\boldsymbol{v_1^h}\|_{X_1^h} \le c\|\boldsymbol{u_1^h}\|_{X_1^h}\|\boldsymbol{v_1^h}\|_{X_1^h}, \\ &\sum_{e\in\Gamma_1^h\cup\Gamma_1} \int_e \frac{\sigma_{1e}}{|e|}[\![\boldsymbol{u_1^h}]\!] \cdot [\![\boldsymbol{v_1^h}]\!]\, ds \\ &\le \Big(\sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_{1e}}{|e|}\|[\![\boldsymbol{u_1^h}]\!]\|^2_{L^2(e)}\Big)^{\frac12}\Big(\sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\sigma_{1e}}{|e|}\|[\![\boldsymbol{v_1^h}]\!]\|^2_{L^2(e)}\Big)^{\frac12} \le c\|\boldsymbol{u_1^h}\|_{X_1^h}\|\boldsymbol{v_1^h}\|_{X_1^h}, \end{aligned}$$

where E^+ and E^- represent the two elements to which the edge e belongs. It follows from HRlder's inequality that,

$$\begin{aligned} &\sum_{E\in\mathcal{T}_2^h} \int_E \big(\boldsymbol{K}^{-1}g_2(|\boldsymbol{u_2^h}|)\boldsymbol{u_2^h} + C_F|\boldsymbol{u_2^h}|\boldsymbol{u_2^h}\big) \cdot \boldsymbol{v_2^h}\, dx \\ &\le \frac{\nu_{20}}{k_{\min}}\|\boldsymbol{u_2^h}\|_{L^2}\|\boldsymbol{v_2^h}\|_{L^2} + C_F\|\boldsymbol{u_2^h}\|^2_{L^3(\Omega_2)}\|\boldsymbol{v_2^h}\|_{L^3(\Omega_2)}. \end{aligned}$$

Hence, combining the above inequalities, we deduce (3.14). □

Lemma 3.2 *If σ_{1e} is sufficiently large, then A^h is coercive, i.e.,*

$$\frac{(A^h(\boldsymbol{v^h}), \boldsymbol{v^h})}{\|\boldsymbol{v^h}\|_{X^h}} \to +\infty \quad as \quad \|\boldsymbol{v^h}\|_{X^h} \to \infty. \tag{3.15}$$

In particular,

$$(A^h(\boldsymbol{v^h}), \boldsymbol{v^h}) \geq c\big(\|\boldsymbol{v_1^h}\|^2_{X_1^h} + \|\boldsymbol{v_2^h}\|^2_{X_2^h}\big) + C_F\|\boldsymbol{v_2^h}\|^3_{L^3(\Omega_2)} \quad (\forall\, \boldsymbol{v^h} \in V^h). \tag{3.16}$$

Proof For all $\boldsymbol{v^h} = (\boldsymbol{v_1^h}, \boldsymbol{v_2^h}) \in \mathring{V}^h$, by the discrete Korn's inequality [4, 28] and (2.4a)

$$\sum_{E\in\mathcal{T}_1^h}\int_E g_1(|\boldsymbol{D}(\boldsymbol{v_1^h})|)\boldsymbol{D}(\boldsymbol{v_1^h}) : \boldsymbol{D}(\boldsymbol{v_1^h})\,dx$$
$$\geq \sum_{E\in\mathcal{T}_1^h}\nu_{1\infty}\|\boldsymbol{D}(\boldsymbol{v_1^h})\|^2_{L^2(E)} \geq \frac{\nu_{1\infty}}{C_1}\,|||\,\nabla\boldsymbol{v_1^h}\,|||^2_{L^2(\Omega_1)} - \nu_{1\infty}\sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{1}{|e|}\|[\![\boldsymbol{v_1}]\!]\|^2_{L^2(e)}.$$

Applying (2.4a), (3.1b) and Young's inequalities,

$$-\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \{\!\{g_1(|\boldsymbol{D}(\boldsymbol{v_1^h})|)\boldsymbol{D}(\boldsymbol{v_1^h})\boldsymbol{n_e}\}\!\}\cdot[\![\boldsymbol{v_1^h}]\!]\,ds$$
$$\geq -\sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{C_t}{2}|e|^{-\frac{1}{2}}\nu_{10}\big(\|\,\nabla\boldsymbol{v_1^h}\|_{L^2(E^+)} + \|\,\nabla\boldsymbol{v_1^h}\|_{L^2(E^-)}\big)\|[\![\boldsymbol{v_1^h}]\!]\|_{L^2(e)}$$
$$\geq -\sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{\nu_{1\infty}}{8C_1}\big(\|\,\nabla\boldsymbol{v_1^h}\|_{L^2(E^+)} + \|\,\nabla\boldsymbol{v_1^h}\|_{L^2(E^-)}\big)^2$$
$$-\sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{C_1C_t^2\nu_{10}^2}{2\nu_{1\infty}|e|}\|[\![\boldsymbol{v_1^h}]\!]\|^2_{L^2(e)}$$
$$\geq -\frac{\nu_{1\infty}}{2C_1}\,|||\,\nabla\boldsymbol{v_1^h}\,|||^2_{L^2(\Omega_1)} - \sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{C_3}{|e|}\|[\![\boldsymbol{v_1}]\!]\|^2_{L^2(e)}.$$

By taking sufficiently large σ_{1e} such that $\sigma_{1e} - \nu_{1\infty} - C_3 > 0$,

$$a_1^h(\boldsymbol{v_1^h}, \boldsymbol{v_1^h}) \geq \frac{\nu_{1\infty}}{2C_1}\,|||\,\nabla\boldsymbol{v_1^h}\,|||^2_{L^2(\Omega_1)} + \sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{\sigma_{1e} - \nu_{1\infty} - C_3}{|e|}\|[\![\boldsymbol{v_1}]\!]\|^2_{L^2(e)} \geq c\|\boldsymbol{v_1^h}\|^2_{X_1^h}.$$

Moreover, by (2.5a), we have

$$a_2^h(\boldsymbol{v_2^h}, \boldsymbol{v_2^h}) \geq \sum_{E\in\mathcal{T}_2^h}\frac{\nu_{2\infty}}{k_{\max}}\|\boldsymbol{v_2^h}\|^2_{L^2(E)} + C_F\sum_{E\in\mathcal{T}_2^h}\|\boldsymbol{v_2^h}\|^3_{L^3(E)}$$
$$+\sum_{e\in\Gamma_2^h\cup\Gamma_2}\frac{\sigma_{2e}}{|e|}\|[\![\boldsymbol{v_2^h}\cdot\boldsymbol{n_e}]\!]\|^2_{L^2(e)} \geq c\|\boldsymbol{v_2^h}\|^2_{X_2^h} + C_F\|\boldsymbol{v_2^h}\|^3_{L^3(\Omega_2)}.$$

Hence, we obtain (3.16), which implies (3.15). □

Lemma 3.3 *We assume the boundedness of* $g_1'(|\boldsymbol{A}|)\boldsymbol{A}$, *i.e.,*

$$\big|g_1'(|\boldsymbol{A}|)\big||\boldsymbol{A}| \leq C \quad (\forall \boldsymbol{A} \in \mathbb{R}^{d\times d}_{\text{sym}}). \tag{3.17}$$

For sufficiently large σ_{1e}, A^h *is monotone, i.e., there is positive constant* c_α *such that, for all* $\boldsymbol{u^h}, \boldsymbol{v^h} \in \mathring{V}^h$,

$$(A^h(\boldsymbol{u^h}) - A^h(\boldsymbol{v^h}), \boldsymbol{u^h} - \boldsymbol{v^h}) \geq c_\alpha\big(\|\boldsymbol{u_1^h} - \boldsymbol{v_1^h}\|^2_{X_1^h} + \|\boldsymbol{u_2^h} - \boldsymbol{v_2^h}\|^2_{X_2^h}\big). \tag{3.18}$$

Proof Let $\boldsymbol{u^h} = (\boldsymbol{u_1^h}, \boldsymbol{u_2^h})$, $\boldsymbol{v^h} = (\boldsymbol{v_1^h}, \boldsymbol{v_2^h}) \in \mathring{V}^h$. By $\left|g_1'(|\boldsymbol{A}|)\right||\boldsymbol{A}| \leq C$ and (3.1b), we calculate as

$$\begin{aligned}
&-\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \{\!\!\{ g_1(|\boldsymbol{D}(\boldsymbol{u_1^h})|)\boldsymbol{D}(\boldsymbol{u_1^h})\boldsymbol{n_e} - g_1(|\boldsymbol{D}(\boldsymbol{v_1^h})|)\boldsymbol{D}(\boldsymbol{v_1^h})\boldsymbol{n_e}\}\!\!\} \cdot [\![\boldsymbol{u_1^h} - \boldsymbol{v_1^h}]\!]\, ds \\
&= -\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e\int_0^1 [\![\boldsymbol{u_1^h} - \boldsymbol{v_1^h}]\!] \\
&\quad\cdot \frac{d}{d\xi}\left[\{\!\!\{ g_1(|\boldsymbol{D}(\boldsymbol{v_1^h} + \xi(\boldsymbol{u_1^h} - \boldsymbol{v_1^h}))|)\boldsymbol{D}(\boldsymbol{v_1^h} + \xi(\boldsymbol{u_1^h} - \boldsymbol{v_1^h}))\boldsymbol{n_e}\}\!\!\}\right] ds \\
&\geq -\sum_{e\in\Gamma_1^h\cup\Gamma_1}(2\nu_{10} - C)\|\{\!\!\{\boldsymbol{D}(\boldsymbol{u_1^h} - \boldsymbol{v_1^h})\boldsymbol{n_e}\}\!\!\}\|_{L^2(e)}\|[\![\boldsymbol{u_1^h} - \boldsymbol{v_1^h}]\!]\|_{L^2(e)} \\
&\geq -c \,|||\, \nabla(\boldsymbol{u_1^h} - \boldsymbol{v_1^h}) \,|||_{L^2(\Omega_1)}^2 - \sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{C_4}{|e|}\|[\![\boldsymbol{u_1^h} - \boldsymbol{v_1^h}]\!]\|_{L^2(e)}^2.
\end{aligned}$$

Taking $\sigma_{1e} > C_4 + C$, together with (2.4c) and Korn's inequality, we obtain

$$\begin{aligned}
&a_1^h(\boldsymbol{u_1^h}, \boldsymbol{u_1^h} - \boldsymbol{v_1^h}) - a_1^h(\boldsymbol{v_1^h}, \boldsymbol{u_1^h} - \boldsymbol{v_1^h}) \\
&\quad\geq c \,|||\, \nabla(\boldsymbol{u_1^h} - \boldsymbol{v_1^h}) \,|||_{L^2(\Omega_1)}^2 + \sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{\sigma_{1e} - C_4 - C}{|e|}\|[\![\boldsymbol{u_1^h} - \boldsymbol{v_1^h}]\!]\|_{L^2(e)}^2 \geq c\|\boldsymbol{u_1^h} - \boldsymbol{v_1^h}\|_{X_1^h}^2.
\end{aligned}$$

It follows from (2.22) and (2.23) that

$$a_2^h(\boldsymbol{u_2^h}, \boldsymbol{u_2^h} - \boldsymbol{v_2^h}) - a_2^h(\boldsymbol{v_2^h}, \boldsymbol{u_2^h} - \boldsymbol{v_2^h}) \geq c\|\boldsymbol{u_2^h} - \boldsymbol{v_2^h}\|_{X_2^h}^2.$$

Hence, we conclude (3.18). □

Theorem 3.1 *Under the assumptions* (2.4), (2.5), (3.17), $\boldsymbol{f_1} \in L^2(\Omega_1)$ *and* $\boldsymbol{f_2} \in L^{\frac{3}{2}}(\Omega_2)$, *there exists a unique solution* $\boldsymbol{u^h} \in \mathring{V}^h$ *of* (3.13). Moreover, there is a unique $p^h \in M^h$ such that $(\boldsymbol{u^h}, p^h)$ solves (3.2), and exists a unique $\lambda^h \in \Lambda^h$ such that $(\boldsymbol{u^h}, p^h, \lambda^h)$ satisfies (3.9). Moreover,

$$\|\boldsymbol{u_1^h}\|_{X_1^h}^2 + \|\boldsymbol{u_2^h}\|_{X_2^h}^2 + \|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)}^3 \leq C\left(\|\boldsymbol{f_1}\|_{L^2(\Omega_1)}^2 + \|\boldsymbol{f_2}\|_{L^{\frac{3}{2}}(\Omega_2)}^{\frac{3}{2}}\right), \tag{3.19a}$$

$$\begin{aligned}
\|p^h\|_{M^h} + \|\lambda^h\|_{-\frac{1}{2},\Lambda^h} \leq C(&\|\boldsymbol{f_1}\|_{L^2(\Omega_1)} + \|\boldsymbol{f_2}\|_{L^{\frac{3}{2}}(\Omega_2)}^{\frac{3}{2}} \\
&+ \|\boldsymbol{u_1^h}\|_{X_1^h} + \|\boldsymbol{u_2^h}\|_{X_2^h} + \|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)}^2).
\end{aligned} \tag{3.19b}$$

Proof Since $A^h : \mathring{V}^h \to (\mathring{V}^h)^*$ is continuous (so that hemicontinuous), coercive, and monotone, there exists a unique $\boldsymbol{u^h} \in \mathring{V}^h$ of (3.13). Substituting $\boldsymbol{v^h} = \boldsymbol{u^h}$ into (3.13) and using (3.16)

$$\begin{aligned}
&c\big(\|\boldsymbol{u_1^h}\|_{X_1^h}^2 + \|\boldsymbol{u_2^h}\|_{X_2^h}^2\big) + C_F\|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)}^3 \leq (A^h(\boldsymbol{u^h}), \boldsymbol{u^h}) = (\boldsymbol{f}, \boldsymbol{u^h}) \\
&\leq \|\boldsymbol{f_1}\|_{L^2(\Omega_1)}\|\boldsymbol{u_1^h}\|_{L^2(\Omega_1)} + \|\boldsymbol{f_2}\|_{L^{\frac{3}{2}}(\Omega_2)}\|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)} \\
&\leq \frac{1}{2c}\|\boldsymbol{f_1}\|_{L^2(\Omega_1)}^2 + \frac{c}{2}\|\boldsymbol{u_1^h}\|_{L^2(\Omega_1)}^2 + (\frac{2}{3})^{\frac{3}{2}}C_F^{-\frac{1}{2}}\|\boldsymbol{f_2}\|_{L^{\frac{3}{2}}(\Omega_2)}^{\frac{3}{2}} + \frac{C_F}{2}\|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)}^3,
\end{aligned} \tag{3.20}$$

which implies (3.19a). By the discrete inf-sup condition (3.6), there is a unique $p^h \in M^h$ such that $(\boldsymbol{u^h}, p^h)$ solves (3.2), and we have the boundedness of p^h:

$$\begin{aligned} \|p^h\|_{M^h} &\le C \sup_{\boldsymbol{v^h} \in V^h} \frac{b^h(\boldsymbol{v^h}, p^h)}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2}\|_{X_2^h, W^{0.3}(\mathrm{div})}} \\ &= C \sup_{\boldsymbol{v^h} \in V^h} \frac{(f, \boldsymbol{v^h}) - a^h(\boldsymbol{u^h}, \boldsymbol{v^h})}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2}\|_{X_2^h, W^{0.3}(\mathrm{div})}}. \end{aligned}$$

In view of

$$\begin{aligned} |a^h(\boldsymbol{u^h}, \boldsymbol{v^h})| \le{}& C\|\boldsymbol{u_1^h}\|_{X_1^h}\|\boldsymbol{v_1^h}\|_{X_1^h} \\ &+ C(\|\boldsymbol{u_2^h}\|_{X_2^h} + \|\boldsymbol{u_2^h}\|^2_{L^3(\Omega_2)})(\|\boldsymbol{v_2^h}\|_{X_2^h} + \|\boldsymbol{v_2^h}\|_{L^3(\Omega_2)}) \\ \le{}& C(\|\boldsymbol{u_1^h}\|_{X_1^h} + \|\boldsymbol{u_2^h}\|_{X_2^h} + \|\boldsymbol{u_2^h}\|^2_{L^3(\Omega_2)})(\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h, W^{0.3}(\mathrm{div})}), \end{aligned}$$

and together with (3.20), we obtain the boundedness of p^h of (3.19b). The unique existence of $\lambda^h \in \Lambda^h$ and (3.19b) follows from (3.8). □

3.3 Picard Iteration for the Discrete Problem

As with Sect. 2.4, we apply the Picard iteration to solve the nonlinear discrete problem (3.13). We set

$$\hat{a}^h(\boldsymbol{w^h}; \boldsymbol{u^h}, \boldsymbol{v^h}) := \hat{a}_1^h(\boldsymbol{w_1^h}; \boldsymbol{u_1^h}, \boldsymbol{v_1^h}) + \hat{a}_2^h(\boldsymbol{w_2^h}; \boldsymbol{u_2^h}, \boldsymbol{v_2^h}) \quad (\forall \boldsymbol{w^h}, \boldsymbol{u^h}, \boldsymbol{v^h} \in \mathring{V}^h),$$

where

$$\begin{aligned} \hat{a}_1^h(\boldsymbol{w_1^h}; \boldsymbol{u_1^h}, \boldsymbol{v_1^h}) :={}& \sum_{E \in \mathcal{T}_1^h} \int_E g_1(|\boldsymbol{D}(\boldsymbol{w_1^h})|)\boldsymbol{D}(\boldsymbol{u_1^h}) : \boldsymbol{D}(\boldsymbol{v_1^h})\, dx \\ &- \sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \{\!\!\{ g_1(|\boldsymbol{D}(\boldsymbol{w_1^h})|)\boldsymbol{D}(\boldsymbol{u_1^h})\boldsymbol{n_e} \}\!\!\} \cdot [\![\boldsymbol{v_1^h}]\!]\, ds \\ &+ \sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \frac{\sigma_{1e}}{|e|} [\![\boldsymbol{u_1^h}]\!] \cdot [\![\boldsymbol{v_1^h}]\!]\, ds + \sum_{l=1}^{d-1} \sum_{e \in \Gamma} \int_e r_l^{-1} (\boldsymbol{u_1^h} \cdot \boldsymbol{t_l})(\boldsymbol{v_1^h} \cdot \boldsymbol{t_l})\, ds, \\ \hat{a}_2^h(\boldsymbol{w_2^h}; \boldsymbol{u_2^h}, \boldsymbol{v_2^h}) :={}& \sum_{E \in \mathcal{T}_2^h} \int_E \left(\boldsymbol{K}^{-1} g_2(|\boldsymbol{w_2^h}|)\boldsymbol{u_2} + C_F |\boldsymbol{w_2^h}| \boldsymbol{u_2^h}\right) \cdot \boldsymbol{v_2^h}\, dx \\ &+ \sum_{e \in \Gamma_2^h \cup \Gamma_2} \int_e \frac{\sigma_{2e}}{|e|} [\![\boldsymbol{u_2^h} \cdot \boldsymbol{n_e}]\!][\![\boldsymbol{v_2^h} \cdot \boldsymbol{n_e}]\!]\, ds. \end{aligned}$$

Note that $\hat{a}_1^h(\boldsymbol{u_1^h}; \boldsymbol{u_1^h}, \boldsymbol{v_1^h}) = a_1^h(\boldsymbol{u_1^h}, \boldsymbol{v_1^h})$ and $\hat{a}_2^h(\boldsymbol{u_2^h}; \boldsymbol{u_2^h}, \boldsymbol{v_2^h}) = a_2^h(\boldsymbol{u_2^h}, \boldsymbol{v_2^h})$. (3.13) is equivalently expressed by:

Find $\boldsymbol{u^h} \in \mathring{V}^h$ such that

$$\hat{a}^h(\boldsymbol{u^h}; \boldsymbol{u^h}, \boldsymbol{v^h}) = (\boldsymbol{f}, \boldsymbol{v^h}) \qquad \forall \boldsymbol{v^h} \in \mathring{V}^h. \tag{3.21}$$

The Picard iteration is stated as follows.

Given $\boldsymbol{u}^{\boldsymbol{h},(0)} = (\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h},(\mathbf{0})}, \boldsymbol{u}_{\mathbf{2}}^{\boldsymbol{h},(\mathbf{0})})$, for $l = 1, 2, \ldots$, find $\boldsymbol{u}^{\boldsymbol{h},(l)} \in \mathring{V}^h$ such that

$$\hat{a}^h(\boldsymbol{u}^{\boldsymbol{h},(l-1)}; \boldsymbol{u}^{\boldsymbol{h},(l)}, \boldsymbol{v}^{\boldsymbol{h}}) = (\boldsymbol{f}, \boldsymbol{v}^{\boldsymbol{h}}) \qquad \forall \boldsymbol{v}^{\boldsymbol{h}} \in \mathring{V}^h. \tag{3.22}$$

Setting $\boldsymbol{\varepsilon}^{\boldsymbol{h},(l)} = (\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}, \boldsymbol{\varepsilon}_{\mathbf{2}}^{\boldsymbol{h},(l)}) := (\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}} - \boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h},(l)}, \boldsymbol{u}_{\mathbf{2}}^{\boldsymbol{h}} - \boldsymbol{u}_{\mathbf{2}}^{\boldsymbol{h},(l)})$, we turn to the convergence of the iteration.

Theorem 3.2 *We set the constants* $C_1 := C_{g_1}^2 \frac{C_t^2 \nu_{1\infty} + 4}{2\nu_{1\infty}^2}$ *and* $C_2 := \left(\frac{k_{\max}}{\nu_{2\infty}}\right)^2 \left(\frac{C_{g2}}{k_{\min}} + C_F\right)^2$. *If* σ_{1e} *is sufficient large and*

$$\max\left(C_1 \|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})\|_{L^\infty(\Omega_1)}^2, C_2 \|\boldsymbol{u}_{\mathbf{2}}^{\boldsymbol{h}}\|_{L^\infty(\Omega_2)}^2\right) =: C_I < 1, \tag{3.23}$$

then the Picard iteration (3.22) *converges. In particular,*

$$\begin{aligned} &\frac{\nu_{1\infty}}{2} \, ||| \, \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}) \, |||_{L^2(\Omega_1)}^2 + \frac{\nu_{2\infty}}{2k_{\max}} \|\boldsymbol{\varepsilon}_{\mathbf{2}}^{\boldsymbol{h},(l)}\|_{L^2(\Omega_2)}^2 \\ &\leq C_I \Big(\frac{\nu_{1\infty}}{2} \, ||| \, \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l-1)}) \, |||_{L^2(\Omega_1)}^2 + \frac{\nu_{2\infty}}{2k_{\max}} \|\boldsymbol{\varepsilon}_{\mathbf{2}}^{\boldsymbol{h},(l-1)}\|_{L^2(\Omega_2)}^2 \Big). \end{aligned} \tag{3.24}$$

Proof Same with the proof of Theorem 2.3, we subtract (3.22) from (3.21), and choose $\boldsymbol{v}^{\boldsymbol{h}} = \boldsymbol{\varepsilon}^{\boldsymbol{h},(l)}$,

$$a^h(\boldsymbol{u}^{\boldsymbol{h}}; \boldsymbol{u}^{\boldsymbol{h}}, \boldsymbol{\varepsilon}^{\boldsymbol{h},(l)}) - \hat{a}^h(\boldsymbol{u}^{\boldsymbol{h},(l-1)}; \boldsymbol{u}^{\boldsymbol{h},(l)}, \boldsymbol{\varepsilon}^{\boldsymbol{h},(l)}) = 0. \tag{3.25}$$

Applying (2.4b), (3.1b) and Young's inequalities, we have

$$\begin{aligned} &\sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \big\{\!\!\big\{ \big(g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})|) - g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h},(l-1)})|) \big) \boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}}) \boldsymbol{n}_{\boldsymbol{e}} \big\}\!\!\big\} \cdot [\![\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}]\!] \, ds \\ &\leq \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{C_{g_1}}{2} C_t |e|^{-\frac{1}{2}} \| \big\{\!\!\big\{ \boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}}) \big\}\!\!\big\} \|_{L^\infty(e)} \\ &\qquad \Big(\|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l-1)})\|_{L^2(E^+)} + \|\boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l-1)})\|_{L^2(E^-)} \Big) \| [\![\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}]\!] \|_{L^2(e)} \\ &\leq \frac{C_{g_1}^2 C_t^2}{4} \|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})\|_{L^\infty(\Omega_1)}^2 \, ||| \, \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l-1)}) \, |||_{L^2(\Omega_1)}^2 + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{1}{|e|} \| [\![\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}]\!] \|_{L^2(e)}^2, \end{aligned}$$

and

$$\begin{aligned} &\sum_{e \in \Gamma_1^h \cup \Gamma_1} \int_e \big\{\!\!\big\{ g_1(|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h},(l-1)})|) \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}) \boldsymbol{n}_{\boldsymbol{e}} \big\}\!\!\big\} \cdot [\![\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}]\!] \, ds \\ &\qquad \leq \frac{\nu_{1\infty}}{4} \, ||| \, \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}) \, |||_{L^2(\Omega_1)}^2 + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{C}{|e|} \| [\![\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}]\!] \|_{L^2(e)}^2, \end{aligned}$$

where $C = \frac{\nu_{10}^2 C_t^2}{\nu_{1\infty}}$. Therefore (3.25) yields

$$\begin{aligned} &\frac{\nu_{1\infty}}{2} \, ||| \, \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}) \, |||_{L^2(\Omega_1)}^2 + \sum_{e \in \Gamma_1^h \cup \Gamma_1} \frac{\sigma_{1e} - 1 - C}{|e|} \| [\![\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l)}]\!] \|_{L^2(e)}^2 \\ &\qquad \leq C_{g_1}^2 \frac{C_t^2 \nu_{1\infty} + 4}{4\nu_{1\infty}} \|\boldsymbol{D}(\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})\|_{L^\infty(\Omega_1)}^2 \, ||| \, \boldsymbol{D}(\boldsymbol{\varepsilon}_{\mathbf{1}}^{\boldsymbol{h},(l-1)}) |||_{L^2(\Omega_1)}^2, \end{aligned}$$

$$\frac{\nu_{2\infty}}{2k_{\max}}\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{\boldsymbol{h},(l)}\|_{L^2(\Omega_2)}^2+\sum_{e\in\Gamma_2^h\cup\Gamma_2}\frac{\sigma_{2e}}{|e|}\|[\![\boldsymbol{\varepsilon}_{\mathbf{2}}^{\boldsymbol{h},(l)}\cdot\boldsymbol{n_e}]\!]\|_{L^2(e)}^2$$
$$\leq\frac{k_{\max}}{2\nu_{2\infty}}(\frac{C_{g2}}{k_{\min}}+C_F)^2\|\boldsymbol{u}_{\mathbf{2}}^{\boldsymbol{h}}\|_{L^\infty(\Omega_2)}^2\|\boldsymbol{\varepsilon}_{\mathbf{2}}^{\boldsymbol{h},(l-1)})\|_{L^2(\Omega_2)}^2.$$

Summing up the above two inequalities and taking sufficiently large σ_{1e} such that $\sigma_{1e}-1-C>0$, we conclude (3.24). □

3.4 The Error Estimates

Theorem 3.3 *Let $(\boldsymbol{u},p,\lambda)$ and $(\boldsymbol{u^h},p^h,\lambda^h)$ be the solutions of* (2.8) *and* (3.9)*, respectively. For $k=1,2$, suppose we have the regularity $\boldsymbol{u}_1\in(H^{k+1}(\Omega_1))\cap W^{1,\infty}(\Omega_1)$ and $\boldsymbol{u}_2\in(H^k(\Omega_2))\cap W^{1,r}(\Omega_2)$ $(r\geq d)$ and $p_i\in H^k(\Omega_i)$ $(i=1,2)$. Then, under the assumptions* (2.4)*,* (2.5) *and* (3.17)*, we have*

$$\|\boldsymbol{u}-\boldsymbol{u^h}\|_{X^h}\leq Ch^k\big(\|\boldsymbol{u_1}\|_{H^{k+1}(\Omega_1)}+\|p_1\|_{H^k(\Omega_1)}+\|\boldsymbol{u_2}\|_{H^k(\Omega_2)}+\|p_2\|_{H^k(\Omega_2)}\big),\quad(3.26a)$$
$$\|p_1-p_1^h\|_{L^2(\Omega_1)}+\|p_2-p_2^h\|_{L^2(\Omega_2)}\leq ch^k(|p_1|_{H^k(\Omega_1)}+|p_2|_{H^k(\Omega_2)})+c\|\boldsymbol{u}-\boldsymbol{u^h}\|_{X^h}.\quad(3.26b)$$

Proof For briefness, we set the notations:

$$\tilde{\boldsymbol{u}}^{\boldsymbol{h}}=(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}},\tilde{\boldsymbol{u}}_{\mathbf{2}}^{\boldsymbol{h}}):=(\Pi_1\boldsymbol{u_1},\Pi_2\boldsymbol{u_2}),\quad\tilde{p}^h:=P_Mp,\quad\tilde{\lambda}^h:=P_\Lambda\lambda.$$

By the triangle inequality and interpolation/projection errors (3.4), (3.5) and (3.11), it suffices to evaluate $\|\tilde{\boldsymbol{u}}^{\boldsymbol{h}}-\boldsymbol{u^h}\|_{X^h}$ and $\|\tilde{p}_i^h-p_i^h\|_{L^2(\Omega_i)}$.

It follows from (3.18) that

$$\begin{aligned}&c_\alpha\big(\|\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}}\|_{X_1^h}^2+\|\tilde{\boldsymbol{u}}_{\mathbf{2}}^{\boldsymbol{h}}-\boldsymbol{u}_{\mathbf{2}}^{\boldsymbol{h}}\|_{X_2^h}^2\big)\leq(A^h(\tilde{\boldsymbol{u}}^{\boldsymbol{h}})-A^h(\boldsymbol{u^h}),\tilde{\boldsymbol{u}}^{\boldsymbol{h}}-\boldsymbol{u^h})\\&=(A^h(\tilde{\boldsymbol{u}}^{\boldsymbol{h}})-A^h(\boldsymbol{u}),\tilde{\boldsymbol{u}}^{\boldsymbol{h}}-\boldsymbol{u^h})+(A^h(\boldsymbol{u})-A^h(\boldsymbol{u^h}),\tilde{\boldsymbol{u}}^{\boldsymbol{h}}-\boldsymbol{u^h}):=\mathrm{I}+\mathrm{II}.\end{aligned}\quad(3.27)$$

The task is to bound I and II. To this end, we decompose I into three parts $\{\mathrm{I}_i\}_{i=1}^3$ and estimates them as follows. By (2.4a), (2.4b) and (3.4),

$$\begin{aligned}\mathrm{I}_1&:=\sum_{E\in\mathcal{T}_1^h}\int_E\big(g_1(|\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}})|)\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}})-g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1})\big):\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})\,dx\\&=\sum_{E\in\mathcal{T}_1^h}\int_E\Big(g_1(|\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}})|)\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u_1})-\big(g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)-g_1(|\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}})|)\big)\boldsymbol{D}(\boldsymbol{u_1})\Big)\\&\qquad:\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})\,dx\leq C_{\infty1}\,|||\,\nabla(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u_1})\,|||_{L^2(\Omega_1)}\,|||\nabla(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})|||_{L^2(\Omega_1)}\\&\leq c\eta^{-1}h^{2k}|\boldsymbol{u_1}|_{H^{k+1}(\Omega_1)}^2+\eta\,|||\,\nabla(\tilde{\boldsymbol{u}}_{\mathbf{1}}^{\boldsymbol{h}}-\boldsymbol{u}_{\mathbf{1}}^{\boldsymbol{h}})|||_{L^2(\Omega_1)}^2,\end{aligned}$$

where $C_{\infty1}=(\nu_{10}+C_{g1}\|\boldsymbol{D}(\boldsymbol{u_1})\|_{L^\infty(\Omega_1)})$, and η is an arbitrary positive constant. Let $I_h\boldsymbol{v_1}\in X_1^h$ be the Lagrange interpolation of $\boldsymbol{v_1}$ satisfying

$$\|I_h\boldsymbol{v_1}-\boldsymbol{v_1}\|_{H^m(E)}\leq ch^{k+1-m}|\boldsymbol{v_1}|_{H^{k+1}(E)}\quad(\forall E\in\mathcal{T}_1^h,\ m=0,1,2,\ k=1,2).\ (3.28)$$

By using (3.1b) and (3.1a), noting that $\{\tilde{\boldsymbol{u}}^{\boldsymbol{h}}\} = \tilde{\boldsymbol{u}}^{\boldsymbol{h}}$, $\{\boldsymbol{u}\} = \boldsymbol{u}$ and $[\![\tilde{\boldsymbol{u}}^{\boldsymbol{h}}]\!] = [\![\boldsymbol{u}]\!] = 0$, we calculate as (the last inequality follows from (3.4) and (3.28))

$$
\begin{aligned}
I_2 :=& \sum_{e\in\Gamma_1^h\cup\Gamma_1} \int_e [\![\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{h}}]\!] \\
&\qquad \cdot \left\{\!\!\left\{ \big(g_1(|\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}})|)\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}}) - g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1})\big)\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}} - \boldsymbol{u_1})\boldsymbol{n_e} \right\}\!\!\right\} ds \\
\leq& \sum_{e\in\Gamma_1^h\cup\Gamma_1} C_{\infty 1}\big(\|\boldsymbol{D}(\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}} - I_h\boldsymbol{u_1})\boldsymbol{n_e}\|_{L^2(e)} + \|\boldsymbol{D}(I_h\boldsymbol{u_1} - \boldsymbol{u_1})\boldsymbol{n_e}\|_{L^2(e)}\big)\|[\![\boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{h}}]\!]\|_{L^2(e)} \\
\leq& c\eta^{-1}\big(|||\, \nabla(\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}} - I_h\boldsymbol{u_1})\, |||_{L^2(\Omega_1)} + \|I_h\boldsymbol{u_1} - \boldsymbol{u_1}\|^2_{H^1(\Omega_1)} + |e|\|I_h\boldsymbol{u_1} - \boldsymbol{u_1}\|_{H^2(\Omega_1)}\big)^2 \\
&+ \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\eta}{|e|}\|[\![\boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{h}}]\!]\|^2_{L^2(e)} \leq c\eta^{-1}h^{2k}|\boldsymbol{u_1}|^2_{H^{k+1}(\Omega_1)} + \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\eta}{|e|}\|[\![\boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{h}}]\!]\|^2_{L^2(e)}.
\end{aligned}
$$

In view of

$$
\begin{aligned}
\|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}}\|_{L^\infty(\Omega_2)} &\leq \|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u_2}\|_{L^\infty(\Omega_2)} + \|\boldsymbol{u_2}\|_{L^\infty(\Omega_2)} \\
&\leq ch^{1-\frac{d}{r}}\|\nabla\boldsymbol{u_2}\|_{W^{1,r}(\Omega_2)} + \|\boldsymbol{u_2}\|_{L^\infty(\Omega_2)} \leq c \quad (r \geq d),
\end{aligned}
$$

we derive (by (2.5a), (2.5b) and (3.5a))

$$
\begin{aligned}
\mathrm{I}_3 :=& a_2^h(\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}}, \tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{h}}) - a_2^h(\boldsymbol{u_2}, \tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{h}}) \\
\leq& \|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u_2}\|_{L^2(\Omega_2)}\|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{h}}\|_{L^2(\Omega_2)}\Big(k_{\max}(\nu_{20} + C_{g2}\|\boldsymbol{u_2}\|_{L^\infty(\Omega_2)}) \\
&\quad + C_F(\|\boldsymbol{u_2}\|_{L^\infty(\Omega_2)} + \|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}}\|_{L^\infty(\Omega_2)})\Big) \\
\leq& c\eta^{-1}h^{2k}|\boldsymbol{u_2}|^2_{H^k(\Omega_2)} + \eta\|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{h}}\|^2_{L^2(\Omega_2)}.
\end{aligned}
$$

Summing up the above estimates of $\{\mathrm{I}_i\}_{i=1}^3$, we achieve

$$
\begin{aligned}
\mathrm{I} \leq& c\eta^{-1}h^{2k}|\boldsymbol{u_1}|^2_{H^{k+1}(\Omega_1)} + c\eta^{-1}h^{2k}|\boldsymbol{u_2}|^2_{H^k(\Omega_2)} + \eta\, |||\, \nabla(\tilde{\boldsymbol{u}}_{\boldsymbol{1}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{h}})|||^2_{L^2(\Omega_1)} \\
&+ \sum_{e\in\Gamma_1^h\cup\Gamma_1} \frac{\eta}{|e|}\|[\![\boldsymbol{u}_{\boldsymbol{1}}^{\boldsymbol{h}}]\!]\|^2_{L^2(e)} + \eta\|\tilde{\boldsymbol{u}}_{\boldsymbol{2}}^{\boldsymbol{h}} - \boldsymbol{u}_{\boldsymbol{2}}^{\boldsymbol{h}}\|^2_{L^2(\Omega_2)}.
\end{aligned}
\tag{3.29}
$$

Now we turn to the estimates of II. Subtracting (3.9) from (2.8), we obtain

$$
\begin{cases}
a^h(\boldsymbol{u}, \boldsymbol{v}^{\boldsymbol{h}}) - a^h(\boldsymbol{u}^{\boldsymbol{h}}, \boldsymbol{v}^{\boldsymbol{h}}) + b^h(\boldsymbol{v}^{\boldsymbol{h}}, p - p^h) + b_I^h(\boldsymbol{v}^{\boldsymbol{h}}, \lambda - \lambda^h) = 0 & \forall \boldsymbol{v}^{\boldsymbol{h}} \in X^h, \\
b^h(\boldsymbol{u}, q^h) - b^h(\boldsymbol{u}^{\boldsymbol{h}}, q^h) = 0 & \forall q^h \in M^h, \\
b_I^h(\boldsymbol{u}, \lambda^h) - b_I^h(\boldsymbol{u}^{\boldsymbol{h}}, \lambda^h) = 0 & \forall \lambda^h \in \Lambda^h.
\end{cases}
\tag{3.30}
$$

Substituting $\boldsymbol{v}^{\boldsymbol{h}} = \tilde{\boldsymbol{u}}^h - \boldsymbol{u}^h$ into (3.30), we have

$$
a^h(\boldsymbol{u}, \tilde{\boldsymbol{u}}^h - \boldsymbol{u}^h) - a^h(\boldsymbol{u}^{\boldsymbol{h}}, \tilde{\boldsymbol{u}}^h - \boldsymbol{u}^h) + b^h(\tilde{\boldsymbol{u}}^h - \boldsymbol{u}^h, p - p^h) + b_I^h(\tilde{\boldsymbol{u}}^h - \boldsymbol{u}^h, \lambda - \lambda^h) = 0,
$$

which yields

$$
\mathrm{II} = b^h(\boldsymbol{u}^h - \tilde{\boldsymbol{u}}^h, p - \tilde{p}^h) + b_I^h(\boldsymbol{u}^h - \tilde{\boldsymbol{u}}^h, \lambda - \tilde{\lambda}^h).
$$

It follows from (3.10a), (3.10b) and (3.11a) that

$$\begin{aligned}\mathrm{II} = &\sum_{e\in\Gamma_1^h\cup\Gamma_1}\int_e \{p_1 - \tilde{p}_1^h\}[\![(\boldsymbol{u_1^h} - \tilde{\boldsymbol{u}}_1^{\boldsymbol{h}})\cdot\boldsymbol{n_e}]\!]\,ds \\ &+ \sum_{e\in\Gamma_2^h\cup\Gamma_2}\int_e \{p_2 - \tilde{p}_2^h\}[\![(\boldsymbol{u_2^h} - \tilde{\boldsymbol{u}}_2^{\boldsymbol{h}})\cdot\boldsymbol{n_e}]\!]\,ds \le c\eta^{-1}h^{2k}|p_1|^2_{H^k(\Omega_1)} \\ &+ c\eta^{-1}h^{2k}|p_2|^2_{H^k(\Omega_2)} + \sum_{e\in\Gamma_1^h\cup\Gamma_1}\frac{\eta}{|e|}\|[\![\boldsymbol{u_1^h}]\!]\|^2_{L^2(e)} + \sum_{e\in\Gamma_2^h\cup\Gamma_2}\frac{\eta}{|e|}\|[\![\boldsymbol{u_2^h}\cdot\boldsymbol{n_e}]\!]\|^2_{L^2(e)}.\end{aligned}$$

Together with (3.29), we get

$$\begin{aligned}\mathrm{I}+\mathrm{II} \le\ &\eta\|\tilde{\boldsymbol{u}}^{\boldsymbol{h}} - \boldsymbol{u^h}\|^2_{X^h} \\ &+ c\eta^{-1}h^{2k}(|\boldsymbol{u_1}|^2_{H^{k+1}(\Omega_1)} + |p_1|^2_{H^k(\Omega_1)} + |\boldsymbol{u_2}|^2_{H^k(\Omega_2)} + |p_2|^2_{H^k(\Omega_2)}).\end{aligned}$$

By (3.27), the triangle inequality and taking $c_\alpha - \eta > 0$, we conclude (3.26a).

It remains to estimate the error of pressure. Applying the inf-sup condition (3.7), (3.30) and (3.10b),

$$\begin{aligned}\|p_1^h - \tilde{p}_1^h\|_{L^2(\Omega_1)} + \|p_2^h - \tilde{p}_2^h\|_{L^2(\Omega_2)} &\le C\sup_{\boldsymbol{v^h}\in V^h}\frac{b^h(\boldsymbol{v^h}, p^h - \tilde{p}^h)}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h,W^{1,2}}} \\ &= C\sup_{\boldsymbol{v^h}\in V^h}\frac{a^h(\boldsymbol{u},\boldsymbol{v^h}) - a^h(\boldsymbol{u^h},\boldsymbol{v^h}) + b^h(\boldsymbol{v^h}, p - \tilde{p}^h)}{\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h,W^{1,2}}}.\end{aligned}$$

The numerator is bounded as follows.

$$\begin{aligned}b^h(\boldsymbol{v^h}, p - \tilde{p}^h) &\le c(\|\boldsymbol{v_1^h}\|_{X_1^h} + \|\boldsymbol{v_2^h}\|_{X_2^h,W^{1,2}})h^k(|p_1|_{H^k(\Omega_1)}|p_2|_{H^k(\Omega_2)}), \\ a^h(\boldsymbol{u},\boldsymbol{v^h}) - a^h(\boldsymbol{u^h},\boldsymbol{v^h}) &\le c \mathrel{|||} \nabla(\boldsymbol{u_1} - \boldsymbol{u_1^h}) \mathrel{|||}_{L^2(\Omega_1)} \|\nabla\boldsymbol{v_1^h}\|_{L^2(\Omega_1)} \\ &+ \sum_{e\in\Gamma_1^h\cup\Gamma_1}\sqrt{|e|}\|\nabla(\boldsymbol{u_1} - \boldsymbol{u_1^h})\|_{L^2(e)}\frac{C_{\infty 1}}{\sqrt{|e|}}\|[\![\boldsymbol{v_1^h}]\!]\|^2_{L^2(e)} \\ &+ \big(c + C_F(\|\boldsymbol{u_2}\|_{L^\infty(\Omega_2)} \\ &+ \|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)})\big)\|\boldsymbol{u_2} - \boldsymbol{u_2^h}\|_{L^2(\Omega_2)}(\|\boldsymbol{v_2^h}\|_{L^2(\Omega_2)} + \|\boldsymbol{v_2^h}\|_{L^6(\Omega_2)}) \\ &+ \sum_{e\in\Gamma_2^h\cup\Gamma_2}\frac{\sigma_{2e}}{|e|}[\![(\boldsymbol{u_2} - \boldsymbol{u_2^h})\cdot\boldsymbol{n_e}]\!][\![\boldsymbol{v_2^h}\cdot\boldsymbol{n_e}]\!]\,ds,\end{aligned}$$

where $\|\boldsymbol{u_2^h}\|_{L^3(\Omega_2)}$ is bounded by Theorem 3.1. By the discrete Sobolev embedding $\|\boldsymbol{v_2^h}\|_{L^6(\Omega_2)} \le C\|\boldsymbol{v_2^h}\|_{X_2^h,W^{1,2}}$, we obtain

$$\|p_1^h - \tilde{p}_1^h\|_{L^2(\Omega_1)} + \|p_2^h - \tilde{p}_2^h\|_{L^2(\Omega_2)} \le ch^k(|p_1|_{H^k(\Omega_1)} + |p_2|_{H^k(\Omega_2)}) + c\|\boldsymbol{u} - \boldsymbol{u^h}\|_{X^h}.$$

Together with the triangle inequality and the projection error (3.11a), we conclude (3.26b).
□

4 Numerical Experiments

We first carry out the numerical experiments using P^k/P^{k-1}-DG element with $k = 1, 2$ respectively, and investigate the experimental convergence rates and the performance of the Picard iteration. Secondly, we present a simulation of the industrial filtration system (cf. [17, 22]) with curved interface Γ. Then the non-constant permeability [16, 32, 33] is considered with a curved interface in the porous medium.

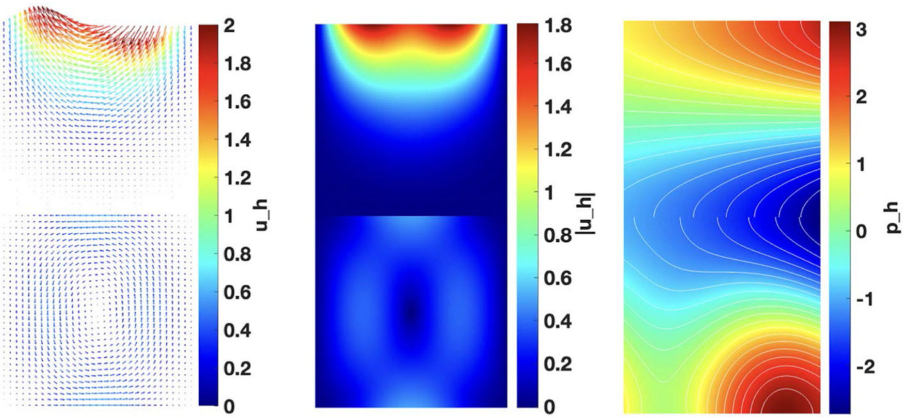
4.1 Example 1: The Experimental Convergence Rates

We set $\Omega_1 = \{(x, y) : \ 0 < x < 1, \ 1 < y < 2\}$ and $\Omega_2 = \{(x, y) : \ 0 < x < 1, \ 0 < y < 1\}$ with the interface $\Gamma = \{(x, 1) : 0 < x < 1\}$ and the boundaries $\Gamma_1 = \partial\Omega_1 \backslash \Gamma$, $\Gamma_2 = \partial\Omega_2 \backslash \Gamma$. Choose the parameters $C_F = 1$, $\boldsymbol{K} = \boldsymbol{I}$, and the viscosity functions $g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)$ and $g_2(|\boldsymbol{u_2}|)$ of the Carreau model with $\nu_{1\infty} = \nu_{2\infty} = 0.001$, $\nu_{10} = \nu_{20} = 0.5$, $G_1(|\boldsymbol{D}(\boldsymbol{u_1})|) = (1 + 0.5|\boldsymbol{D}(\boldsymbol{u_1})|^2)^{-0.25}$ and $G_2(|\boldsymbol{u_2}|) = (1 + 0.5|\boldsymbol{u_2}|^2)^{-0.25}$. The exact solution is stated as follows.

$$\boldsymbol{u_1}(x, y) = \big(-\frac{3}{2}\sin^2(\pi x)(y-1)^2,\ \pi\sin(\pi x)\cos(\pi x)(y-1)^3\big)^t,$$
$$\boldsymbol{u_2}(x, y) = \big((\frac{1}{2} - y)\sin(\pi x)^2,\ \pi\sin(\pi x)\cos(\pi x)y(y-1)\big)^t,$$
$$p_1(x, y) = e^x\cos(\pi y), \quad p_2(x, y) = e^x\cos(\pi y) - (g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)\boldsymbol{D}(\boldsymbol{u_1})\boldsymbol{n_{12}}) \cdot \boldsymbol{n_{12}},$$

where $\boldsymbol{n_{12}} = (0, -1)^t$. We replace the homogeneous Dirichlet boundary condition (2.1c) with the inhomogeneous one. The solutions $\boldsymbol{u_1}$ and $\boldsymbol{u_2}$ are divergence-free in Ω_1 and Ω_2, respectively. In particular, the BJS condition (2.3c) is satisfied with $r_l = 1$.

Taking $\sigma_{1e} = 0.5$ and $\sigma_{2e} = 1$, we carry out the simulation using P^1/P^0-DG element and P^2/P^1-DG element, respectively. The numerical solutions with P^2/P^1-DG element and $h = \frac{1}{64}$ are plotted in Fig. 1. The experimental errors and convergence rates are presented in Tables 2 and 3, where we observe the $O(h^k)$-convergence of the errors for both the velocity and pressure in two subregions. The experimental results confirm the theoretical



(1) $\boldsymbol{u^h}$ (2) $|\boldsymbol{u^h}|$ (3) p^h

Fig. 1 The numerical solution $(\boldsymbol{u^h}, p^h)$ with P^2/P^1-DG element and $h = \frac{1}{64}$

Table 2 The experimental errors of P^1/P^0-DG element

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$
$\|\boldsymbol{u_1} - \boldsymbol{u_1^h}\|_{X_1^h}$	4.31e−01	2.14e−01	1.07e−01	5.32e−02
Rate	–	1.01	1.01	1.00
$\|p_1 - p_1^h\|_{L^2(\Omega_1)}$	6.44e−02	3.21e−02	1.60e−02	8.01e−03
Rate	–	1.00	1.00	1.00
$\|\boldsymbol{u_2} - \boldsymbol{u_2^h}\|_{X_2^h}$	1.54e−01	7.65e−02	3.81e−02	1.90e−02
Rate	–	1.01	1.01	1.00
$\|p_2 - p_2^h\|_{L^2(\Omega_2)}$	1.20e−01	6.00e−02	2.99e−02	1.50e−02
Rate	–	1.00	1.00	1.00
$\|\lambda - \lambda^h\|_{L^2(\Gamma)}$	2.63e−03	6.49e−04	1.35e−04	4.07e−05
Rate	–	2.02	2.27	1.73

Table 3 The experimental errors of P^2/P^1-DG element

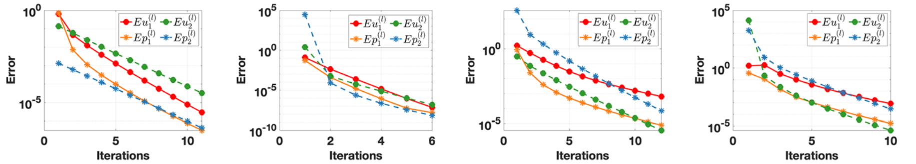
h	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
$\|\boldsymbol{u_1} - \boldsymbol{u_1^h}\|_{X_1^h}$	4.01e−01	1.02e−01	2.53e−02	6.33e−03
Rate	–	1.98	2.00	2.00
$\|p_1 - p_1^h\|_{L^2(\Omega_1)}$	1.01e−01	2.20e−02	5.00e−03	1.18e−03
Rate	–	2.27	2.14	2.08
$\|\boldsymbol{u_2} - \boldsymbol{u_2^h}\|_{X_2^h}$	1.16e−01	2.88e−02	7.15e−03	1.87e−03
Rate	–	2.00	2.01	1.93
$\|p_2 - p_2^h\|_{L^2(\Omega_2)}$	4.86e−02	9.37e−03	2.41e−03	6.27e−04
Rate	–	2.38	1.96	1.94
$\|\lambda - \lambda^h\|_{L^2(\Gamma)}$	3.29e−02	5.08e−03	8.20e−04	1.47e−04
Rate	–	2.70	2.63	2.48

error estimates obtained by Theorem 3.3. We also compute the experimental L^2-error of the Lagrange multiplier λ_h on the interface (see Tables 2 and 3), where the exact λ is obtained by using the formula (2.9).

Furthermore, at l-th iteration step ($l = 1, 2, \ldots$), we compute the errors $Eu_i^{(l)} := \|\boldsymbol{u_i}^{\boldsymbol{h},(l)} - \boldsymbol{u_i}^{\boldsymbol{h},(l-1)}\|_{X_i^h}$ and $Ep_i^{(l)} := \|p_i^{h,(l)} - p_i^{h,(l-1)}\|_{L^2(\Omega_i)}$ $(i = 1, 2)$, and plot them in Fig. 2(1) in log-scale. We see the iteration error decreases exponentially fast, indicating the Picard iteration's good applicability.

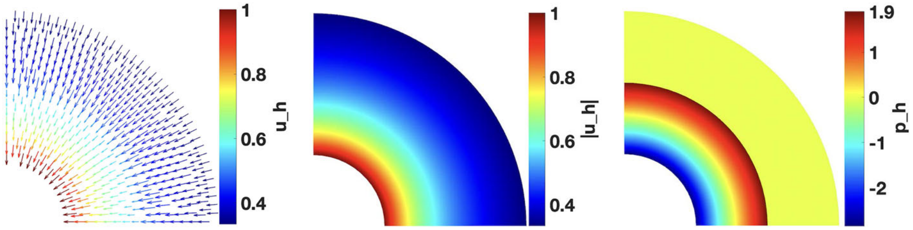
4.2 Example 2: The Dead-end Filter Domain

We consider the region of a concentric quarter circular. The entire domain is divided into the Stokes region $\Omega_1 = \{(x, y) : 2 < x^2 + y^2 < 3, x > 0, y > 0\}$ and the Darcy region $\Omega_2 = \{(x, y) : 1 < x^2 + y^2 < 2, x > 0, y > 0\}$ with the interface $\Gamma = \{(x, y) : x^2 + y^2 =$



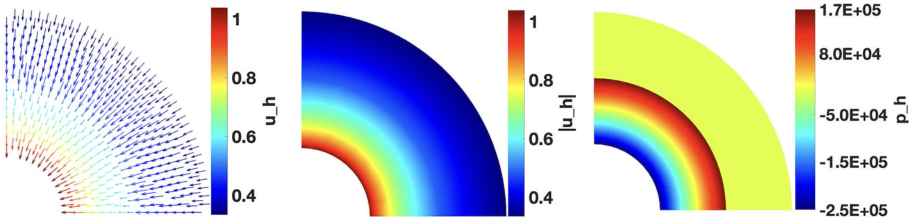
(1) Example 1 (2) Example 2 (3) Example 3 with periodic k_{pre}^{-1} (4) Example 3 with random k_{pre}^{-1}

Fig. 2 The convergence of the Picard iteration



(1) $\boldsymbol{u^h}$ (2) $|\boldsymbol{u^h}|$ (3) p^h

Fig. 3 The velocity and pressure with permeability $\boldsymbol{K} = I$ and $C_F = 1$



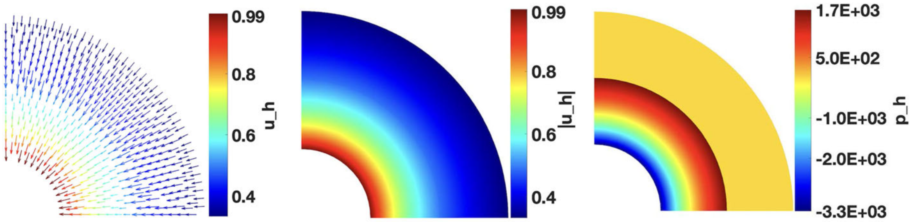
(1) $\boldsymbol{u^h}$ (2) $|\boldsymbol{u^h}|$ (3) p^h

Fig. 4 The velocity and pressure with permeability $\boldsymbol{K} = 10^{-5}I$ and $C_F = 1$

2}, and the boundaries $\Gamma_1 = \partial\Omega_1\backslash\Gamma$, $\Gamma_2 = \partial\Omega_2\backslash\Gamma$, $\Gamma_{21} = \{(x, y) : x^2 + y^2 = 1\}$ and $\Gamma_{22} = \Gamma_2\backslash\Gamma_{21}$. The following boundary conditions are imposed.

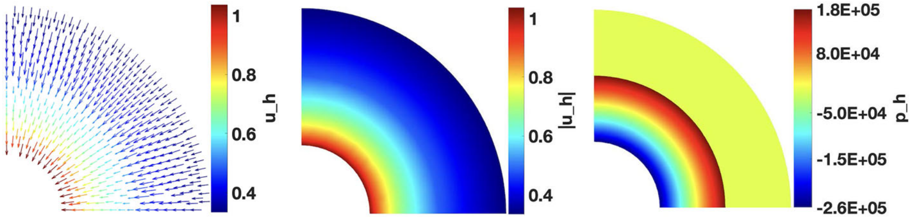
$$\boldsymbol{u_1}(x, y) = \left(-\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2}\right)^{\top} \quad \text{on } \Gamma_1,$$
$$\boldsymbol{u_2} \cdot \boldsymbol{n} = 1 \quad \text{on } \Gamma_{21}, \quad \boldsymbol{u_2} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma_{22}.$$

We take $g_1(|\boldsymbol{D}(\boldsymbol{u_1})|)$ and $g_2(|\boldsymbol{u_2}|)$ of the Carreau model with $G_1(|\boldsymbol{D}(\boldsymbol{u_1})|) = (1 + 0.5|\boldsymbol{D}(\boldsymbol{u_1})|^2)^{-0.425}$, $G_2(|\boldsymbol{u_2}|) = (1+0.5|\boldsymbol{u_2}|^2)^{-0.425}$, $\nu_{1\infty} = 0.001$, $\nu_{10} = 0.1$, $\nu_{2\infty} = 1$ and $\nu_{20} = 10$. We adopt P^2/P^1-DG element with $\sigma_{1e} = 10$ and $\sigma_{2e} = 10$, and plot the numerical solutions with different permeability $\boldsymbol{K}$ and Forchheimer coefficient C_F in Figs. 3, 4, 5, 6, 7. The velocity fields of these four cases are almost the same. In contrast, the magnitude of the pressure behaves differently, which depends on the permeability $\boldsymbol{K}$ and the Forchheimer coefficient C_F. The filtration pressure increases dramatically when $\boldsymbol{K} \ll 1$ or $C_F \gg 1$. In this example, we also observe exponentially fast decreasing errors of the Picard iteration (see Fig. 2(2) with $\boldsymbol{K} = 10^{-5}I$ and $C_F = 1$).



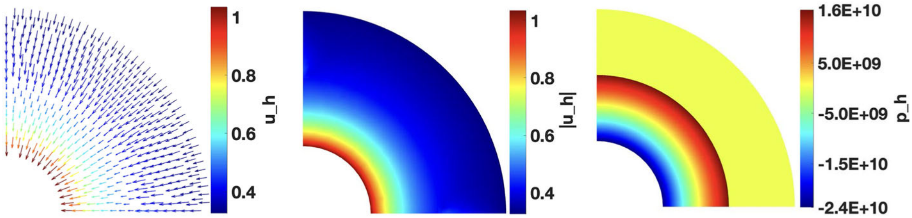
(1) $\boldsymbol{u}^{\boldsymbol{h}}$ (2) $|\boldsymbol{u}^{\boldsymbol{h}}|$ (3) p^h

Fig. 5 The velocity and pressure with permeability $\boldsymbol{K} = I$ and $C_F = 10^4$



(1) $\boldsymbol{u}^{\boldsymbol{h}}$ (2) $|\boldsymbol{u}^{\boldsymbol{h}}|$ (3) p^h

Fig. 6 The velocity and pressure with permeability $\boldsymbol{K} = 10^{-5} I$ and $C_F = 10^4$



(1) $\boldsymbol{u}^{\boldsymbol{h}}$ (2) $|\boldsymbol{u}^{\boldsymbol{h}}|$ (3) p^h

Fig. 7 The velocity and pressure with permeability $\boldsymbol{K} = 10^{-10} I$ and $C_F = 10^4$

4.3 Example 3: The Varying Permeability Models

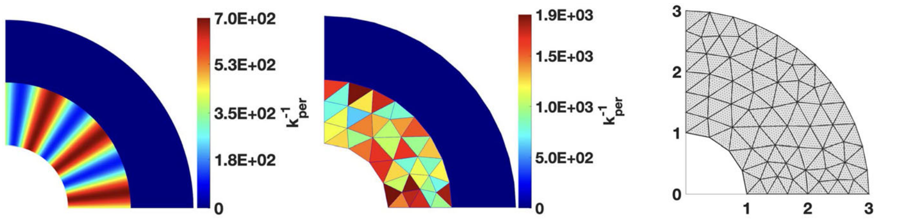
In the previous example, the permeability is chosen as a constant. To be more realistic, in this subsection, we explore the velocity and pressure behavior in the coupled domain for non-constant permeability. Particularly, we use the same Stokes and Darcy region, boundary conditions and the non-Newtonian models configuration as in Example 2.

First, we carry out the simulation with a periodic permeability $K = k_{per} I$ with

$$k_{per}^{-1} = 300\left(1 + \sin\left(12 \arcsin\left(\frac{y}{\sqrt{x^2 + y^2}}\right)\right)\right) + 100.$$

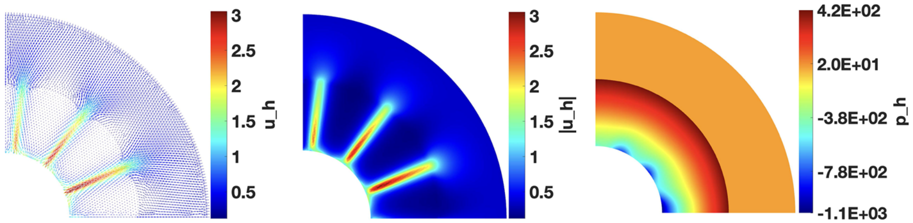
k_{per}^{-1} is plotted in Fig. 8(1). We utilize the P^2/P^1-DG element to calculate $\boldsymbol{u}^{\boldsymbol{h}}$ and p_h, and observe that the velocity and pressure behave periodically as permeability (see Fig. 9).

Next, we randomly selecte a value for k_{per}^{-1} between 500 and 2000 in a coarse mesh (see Fig. 8(2)). Note that to ensure the accuracy of the calculation, we compute the solution on a



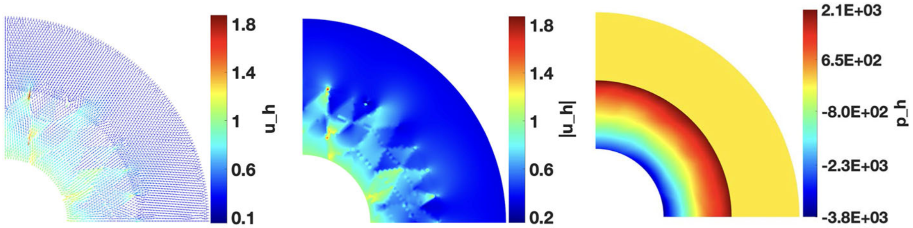
(1) k_{per}^{-1}: a periodic permeability (2) k_{per}^{-1}: a random permeability (3) A refined mesh for the random permeability

Fig. 8 The periodic and random permeability fields and a refined mesh for the case of random permeability



(1) $\boldsymbol{u}^{\boldsymbol{h}}$ (2) $|\boldsymbol{u}^{\boldsymbol{h}}|$ (3) p^h

Fig. 9 The velocity and pressure with a periodic permeability



(1) $\boldsymbol{u}^{\boldsymbol{h}}$ (2) $|\boldsymbol{u}^{\boldsymbol{h}}|$ (3) p^h

Fig. 10 The velocity and pressure with a random permeability

refined mesh (see Fig. 8(3)). Figure 10 displays the simulation results, which show that the fluid tends to flow towards with higher permeability in the porous medium region.

In addition, the iteration errors for both the periodic and random permeability cases are plotted in Fig. 2(3)(4), indicating the good applicability of the Picard iteration.

Concluding Remark

In this remark, we summarize the assumptions made during our theoretical analysis. To establish the well-posedness of the non-Newtonian Stokes-Darcy-Forchheimer model (see Theorems 2.1), we assume the positivity, boundedness, Lipschitz continuity, and strong monotonicity of $g_1(\cdot)$ and $g_2(\cdot)$ (as described in (2.4) and (2.5)). We have discussed these assumptions in Remark 2.3, and can confirm that they are satisfied for various non-Newtonian models, such as the Carreau model, Cross model, Powell-Eyring model, and so on, which are listed in Table 1. In order to establish the well-posedness of the P^k/P^{k-1}-DG approximation problem (3.2) (as stated in Theorem 3.1), it is necessary to take sufficiently large coefficients σ_{1e} and σ_{2e} for the stabilization terms (as explained in Lemmas 3.2 and 3.3). Additionally, we assume that $g_1'(|\boldsymbol{A}|)\boldsymbol{A}$ is bounded (see Lemma 3.3), which is true for the models with parameters listed in Table 1.

Furthermore, to obtain the error analysis of the DG method (see Theorem 3.3), we make the regularity assumption $\boldsymbol{u}_1 \in (H^{k+1}(\Omega_1)) \cap W^{1,\infty}(\Omega_1)$ and $\boldsymbol{u}_2 \in (H^k(\Omega_2)) \cap W^{1,r}(\Omega_2)$ $(r \geq d)$ and $p_i \in H^k(\Omega_i)$ $(i = 1, 2)$. To guarantee the convergence of the Picard iteration of the continuous (resp. discrete) problem (see Theorem (2.3) (resp. 3.2)), we introduce a sufficient condition (2.44) (resp. (3.23)), which takes into account the $(W^{1,\infty}, L^\infty)$-norm of $(\boldsymbol{u_1}, \boldsymbol{u_2})$ (or $(\boldsymbol{u_1^h}, \boldsymbol{u_2^h})$). However, validating this condition can be challenging. Despite this, we find that the Picard iteration is effective in simulating various types of permeability, as demonstrated in our numerical examples.

Data availibility The authors confirm that the data supporting the findings of this study are available within the article.

Declarations

Conflict of interest The authors have no conflicts of interest to declare. All co-authors have seen and agree with the contents of the manuscript, and there is no financial interest to report.

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