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Numerical Estimation of the Inverse Eigenvalue Problem for a Weighted Helmholtz Equation

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Abstract
The inverse eigenvalue problem for a weighted Helmholtz equation is investigated. Based on the finite spectral data, the density function is estimated. The inverse problem is formulated as a least squared functional with respect to the density function, with a L^2 regularity term. The continuity of the eigenpairs with respect to the density is proved. Mathematical properties of the continuous and the discrete optimization problems are established. A conjugate gradient algorithm is proposed. Numerical results for $1D$ and $2D$ inverse eigenvalue problem of the weighted Helmholtz equation are presented to illustrate the effectiveness and efficiency of the proposed algorithm.

Keywords Inverse eigenvalue problem · Weighted Helmholtz equation · Conjugate gradient algorithm · Finite element method

1 Introduction

An inverse eigenvalue problem concerns the reconstruction or identification of the parameters in the governing differential equation from the prescribed spectral data. It arises in various applications, such as control design, system identification, seismic tomography, principal component analysis, exploration and remote sensing, antenna array processing, geophysics, molecular spectroscopy, particle physics, structure analysis, circuit theory, mechanical system simulation, and so on [1].

The inverse eigenvalue problem for a weighted Helmholtz equation is a kind of the classical inverse Sturm-Liouville problem [2–5]. Given the first finite smallest eigenvalues, the

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density function in the weighted Helmholtz equation is recovered. McCarthy uses projection of the boundary value problem and its coefficients onto appropriate vector spaces, which leads to a matrix inverse problem [6, 7]. Andrew proposes a new algorithm for solving the inverse Sturm-Liouville problem of reconstructing a symmetric potential from eigenvalues [8]. Drignei deals with the recovery of the potential coefficient of a Sturm-Liouville operator from three known sequences of eigenvalues corresponding respectively to three sets of Dirichlet boundary conditions [9]. Jiang et al. investigate the inverse second-order Sturm-Liouville problem and the inverse fourth-order Sturm-Liouville problem, and derive trace formulas showing relations between the unknown coefficients and eigenvalues explicitly for both problems [10]. Gao et al. propose a new iterative method to recover the impedance of Sturm-Liouville problem from the finite eigenvalues [11]. Based on natural eigenfrequencies, Zhang et al. investigate the damage identification of elastic vibration structure, where level set method is introduced to represent two different material regions [12]. For the same objective functional, Zhang et al. propose the piecewise constant level set method to represent the shape and topology of the damaged region [13]. Lee and Shin introduce a frequency response function-based structural damage identification method for beam structures [14].

The finite element method is used to solve the eigenvalue problem. There are many excellent works on it, and we refer to [15, 16] and references cited therein. The refined estimates for Galerkin approximations of the eigenvalues and eigenvectors of selfadjoint eigenvalue problem is investigated in [17, 18]. The error estimates for the generalised Dirichlet eigenvalue problem with stochastic coefficients is presented in [19].

This paper is organized as follows. In Sect. 2, the mathematical formulations of the inverse eigenvalue problem of the weighted Helmholtz equation is described. In Sect. 3, the properties of existence, stability and Fr$\acute{e}$chet derivative of the continuous optimization problems are established. In Sect. 4, the properties of existence and the convergence of the discrete optimization problems are given. A conjugate gradient method is proposed in Sect. 5. $1D$ and $2D$ numerical results of inverse eigenvalue problem of the weighted Helmholtz equation are presented In Sect. 6.

2 Problem Statement

Consider the weighted Helmholtz equation

$$-\Delta u = \lambda \rho u, \quad \text{in } \Omega, \tag{1}$$

$$u = 0, \quad \text{on } \partial\Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 1, 2)$ is a bounded and connected domain and $\partial\Omega$ is the boundary of the domain. $\rho(x)$ is the density function and is assumed to satisfy the condition

$$0 < \rho_0 \leq \rho(x) \leq \rho_1 \quad \text{in } \Omega, \tag{3}$$

where ρ_0 and ρ_1 are two constants. (λ, u) is the eigenpair of the minus Laplace operator $-\Delta$ with density function $\rho(x)$. The weak formulation of (1) and (2) is given by

$$\int_\Omega \nabla u \cdot \nabla v dx = \lambda \int_\Omega \rho u v dx, \quad \forall v \in H_0^1(\Omega). \tag{4}$$

By rearranging, (4) admits a countable sequence of real eigenvalues [see 18]

$$0 < \lambda_1(\rho) \leq \lambda_2(\rho) \leq \lambda_3(\rho) \leq \cdots \tag{5}$$

and the corresponding eigenfunctions

$$u_1(\rho),\ u_2(\rho),\ u_3(\rho)\ldots. \tag{6}$$

The i-th eigenpair $(\lambda_i(\rho), u_i(\rho))$ is obtained by the min-max principle [18]

$$\begin{aligned}\lambda_i(\rho) &= \min_{V_i\subset H_0^1(\Omega),\ \dim(V_i)=i}\ \max_{u\in V_i} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \rho u^2 dx} \\ &= \max_{u\in span\{u_1,u_2,\ldots,u_i\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \rho u^2 dx} \\ &= \frac{\int_\Omega |\nabla u_i(\rho)|^2 dx}{\int_\Omega \rho u_i^2(\rho) dx},\end{aligned} \tag{7}$$

where $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ consisting of functions which vanish at the boundary of Ω in the sense of trace. Notice that the eigenfunction $u_i(\rho)$ in (7) is not unique, since for a nonzero constant C, $Cu_i(\rho)$ is also the eigenfunction corresponding to $\lambda_i(\rho)$. When $\lambda_i(\rho)$ is multiple, $u_i(\rho)$ is assigned one of the eigenfunctions corresponding to $\lambda_i(\rho)$. For any $\lambda_i(\rho)$ we let

$$M(\lambda_i(\rho)) = \{u :\ u \text{ is an eigenfunction of(4) corresponding to } \lambda_i(\rho)\}. \tag{8}$$

The eigenfunction $u_i(\rho)$ in (7) is deemed to be one of the eigenfunctions corresponding to $\lambda_i(\rho)$, that is, $u_i(\rho) \in M(\lambda_i(\rho))$. The eigenfunctions in (6) are normalized and orthogonalized to satisfy

$$\int_\Omega \nabla u_i(\rho)\cdot\nabla u_j(\rho)dx = \lambda_i(\rho)\int_\Omega \rho u_i(\rho)u_j(\rho)dx = \delta_{ij},\ i, j = 1,\ 2,\ \ldots. \tag{9}$$

It is known that an inverse eigenvalue problem, especially for the real-valued case, may not necessarily have an exact solution [1]. It is also known that the spectral information, in practice, is often obtained by experimental devices and thus inevitably contaminated with measurement errors. That is, there are situations where an approximate solution best in the sense of least squares would be satisfactory. In order to deal with the instability of the inverse problem, a L^2 regularity term is added. The inverse eigenvalue problem is reformulated as the following constrained optimization problem:

$$\min_{\rho\in\mathcal{A}} F(\rho) = \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega \rho^2 dx \tag{10}$$

where $\widehat{\lambda}_i$, $(i = 1, 2, \ldots, N)$ are the measured data of the first N eigenvalues, ε is the regularity parameter and $\mathcal{A}$ is the admissible set of the density function, that is,

$$\mathcal{A} = \{\rho(x) \in L^\infty(\Omega) :\ \rho_0 \le \rho(x) \le \rho_1 \text{ a.e.} x \in \Omega\}. \tag{11}$$

3 Existence and Stability of the Optimization Problem

In this section, we present the properties of existence, stability and Fréchet derivative of the continuous optimization problem (10) and (4).

Lemma 3.1 *For any $\rho \in \mathcal{A}$, we have*

$$\lambda_i(\rho_1) \le \lambda_i(\rho) \le \lambda_i(\rho_0),\ i = 1, 2, \ldots, N. \tag{12}$$

Proof For $i = 1, 2, \ldots, N$, by (7), we set $u_i^0,\ u_i^1 \in V_i \subset H_0^1(\Omega)$ and $dim(V_i) = i$ such that

$$\lambda_i(\rho_0) = \min_{V_i \subset H_0^1(\Omega),\ dim(V_i)=i} \max_{u \in V_i} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \rho_0 u^2 dx} = \frac{\int_\Omega |\nabla u_i^0|^2 dx}{\int_\Omega \rho_0 {u_i^0}^2 dx}, \tag{13}$$

and

$$\lambda_i(\rho_1) = \min_{V_i \subset H_0^1(\Omega),\ dim(V_i)=i} \max_{u \in V_i} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega \rho_1 u^2 dx} = \frac{\int_\Omega |\nabla u_i^1|^2 dx}{\int_\Omega \rho_1 {u_i^1}^2 dx}. \tag{14}$$

Consequently, by (3), (7), (13) and (14), we have

$$\begin{aligned} \lambda_i(\rho_1) &= \frac{\int_\Omega |\nabla u_i^1|^2 dx}{\int_\Omega \rho_1 {u_i^1}^2 dx} \le \frac{\int_\Omega |\nabla u_i|^2 dx}{\int_\Omega \rho_1 u_i^2 dx} \le \lambda_i(\rho) = \frac{\int_\Omega |\nabla u_i|^2 dx}{\int_\Omega \rho {u_i}^2 dx} \\ &\le \frac{\int_\Omega |\nabla u_i^0|^2 dx}{\int_\Omega \rho {u_i^0}^2 dx} \le \frac{\int_\Omega |\nabla u_i^0|^2 dx}{\int_\Omega \rho_0 {u_i^0}^2 dx} = \lambda_i(\rho_0). \end{aligned} \tag{15}$$

□

Lemma 3.2 *For $i = 1, 2, \ldots, N$, assume that $(\lambda_i(\rho), u_i(\rho))$ is the i-th eigenpair of (4), and $(\tilde{\lambda}_i, \tilde{u}_i)$ is the eigenpair of (4) replacing ρ by $\tilde{\rho}$, then*

$$\begin{aligned} &\int_\Omega |\nabla(u_i(\rho) - \tilde{u}_i)|^2\, dx - \tilde{\lambda}_i \int_\Omega \tilde{\rho}(u_i(\rho) - \tilde{u}_i)^2 dx \\ &= (\lambda_i(\rho) - \tilde{\lambda}_i) \int_\Omega \rho u_i^2(\rho) dx + \tilde{\lambda}_i \int_\Omega (\rho - \tilde{\rho}) u_i^2(\rho) dx. \end{aligned} \tag{16}$$

Proof For $i = 1, 2, \ldots, N$, since $(\lambda_i(\rho), u_i(\rho))$ satisfies (4) and $(\tilde{\lambda}_i, \tilde{u}_i)$ satisfies (4) by replacing ρ by $\tilde{\rho}$, we obtain

$$\begin{aligned} &\int_\Omega |\nabla(u_i(\rho) - \tilde{u}_i)|^2\, dx - \tilde{\lambda}_i \int_\Omega \tilde{\rho}(u_i(\rho) - \tilde{u}_i)^2 dx \\ &= \int_\Omega |\nabla u_i(\rho)|^2 dx - 2\int_\Omega \nabla u_i(\rho) \cdot \nabla \tilde{u}_i dx + \int_\Omega |\nabla \tilde{u}_i|^2 dx - \tilde{\lambda}_i \int_\Omega \tilde{\rho} u_i^2(\rho) dx \\ &\quad + 2\tilde{\lambda}_i \int_\Omega \tilde{\rho} u_i(\rho) \tilde{u}_i dx - \tilde{\lambda}_i \int_\Omega \tilde{\rho} \tilde{u}_i^2 dx \\ &= \lambda_i(\rho) \int_\Omega \rho u_i^2(\rho) dx - \tilde{\lambda}_i \int_\Omega \tilde{\rho} u_i^2(\rho) dx \\ &= (\lambda_i(\rho) - \tilde{\lambda}_i) \int_\Omega \rho u_i^2(\rho) dx + \tilde{\lambda}_i \int_\Omega (\rho - \tilde{\rho}) u_i^2(\rho) dx. \end{aligned}$$

□

Replacing $\rho \in \mathcal{A}$ in (4)–(9) by $\rho^n,\ \rho^* \in \mathcal{A}$, the eigenpairs $(u_i(\rho^n), \lambda_i(\rho^n))$ and $(u_i(\rho^*), \lambda_i(\rho^*)),\ i = 1,\ 2,\ \ldots,$ are obtained, respectively.

Lemma 3.3 *For $i = 1, 2, \ldots, N$, let $\rho^n,\ \rho^* \in \mathcal{A}$ and $\rho^n \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$ as $n \to \infty$, then $\lambda_i(\rho^n) \to \lambda_i(\rho^*)$. Moreover, there exists a subsequence $u_i(\rho^n) \in M(\lambda_i(\rho^n))$, and some $u_i(\rho^*) \in M(\lambda_i(\rho^*))$, satisfying $u_i(\rho^n) \to u_i(\rho^*)$ in $H^1(\Omega)$.*

Proof For $i = 1, 2, \ldots, N$, by Lemma 3.1, the boundedness of $\lambda_i(\rho^n)$ implies that there exists a subsequence, also denoted by $\lambda_i(\rho^n)$, such that

$$\lim_{n\to\infty} \lambda_i(\rho^n) = \lambda_i^*. \tag{17}$$

Notice that by (9) (replacing ρ by ρ^n) the eigenfunction $u_i(\rho^n) \in M(\lambda_i(\rho^n))$ satisfies

$$\int_\Omega |\nabla u_i(\rho^n)|^2 dx = 1. \tag{18}$$

By Poincaré inequality, we have

$$\|u_i(\rho^n)\|_{H^1(\Omega)} < C, \tag{19}$$

where C is a constant independent of ρ^n. Thus, there exists a subsequence, also denoted by $u_i(\rho^n)$, such that

$$u_i(\rho^n) \rightharpoonup u_i^* \text{ in } H^1(\Omega) \text{ and } u_i(\rho^n) \to u_i^* \text{ in } L^2(\Omega). \tag{20}$$

We need to prove that (λ_i^*, u_i^*) is the eigenpair of (4) corresponding to ρ^* for $i = 1, 2, \ldots, N$. Notice that the eigenpair $(\lambda_i(\rho^n),\ u_i(\rho^n))$ corresponding to ρ^n satisfies (4) (replacing ρ by ρ^n), that is,

$$\int_\Omega \nabla u_i(\rho^n) \cdot \nabla v dx = \lambda_i(\rho^n) \int_\Omega \rho^n u_i(\rho^n) v dx,\ \forall\, v \in H_0^1(\Omega),\ i = 1, 2, \ldots, N. \tag{21}$$

The RHS item of equation of (21) could be rewritten by

$$\begin{aligned}\lambda_i(\rho^n) \int_\Omega \rho^n u_i(\rho^n) v dx =& (\lambda_i(\rho^n) - \lambda_i^*) \int_\Omega \rho^n u_i(\rho^n) v dx + \lambda_i^* \int_\Omega \rho^n (u_i(\rho^n) - u_i^*) v dx \\ &+ \lambda_i^* \int_\Omega (\rho^n - \rho^*) u_i^* v dx + \lambda_i^* \int_\Omega \rho^* u_i^* v dx.\end{aligned} \tag{22}$$

By (11), (17), (19), (20) and $\rho^n \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$, we have

$$\lim_{n\to\infty} \lambda_i(\rho^n) \int_\Omega \rho^n u_i(\rho^n) v dx = \lambda_i^* \int_\Omega \rho^* u_i^* v dx. \tag{23}$$

By (20), we have

$$\lim_{n\to\infty} \int_\Omega \nabla u_i(\rho^n) \cdot \nabla v dx = \int_\Omega \nabla u_i^* \cdot \nabla v dx \tag{24}$$

Combining (23) and (24), and taking the limitation of equation (21), we have

$$\int_\Omega \nabla u_i^* \cdot \nabla v dx = \lambda_i^* \int_\Omega \rho^* u_i^* v dx,\ \forall\, v \in H_0^1(\Omega),\ i = 1, 2, \ldots, N. \tag{25}$$

Therefore, it concludes that (λ_i^*, u_i^*) $(i = 1, 2, \ldots, N)$ are eigenpairs of (4) corresponding to ρ^*.

Furthermore, replacing $\rho,\ u_i(\rho),\ \lambda_i(\rho)$ by $\rho^n,\ u_i(\rho^n),\ \lambda_i(\rho^n)$, and replacing $\tilde{\rho},\ \tilde{u}_i,\ \tilde{\lambda}_i$ by $\rho^*,\ u_i^*,\ \lambda_i^*$ in Lemma 3.2, respectively, we have

$$\begin{aligned}
&\int_\Omega |\nabla(u_i(\rho^n) - u_i^*)|^2 dx \\
&\quad = \lambda_i^* \int_\Omega \rho^*(u_i(\rho^n) - u_i^*)^2 dx + (\lambda_i(\rho^n) - \lambda_i^*) \int_\Omega \rho^n u_i^2(\rho^n) dx \\
&\qquad + \lambda_i^* \int_\Omega (\rho^n - \rho^*) u_i^2(\rho^n) dx \\
&\quad \to 0 \quad \text{as } n \to \infty,
\end{aligned} \tag{26}$$

with (11), (20), (17), (19) and $\rho^n \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$. Combining (20) and (26), we have

$$u_i(\rho^n) \to u_i^* \text{ in } H^1(\Omega). \tag{27}$$

By the orthogonality (9) (replacing ρ by ρ^n), (27), (17) and $\rho^n \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$, the orthogonality of u_i^* is obtained by

$$\int_\Omega \nabla u_i^* \cdot \nabla u_j^* dx = \lambda_i^* \int_\Omega \rho^* u_i^* u_j^* dx = \delta_{ij},\ i, j = 1,\ 2,\ \ldots\ N. \tag{28}$$

Finally, we prove (λ_i^*, u_i^*) $(i = 1, 2, \ldots, N)$ are the i-th eigenpairs for ρ^* by induction. For $i = 1$, by (7) (replacing ρ by ρ^*) we have

$$\begin{aligned}
\lambda_1(\rho^*) &= \frac{\int_\Omega |\nabla u_1(\rho^*)|^2\, dx}{\int_\Omega \rho^* |u_1(\rho^*)|^2\, dx} \\
&\le \frac{\int_\Omega |\nabla u_1^*|^2\, dx}{\int_\Omega \rho^* |u_1^*|^2\, dx} = \lambda_1^*
\end{aligned} \tag{29}$$

On the other hand, by (17), (7) (replacing ρ by ρ^n) and $\rho^n \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$, we have

$$\begin{aligned}
\lambda_1^* &= \lim_{n\to\infty} \lambda_1(\rho^n) \\
&= \lim_{n\to\infty} \frac{\int_\Omega |\nabla u_1(\rho^n)|^2\, dx}{\int_\Omega \rho^n u_1^2(\rho^n)\, dx} \\
&\le \lim_{n\to\infty} \frac{\int_\Omega |\nabla u_1(\rho^*)|^2\, dx}{\int_\Omega \rho^n u_1^2(\rho^*)\, dx} \\
&= \frac{\int_\Omega |\nabla u_1(\rho^*)|^2\, dx}{\int_\Omega \rho^* u_1^2(\rho^*)\, dx} = \lambda_1(\rho^*).
\end{aligned} \tag{30}$$

Combining (29) and (30), we obtain $\lambda_1^* = \lambda_1(\rho^*)$. The eigenfunction u_1^* corresponding to λ_1^* is also the eigenfunction corresponding to $\lambda_1(\rho^*)$. That is, $u_1^* \in M(\lambda_1(\rho^*))$. Thus, we set $u_1(\rho^*) \in M(\lambda_1(\rho^*))$ to satisfy $u_1^* = u_1(\rho^*)$.

For $i = 1, 2, \ldots, k$, assuming that

$$\lambda_i^* = \lambda_i(\rho^*), \tag{31}$$

$$u_i^* = u_i(\rho^*), \tag{32}$$

we need to prove that $\lambda^*_{k+1} = \lambda_{k+1}(\rho^*)$ and $u^*_{k+1} = u_{k+1}(\rho^*)$. By (7) and (9) (replacing ρ by ρ^*), (28) and (32), we have

$$\lambda_{k+1}(\rho^*) = \frac{\int_\Omega |\nabla u_{k+1}(\rho^*)|^2\,dx}{\int_\Omega \rho^* u^2_{k+1}(\rho^*)\,dx}$$
$$= \min_{V_{k+1}\subset H^1_0(\Omega),\dim(V_{k+1})=k+1}\ \max_{u\in V_{k+1}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \rho^* u^2\,dx}$$
$$\le \max_{u\in span\{u^*_1,u^*_2,\dots,u^*_k,u^*_{k+1}\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \rho^* u^2\,dx} = \frac{\int_\Omega |\nabla u^*|^2\,dx}{\int_\Omega \rho^* u^{*2}\,dx}. \tag{33}$$

Let $u^* = \alpha_1 u^*_1 + \alpha_2 u^*_2 + \dots + \alpha_k u^*_k + \alpha_{k+1} u^*_{k+1}$. By the orthogonality (28), we have

$$\int_\Omega |\nabla u^*|^2\,dx = \sum_{j=1}^{k+1}\int_\Omega \alpha_j^2 |\nabla u^*_j|^2\,dx = \sum_{j=1}^{k+1}\alpha_j^2\lambda^*_j\int_\Omega \rho^* u^{*2}_j\,dx, \tag{34}$$

$$\int_\Omega \rho^* u^{*2}\,dx = \sum_{j=1}^{k+1}\alpha_j^2\int_\Omega \rho^* u^{*2}_j\,dx. \tag{35}$$

By (5) (replacing ρ by ρ^*) and (31), we have

$$\lambda^*_1 \le \lambda^*_2 \le \dots \le \lambda^*_k. \tag{36}$$

Then, we use reduction to absurdity to prove $\lambda^*_{k+1} \ge \lambda^*_k$. If the claim is negated to assume that

$$\lambda^*_{k+1} < \lambda^*_k, \tag{37}$$

combining with (34), (35) and (36), we have

$$\int_\Omega |\nabla u^*|^2\,dx = \sum_{j=1}^{k+1}\alpha_j^2\lambda^*_j\int_\Omega \rho^* u^{*2}_j\,dx$$
$$< \lambda^*_k\sum_{j=1}^{k+1}\alpha_j^2\int_\Omega \rho^* u^{*2}_j\,dx$$
$$= \lambda^*_k\int_\Omega \rho^* u^{*2}\,dx. \tag{38}$$

By (33) and (38), we can get $\lambda_{k+1}(\rho^*) < \lambda^*_k = \lambda_k(\rho^*)$, which incurs the contradiction. Therefore, we conclude that

$$\lambda^*_{k+1} \ge \lambda^*_k. \tag{39}$$

Combining with (34), (35), (36) and (39), we obtain

$$\int_\Omega |\nabla u^*|^2\,dx = \sum_{j=1}^{k+1}\alpha_j^2\lambda^*_j\int_\Omega \rho^* u^{*2}_j\,dx$$
$$\le \lambda^*_{k+1}\sum_{j=1}^{k+1}\alpha_j^2\int_\Omega \rho^* u^{*2}_j\,dx = \lambda^*_{k+1}\int_\Omega \rho^* u^{*2}\,dx. \tag{40}$$

Substituting (40) into (33), we obtain

$$\lambda_{k+1}(\rho^*) \leq \lambda_{k+1}^*. \tag{41}$$

On the other hand, we need to prove that $\lambda_{k+1}(\rho^*) \geq \lambda_{k+1}^*$. First, we can conclude that

$$\dim span \left\{u_1(\rho^n), u_2(\rho^n), ..., u_k(\rho^n), u_{k+1}(\rho^*)\right\} = k+1. \tag{42}$$

If the claim is negated to assume that

$$\dim span \left\{u_1(\rho^n), u_2(\rho^n), ..., u_k(\rho^n), u_{k+1}(\rho^*)\right\} = k,$$

since $u_1(\rho^n), u_2(\rho^n), ..., u_k(\rho^n)$ are orthogonal by (9) (replacing ρ by ρ^n), then

$$u_{k+1}(\rho^*) = \gamma_1 u_1(\rho^n) + \gamma_2 u_2(\rho^n) + ... + \gamma_k u_k(\rho^n). \tag{43}$$

Thus,

$$\int_\Omega \nabla u_{k+1}(\rho^*)\nabla u_i(\rho^*)dx = \sum_{j=1}^k \gamma_j \int_\Omega \nabla u_j(\rho^n)\nabla u_i(\rho^*)dx,\ i = 1, 2, \ldots, k. \tag{44}$$

Taking the limitation of (44) as $n \to \infty$, with (27), (32) and (9) (replacing ρ by ρ^*), we obtain

$$\begin{aligned} 0 &= \int_\Omega \nabla u_{k+1}(\rho^*)\nabla u_i(\rho^*)dx \\ &= \sum_{j=1}^k \gamma_j \int_\Omega \nabla u_j(\rho^*)\nabla u_i(\rho^*)dx \\ &= \gamma_i,\ i = 1, 2, \ldots, k. \end{aligned} \tag{45}$$

Therefore, $u_{k+1}(\rho^*) \equiv 0$, which contradicts the definition of a nontrivial eigenfunction.

By (7) (replacing ρ by ρ^n) and (42), we have

$$\begin{aligned} \lambda_{k+1}(\rho^n) &= \frac{\int_\Omega |\nabla u_{k+1}(\rho^n)|^2\,dx}{\int_\Omega \rho^n u_{k+1}^2(\rho^n)\,dx} \\ &= \min_{V_{k+1}\subset H_0^1(\Omega), \dim V_{k+1}=k+1} \max_{u\in V_{k+1}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \rho^n u^2\,dx} \\ &\leq \max_{u\in span\{u_1(\rho^n), u_2(\rho^n), \ldots, u_k(\rho^n), u_{k+1}(\rho^*)\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega \rho^n u^2\,dx} \\ &= \frac{\int_\Omega |\nabla u^n|^2\,dx}{\int_\Omega \rho^n u^{n2}\,dx}. \end{aligned} \tag{46}$$

Let $u^n = \beta_1 u_1(\rho^n) + \beta_2 u_2(\rho^n) +, ..., \beta_k u_k(\rho^n) + \beta_{k+1} u_{k+1}(\rho^*)$. By (9) and (4) (replacing ρ by ρ^n and ρ^*, respectively), (11), (17), (27), (31), (32) and $\rho^n \overset{*}{\rightharpoonup} \rho^*\ in\ L^\infty(\Omega)$, we have

$$\begin{aligned} \int_\Omega |\nabla u^n|^2\,dx &= \sum_{j=1}^k \beta_j^2 \int_\Omega |\nabla u_j(\rho^n)|^2\,dx + \beta_{k+1}^2 \int_\Omega |\nabla u_{k+1}(\rho^*)|^2\,dx \\ &\quad + 2\sum_{j=1}^k \beta_j\beta_{k+1} \int_\Omega \nabla u_j(\rho^n)\nabla u_{k+1}(\rho^*)\,dx \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} \beta_j^2 \lambda_j(\rho^n) \int_\Omega \rho^n u_j^2(\rho^n)\,dx + \beta_{k+1}^2 \lambda_{k+1}(\rho^*) \int_\Omega \rho^* u_{k+1}^2(\rho^*)\,dx \\
&\quad + 2\sum_{j=1}^{k} \beta_j \beta_{k+1} \lambda_j(\rho^n) \int_\Omega \rho^n u_j(\rho^n) u_{k+1}(\rho^*)\,dx \\
&\to \sum_{j=1}^{k} \beta_j^2 \lambda_j(\rho^*) \int_\Omega \rho^* u_j^2(\rho^*)\,dx + \beta_{k+1}^2 \lambda_{k+1}(\rho^*) \int_\Omega \rho^* u_{k+1}^2(\rho^*)\,dx \\
&\quad + 2\sum_{j=1}^{k} \beta_j \beta_{k+1} \lambda_j(\rho^*) \int_\Omega \rho^* u_j(\rho^*) u_{k+1}(\rho^*)\,dx, \quad \text{as } n \to \infty
\end{aligned} \tag{47}
$$

and

$$
\begin{aligned}
\int_\Omega \rho^n u^{n2}\,dx &= \sum_{j=1}^{k} \beta_j^2 \int_\Omega \rho^n u_j^2(\rho^n) dx + \beta_{k+1}^2 \int_\Omega \rho^n u_{k+1}^2(\rho^*) dx \\
&\quad + 2\sum_{j=1}^{k} \beta_j \beta_{k+1} \int_\Omega \rho^n u_j(\rho^n) u_{k+1}(\rho^*) dx \\
&\to \sum_{j=1}^{k} \beta_j^2 \int_\Omega \rho^* u_j^2(\rho^*) dx + \beta_{k+1}^2 \int_\Omega \rho^* u_{k+1}^2(\rho^*) dx \\
&\quad + 2\sum_{j=1}^{k} \beta_j \beta_{k+1} \int_\Omega \rho^* u_j(\rho^*) u_{k+1}(\rho^*) dx, \quad \text{as } n \to \infty.
\end{aligned} \tag{48}
$$

By (5) (replacing ρ by ρ^*) and (47), we have

$$
\begin{aligned}
&\sum_{j=1}^{k} \beta_j^2 \lambda_j(\rho^*) \int_\Omega \rho^* u_j^2(\rho^*)\,dx + \beta_{k+1}^2 \lambda_{k+1}(\rho^*) \int_\Omega \rho^* u_{k+1}^2(\rho^*)\,dx \\
&\quad + 2\sum_{j=1}^{k} \beta_j \beta_{k+1} \lambda_j(\rho^*) \int_\Omega \rho^* u_j(\rho^*) u_{k+1}(\rho^*)\,dx \\
&\le \lambda_{k+1}(\rho^*) \Big[\sum_{j=1}^{k} \beta_j^2 \int_\Omega \rho^* u_j^2(\rho^*)\,dx + \beta_{k+1}^2 \int_\Omega \rho^* u_{k+1}^2(\rho^*)\,dx \\
&\quad + 2\sum_{j=1}^{k} \beta_j \beta_{k+1} \int_\Omega \rho^* u_j(\rho^*) u_{k+1}(\rho^*)\,dx \Big].
\end{aligned} \tag{49}
$$

Combining (17), (46), (47), (48) and (49), it implies that

$$
\lambda_{k+1}^* = \lim_{n\to\infty} \lambda_{k+1}(\rho^n) \le \lambda_{k+1}(\rho^*). \tag{50}
$$

With (41) and (50), we conclude that $\lambda_{k+1}^* = \lambda_{k+1}(\rho^*)$. Moreover, the eigenfunction u_{k+1}^* corresponding to λ_{k+1}^* is also the eigenfunction corresponding to $\lambda_{k+1}(\rho^*)$. That is, $u_{k+1}^* \in M(\lambda_{k+1}(\rho^*))$. We set $u_{k+1}(\rho^*) \in M(\lambda_{k+1}(\rho^*))$ to satisfy $u_{k+1}^* = u_{k+1}(\rho^*)$. By induction, we conclude that $\lambda_i^* = \lambda_i(\rho^*)$ and $u_i^* = u_i(\rho^*)$, for $i = 1, 2, \ldots, N$. Combined with (17) and (27), the proof is complete. □

Theorem 3.1 *There exists at least one minimizer to the optimization problem* (10) *and* (4).

Proof By Lemma 3.1, we conclude that the minimization of $F(\rho)$ is finite over the admissible set $\mathcal{A}$. Therefore, there exists a sequence $\rho^n \in \mathcal{A}$ such that

$$\lim_{n\to\infty} F(\rho^n) = \inf_{\rho\in\mathcal{A}} F(\rho). \tag{51}$$

The uniform boundedness of the sequence ρ^n in $L^\infty(\Omega)$ implies that there exists a subsequence, also denoted by ρ^n, and some $\rho^* \in \mathcal{A}$ such that

$$\rho^n \overset{*}{\rightharpoonup} \rho^* \ \text{ in } L^\infty(\Omega). \tag{52}$$

By Lemma 3.3, we have

$$\lambda_i(\rho^n) \to \lambda_i(\rho^*), \ \text{ for } i = 1, 2, \ldots, N. \tag{53}$$

Now the convergence of $\lambda_i(\rho^n)$ and the lower semicontinuity of a norm imply that

$$\begin{aligned} F(\rho^*) &= \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^*) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^*)^2 dx \\ &\le \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^n) - \widehat{\lambda}_i)^2 + \liminf_{n\to\infty} \frac{\varepsilon}{2}\int_\Omega (\rho^n)^2 dx \\ &\le \liminf_{n\to\infty} \Big(\frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^n) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^n)^2 dx\Big) \\ &= \liminf_{n\to\infty} F(\rho^n) \le \inf_{\rho\in\mathcal{A}} F(\rho). \end{aligned} \tag{54}$$

Therefore ρ^* is a minimizer of $F(\rho)$. □

Next theorem shows the stability of the optimization problem with respect to the data perturbation.

Theorem 3.2 *For* $i = 1, 2, \ldots, N$, *let* $\widehat{\lambda}_i^n$ *be a sequence such that*

$$\widehat{\lambda}_i^n \to \widehat{\lambda}_i$$

and ρ^n *be the minimizer of the optimization problem* (10) *and* (4) *with* $\widehat{\lambda}_i$ *replaced by* $\widehat{\lambda}_i^n$. *Then there exists a subsequence of* ρ^n *weak*$-*$ *converging in* $L^\infty(\Omega)$ *to a minimizer of the optimization problem* (10) *and* (4).

Proof Since ρ^n is the minimizer of the optimization problem (10) and (4) with $\widehat{\lambda}_i$ replaced by $\widehat{\lambda}_i^n$, for all $\rho \in \mathcal{A}$, we have that

$$\frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^n) - \widehat{\lambda}_i^n)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^n)^2 dx \le \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho) - \widehat{\lambda}_i^n)^2 + \frac{\varepsilon}{2}\int_\Omega \rho^2 dx \tag{55}$$

The uniform boundedness of the sequence ρ^n in $L^\infty(\Omega)$ implies that there exists a subsequence, also denoted by ρ^n, and some $\rho^* \in \mathcal{A}$ such that

$$\rho^n \overset{*}{\rightharpoonup} \rho^* \text{ in } L^\infty(\Omega). \tag{56}$$

By Lemma 3.3, there exists a subsequence of $\lambda_i(\rho^n)$, also denoted by $\lambda_i(\rho^n)$, such that

$$\lim_{n\to\infty} \lambda_i(\rho^n) = \lambda_i(\rho^*), \; i = 1, 2, \ldots, N. \tag{57}$$

Thus, by the convergence of $\widehat{\lambda}_i^n \to \widehat{\lambda}_i$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{N} (\lambda_i(\rho^n) - \widehat{\lambda}_i^n)^2 = \sum_{i=1}^{N} (\lambda_i(\rho^*) - \widehat{\lambda}_i)^2.$$

Now, for all $\rho \in \mathcal{A}$, we have that

$$\begin{aligned} F(\rho^*) &= \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^*) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^*)^2 dx \\ &\le \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^n) - \widehat{\lambda}_i^n)^2 + \liminf_{n\to\infty} \frac{\varepsilon}{2}\int_\Omega (\rho^n)^2 dx \\ &\le \liminf_{n\to\infty} \Big(\frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^n) - \widehat{\lambda}_i^n)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^n)^2 dx\Big) \\ &\le \liminf_{n\to\infty} \Big(\frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho) - \widehat{\lambda}_i^n)^2 + \frac{\varepsilon}{2}\int_\Omega \rho^2 dx\Big) \\ &= \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega \rho^2 dx = F(\rho). \end{aligned} \tag{58}$$

Therefore ρ^* is a minimizer of (10) and (4). □

Theorem 3.3 *For $\rho \in \mathcal{A}$ and $i = 1, 2, \ldots, N$, if eigenvalues $\lambda_i(\rho)$ of (4) are simple, then the functional $F\colon \mathcal{A} \subset L^\infty(\Omega) \to \mathbb{R}$ is Fréchet differentiable and its Fréchet derivative $F'(\rho)$ at $\rho \in \mathcal{A}$ is given by*

$$F'(\rho)[\gamma] = -\sum_{i=1}^{N}(\lambda_i(\rho) - \widehat{\lambda}_i)\frac{\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx}{\int_\Omega \rho u_i^2(\rho)dx} + \varepsilon\int_\Omega \rho\gamma dx. \tag{59}$$

Proof Replacing ρ by $(\rho+\gamma)$ in (7), we define the i-th eigenpair $(\lambda_i(\rho+\gamma), u_i(\rho+\gamma))$ for $(\rho+\gamma)$ by

$$\lambda_i(\rho+\gamma) = \frac{\int_\Omega |\nabla u_i(\rho+\gamma)|^2 dx}{\int_\Omega (\rho+\gamma)u_i^2(\rho+\gamma)dx}, \; i = 1, 2, \ldots, N. \tag{60}$$

Similar to the proof of Lemma 3.3, we can show that as $\gamma \to 0$ in $L^\infty(\Omega)$,

$$\lambda_i(\rho+\gamma) \to \lambda_i(\rho), \tag{61}$$

$$u_i(\rho+\gamma) \to u_i(\rho) \; in \; H^1(\Omega), \; i = 1, 2, \ldots, N, \tag{62}$$

where $u_i(\rho) \in M(\lambda_i(\rho))$. Combined with (7) and (60), we have

$$\begin{aligned} \lambda_i(\rho+\gamma) - \lambda_i(\rho) &= \frac{\int_\Omega |\nabla u_i(\rho+\gamma)|^2 dx}{\int_\Omega (\rho+\gamma)u_i^2(\rho+\gamma)dx} - \frac{\int_\Omega |\nabla u_i(\rho)|^2 dx}{\int_\Omega \rho u_i^2(\rho)dx} \\ &= \frac{\int_\Omega |\nabla u_i(\rho+\gamma)|^2 dx \int_\Omega \rho u_i^2(\rho)dx - \int_\Omega (\rho+\gamma)u_i^2(\rho+\gamma)dx \int_\Omega |\nabla u_i(\rho)|^2 dx}{\int_\Omega (\rho+\gamma)u_i^2(\rho+\gamma)dx \int_\Omega \rho u_i^2(\rho)dx} \end{aligned} \tag{63}$$

With (4), (7) and (60), the numerator of (63) could be evaluated by

$$
\begin{aligned}
&\int_\Omega |\nabla u_i(\rho+\gamma)|^2 dx \int_\Omega \rho u_i^2(\rho) dx - \int_\Omega (\rho+\gamma) u_i^2(\rho+\gamma) dx \int_\Omega |\nabla u_i(\rho)|^2 dx \\
&\quad = \int_\Omega (|\nabla u_i(\rho+\gamma)|^2 - |\nabla u_i(\rho)|^2) dx \int_\Omega \rho u_i^2(\rho) dx \\
&\qquad - \int_\Omega ((\rho+\gamma) u_i^2(\rho+\gamma) - \rho u_i^2(\rho)) dx \int_\Omega |\nabla u_i(\rho)|^2 dx \\
&\quad = \int_\Omega \nabla(u_i(\rho+\gamma) - u_i(\rho)) \cdot \nabla(u_i(\rho+\gamma) + u_i(\rho)) dx \int_\Omega \rho u_i^2(\rho) dx \\
&\qquad - \int_\Omega (\gamma u_i^2(\rho+\gamma) + \rho(u_i(\rho+\gamma) - u_i(\rho))(u_i(\rho+\gamma) + u_i(\rho))) dx \int_\Omega |\nabla u_i(\rho)|^2 dx \\
&\quad = \lambda_i(\rho+\gamma) \int_\Omega (\rho+\gamma)(u_i(\rho+\gamma) - u_i(\rho)) u_i(\rho+\gamma) dx \int_\Omega \rho u_i^2(\rho) dx \\
&\qquad + \lambda_i(\rho) \int_\Omega \rho(u_i(\rho+\gamma) - u_i(\rho)) u_i(\rho) dx \int_\Omega \rho u_i^2(\rho) dx \\
&\qquad - \int_\Omega \gamma u_i^2(\rho+\gamma) dx \lambda_i(\rho) \int_\Omega \rho u_i^2(\rho) dx \\
&\qquad - \int_\Omega \rho(u_i(\rho+\gamma) - u_i(\rho))(u_i(\rho+\gamma) + u_i(\rho)) dx \lambda_i(\rho) \int_\Omega \rho u_i^2(\rho) dx \\
&\quad = \Big[(\lambda_i(\rho+\gamma) - \lambda_i(\rho)) \int_\Omega (\rho+\gamma)(u_i(\rho+\gamma) - u_i(\rho)) u_i(\rho+\gamma) dx \\
&\qquad - \lambda_i(\rho) \int_\Omega \gamma u_i(\rho)(u_i(\rho+\gamma) - u_i(\rho)) dx - \lambda_i(\rho) \int_\Omega \gamma u_i^2(\rho) dx\Big] \int_\Omega \rho u_i^2(\rho) dx
\end{aligned} \tag{64}
$$

and the denominator of (63) could be evaluated by

$$
\begin{aligned}
&\int_\Omega (\rho+\gamma) u_i^2(\rho+\gamma) dx \int_\Omega \rho u_i^2(\rho) dx \\
&\quad = \Big[\int_\Omega \gamma u_i^2(\rho+\gamma) dx + \int_\Omega \rho(u_i(\rho+\gamma) - u_i(\rho))(u_i(\rho+\gamma) + u_i(\rho)) dx \\
&\qquad + \int_\Omega \rho u_i^2(\rho) dx\Big] \int_\Omega \rho u_i^2(\rho) dx.
\end{aligned} \tag{65}
$$

Combining (64) and (65), (63) could be simplified by

$$
\begin{aligned}
&\lambda_i(\rho+\gamma) - \lambda_i(\rho) \\
&\quad = \frac{-\lambda_i(\rho) \int_\Omega \gamma u_i(\rho)(u_i(\rho+\gamma) - u_i(\rho)) dx - \lambda_i(\rho) \int_\Omega \gamma u_i^2(\rho) dx}{\int_\Omega \rho u_i^2(\rho) dx + \int_\Omega \rho(u_i(\rho+\gamma) - u_i(\rho)) u_i(\rho) dx + \int_\Omega \gamma u_i(\rho) u_i(\rho+\gamma) dx}.
\end{aligned} \tag{66}
$$

Thus,

$$
\begin{aligned}
&\lambda_i(\rho+\gamma)-\lambda_i(\rho)+\frac{\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx}{\int_\Omega \rho u_i^2(\rho)dx}\\
&=\Big[-\lambda_i(\rho)\int_\Omega \gamma u_i(\rho)(u_i(\rho+\gamma)-u_i(\rho))dx)\int_\Omega \rho u_i^2(\rho)dx\\
&\quad+\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx\big(\int_\Omega \rho(u_i(\rho+\gamma)-u_i(\rho))u_i(\rho)dx+\int_\Omega \gamma u_i(\rho)u_i(\rho+\gamma)dx\big)\Big]\\
&\quad\Big/\Big[\big(\int_\Omega \rho u_i^2(\rho)dx+\int_\Omega \rho(u_i(\rho+\gamma)-u_i(\rho))u_i(\rho)dx\\
&\quad+\int_\Omega \gamma u_i(\rho)u_i(\rho+\gamma)dx\big)\int_\Omega \rho u_i^2(\rho)dx\Big]. \qquad (67)
\end{aligned}
$$

By (9) and (62), we have $\|u_i(\rho)\|_{H^1(\Omega)} < C$ and $\|u_i(\rho+\gamma)\|_{H^1(\Omega)} < C$ when $\|\gamma\|_{L^\infty(\Omega)}$ is small enough. Therefore, the denominator of (67) could be bounded. Combined with Lemma 3.1, (67) is evaluated by

$$
\begin{aligned}
&|\lambda_i(\rho+\gamma)-\lambda_i(\rho)+\frac{\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx}{\int_\Omega \rho u_i^2(\rho)dx}|\\
&\le C|\lambda_i(\rho)\int_\Omega \gamma u_i(\rho)(u_i(\rho+\gamma)-u_i(\rho))dx)\int_\Omega \rho u_i^2(\rho)dx|\\
&\quad+|\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx\big(\int_\Omega \rho(u_i(\rho+\gamma)-u_i(\rho))u_i(\rho)dx+\int_\Omega \gamma u_i(\rho)u_i(\rho+\gamma)dx\big)|\\
&\le C\|\gamma\|_{L^\infty(\Omega)}\big(\|u_i(\rho+\gamma)-u_i(\rho)\|_{H^1(\Omega)}+\|\gamma\|_{L^\infty(\Omega)}\big).
\end{aligned}
$$

Thus, by (62), we have

$$
\frac{|\lambda_i(\rho+\gamma)-\lambda_i(\rho)+\frac{\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx}{\int_\Omega \rho u_i^2(\rho)dx}|}{\|\gamma\|_{L^\infty(\Omega)}} \to 0 \text{ as } \gamma\to 0 \text{ in } L^\infty(\Omega).
$$

The Fre$\acute{}$chet derivative $\lambda_i'(\rho)$ at $\rho\in\mathcal{A},\ i=1,2,\dots,N$, is given by

$$
\lambda_i'(\rho)[\gamma]=-\frac{\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx}{\int_\Omega \rho u_i^2(\rho)dx}. \qquad (68)
$$

Consequently,

$$
F'(\rho)[\gamma]=-\sum_{i=1}^N(\lambda_i(\rho)-\widehat{\lambda}_i)\frac{\lambda_i(\rho)\int_\Omega \gamma u_i^2(\rho)dx}{\int_\Omega \rho u_i^2(\rho)dx}+\varepsilon\int_\Omega \rho\gamma dx. \qquad (69)
$$

□

4 Finite-Element Approximation and Its Convergence

The finite-element method is applied to discretize the optimization problem (10) and (4). The domain Ω is partitioned into regular elements $\mathcal{T}_h$. The linear finite-element space V_h is defined by

$$
V_h=\{\phi_h\in C^0(\Omega):\ \phi_h|_{T_i}\in P_1(T_i),\ \forall T_i\in\mathcal{T}_h\} \qquad (70)
$$

where $P_1(T_i)$ denotes the space of linear (bilinear) polynomials on the element T_i. The discrete admissible set of the ρ_h is defined by

$$\mathcal{A}_h = \{\rho_h \in V_h : \ \rho_0 \leq \rho_h(x) \leq \rho_1, \forall x \in \Omega\} \tag{71}$$

and $\mathcal{A}_h \subset \mathcal{A}$. Then the optimization problem (10) and (4) is approximated by the discrete minimization optimization

$$\min_{\rho_h \in \mathcal{A}_h} F_h(\rho_h) = \frac{1}{2}\sum_{i=1}^{N}(\lambda_{i,h}(\rho_h) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega \rho_h^2 dx, \tag{72}$$

where the eigenpairs $(\lambda_{i,h}(\rho_h), u_{i,h}(\rho_h)),\ i = 1, 2, \ldots, m = \dim V_h$, satisfy the following weak formulations

$$\int_\Omega \nabla u_{i,h}(\rho_h) \cdot \nabla v_h dx = \lambda_{i,h}(\rho_h) \int_\Omega \rho_h u_{i,h}(\rho_h) v_h dx, \quad \forall v_h \in V_h,\ i = 1, 2, \ldots, m. \tag{73}$$

The i-th eigenpair $(\lambda_{i,h}(\rho_h), u_{i,h}(\rho_h))$ is obtained by the min-max principle [see 18]

$$\begin{aligned} \lambda_{i,h}(\rho_h) &= \min_{V_i \subset V_h,\ \dim(V_i)=i}\ \max_{u_h \in V_i} \frac{\int_\Omega |\nabla u_h|^2 dx}{\int_\Omega \rho_h u_h^2 dx} \\ &= \max_{u_h \in span\{u_{1,h}(\rho_h), u_{2,h}(\rho_h), \ldots, u_{i,h}(\rho_h)\}} \frac{\int_\Omega |\nabla u_h|^2 dx}{\int_\Omega \rho_h u_h^2 dx} \\ &= \frac{\int_\Omega |\nabla u_{i,h}(\rho_h)|^2 dx}{\int_\Omega \rho_h u_{i,h}^2(\rho_h) dx}, \quad \forall u_{i,h} \in M_h(\lambda_{i,h}(\rho_h)), \end{aligned} \tag{74}$$

where

$$M_h(\lambda_{i,h}(\rho_h)) = \{u :\ u \text{ is an eigenfunction of(73) corresponding to } \lambda_{i,h}(\rho_h)\}. \tag{75}$$

By rearranging, (73) admits a sequence of real eigenvalues

$$0 < \lambda_{1,h}(\rho_h) \leq \lambda_{2,h}(\rho_h) \leq \cdots \leq \cdots \leq \lambda_{m,h}(\rho_h), \tag{76}$$

and the corresponding eigenfunctions

$$u_{1,h}(\rho_h),\ u_{2,h}(\rho_h) \ldots,\ u_{m,h}(\rho_h) \tag{77}$$

which can be chosen to satisfy

$$\begin{aligned} \int_\Omega \nabla u_{i,h}(\rho_h) \cdot \nabla u_{j,h}(\rho_h) dx &= \lambda_{i,h}(\rho_h) \int_\Omega \rho_h u_{i,h}(\rho_h) u_{j,h}(\rho_h) dx \\ &= \delta_{ij},\ i, j = 1,\ 2,\ \ldots,\ m. \end{aligned} \tag{78}$$

Analogous to Lemma 3.1, the boundedness of $\lambda_{i,h}(\rho_h)$ could be obtained as follows:

Lemma 4.1 *For any $\rho_h \in \mathcal{A}_h$, we have*

$$\lambda_{i,h}(\rho_1) \leq \lambda_{i,h}(\rho_h) \leq \lambda_{i,h}(\rho_0),\ i = 1, 2, \ldots, m. \tag{79}$$

Lemma 4.2 *For $i = 1, 2, \ldots, m$, let $\rho_h^n,\ \rho_h^* \in \mathcal{A}_h$ and $\rho_h^n \to \rho_h^*$ in any norm as $n \to \infty$, then $\lambda_{i,h}(\rho_h^n) \to \lambda_{i,h}(\rho_h^*)$. Moreover, there exists a subsequence $u_{i,h}(\rho_h^n) \in M_h(\lambda_{i,h}(\rho_h^n))$, and some $u_{i,h}(\rho_h^*) \in M_h(\lambda_{i,h}(\rho_h^*))$, satisfying $u_{i,h}(\rho_h^n) \to u_{i,h}(\rho_h^*)$ in $H^1(\Omega)$, as $n \to \infty$.*

Proof The proof is analogue to Lemma 3.3, replacing ρ^n, ρ^*, $\lambda_i(\rho^n)$, $u_i(\rho^n)$ by ρ_h^n, ρ_h^*, $\lambda_{i,h}(\rho_h^n)$, $u_{i,h}(\rho_h^n)$, respectively. □

Theorem 4.1 *There exists at least one minimizer to the optimization problem* (72) *and* (73).

Proof By Lemma 4.1, we conclude that the minimization of $F_h(\rho_h)$ is finite over the admissible set $\mathcal{A}_h$. Therefore, there exists a sequence $\rho_h^n \in \mathcal{A}_h$ such that

$$\lim_{n\to\infty} F_h(\rho_h^n) = \inf_{\rho_h \in \mathcal{A}_h} F_h(\rho_h). \tag{80}$$

The uniform boundedness of the sequence ρ_h^n in $L^\infty(\Omega)$ implies that there exists a subsequence, also denoted by ρ_h^n, and some ρ_h^* such that

$$\rho_h^n \to \rho_h^*, \quad \text{in any norm as n} \to \infty. \tag{81}$$

By Lemma 4.2, and the lower semicontinuity of a norm, we obtain

$$\begin{aligned}
F_h(\rho_h^*) &= \frac{1}{2}\sum_{i=1}^N (\lambda_{i,h}(\rho_h^*) - \widehat{\lambda}_{i,h})^2 + \frac{\varepsilon}{2}\int_\Omega (\rho_h^*)^2 dx \\
&\le \lim_{n\to\infty} \frac{1}{2}\sum_{i=1}^N (\lambda_{i,h}(\rho_h^n) - \widehat{\lambda}_{i,h})^2 + \liminf_{n\to\infty} \frac{\varepsilon}{2}\int_\Omega (\rho_h^n)^2 dx \\
&\le \liminf_{n\to\infty} \left(\frac{1}{2}\sum_{i=1}^N (\lambda_{i,h}(\rho_h^n) - \widehat{\lambda}_{i,h})^2 + \frac{\varepsilon}{2}\int_\Omega (\rho_h^n)^2 dx\right) \\
&= \liminf_{n\to\infty} F_h(\rho_h^n) \le \inf_{\rho_h \in \mathcal{A}_h} F_h(\rho_h).
\end{aligned}$$

Therefore, ρ_h^* is a minimizer of $F_h(\rho_h)$. □

Lemma 4.3 *Assume that* $(\lambda_{i,h}(\rho_h), u_{i,h}(\rho_h))$ $(i = 1, 2, \ldots, M)$ *are the eigenpairs of* (73), *and* $(\tilde{\lambda}_i, \tilde{u}_i)$ *are the eigenpairs of* (4) *replacing* ρ *by* $\tilde{\rho}$, *then*

$$\begin{aligned}
&\int_\Omega |\nabla(u_{i,h}(\rho_h) - \tilde{u}_i)|^2\, dx - \tilde{\lambda}_i \int_\Omega \tilde{\rho}(u_{i,h}(\rho_h) - \tilde{u}_i)^2 dx \\
&= (\lambda_{i,h}(\rho_h) - \tilde{\lambda}_i)\int_\Omega \rho_h u_{i,h}^2(\rho_h) dx + \tilde{\lambda}_i \int_\Omega (\rho_h - \tilde{\rho}) u_{i,h}^2(\rho_h) dx.
\end{aligned} \tag{82}$$

Proof The proof is analogue to Lemma 3.2, replacing ρ, $\lambda_i(\rho)$, $u_i(\rho)$ by ρ_h, $\lambda_{i,h}(\rho_h)$, $u_{i,h}(\rho_h)$, respectively. □

We introduce the standard interpolation operator $\mathcal{I}_h : W^{1,\infty}(\Omega) \to V_h$ and the projection operator $\mathcal{R}_h : H^1(\Omega) \to V_h$ defined by

$$\int_\Omega \nabla \mathcal{R}_h w \cdot \nabla \phi_h dx = \int_\Omega \nabla w \cdot \nabla \phi_h dx, \ \forall\, w \in H^1(\Omega),\ \phi_h \in V_h.$$

It is well known [see 20, 21] that, for any $p > d = dim(\Omega)$, we have

$$\lim_{h\to 0} \|\mathcal{I}_h w - w\|_{W^{1,p}(\Omega)} = 0, \ \forall\, w \in W^{1,p}(\Omega) \tag{83}$$

and

$$\lim_{h\to 0} \|\mathcal{R}_h w - w\|_{H^1(\Omega)} = 0, \ \forall\, w \in H^1(\Omega). \tag{84}$$

Lemma 4.4 *Let $\rho_h \in \mathcal{A}_h$, $\rho^* \in \mathcal{A}$ in $L^\infty(\Omega)$, if $\rho_h \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$ as $h \to 0$, then $\lambda_{i,h}(\rho_h) \to \lambda_i(\rho^*)$. Moreover, there exists a subsequence $u_{i,h}(\rho_h) \in M_h(\lambda_{i,h}(\rho_h))$, and some $u_i(\rho^*) \in M(\lambda_i(\rho^*))$, satisfying $u_{i,h}(\rho_h) \to u_i(\rho^*)$ in $H^1(\Omega)$, as $h \to 0$.*

Proof By Lemma 4.1, for $i = 1, 2, \dots, N$, we have

$$\lambda_{i,h}(\rho_1) \le \lambda_{i,h}(\rho_h) \le \lambda_{i,h}(\rho_0).$$

The finite-element solutions $\lambda_{i,h}(\rho_0)$ and $\lambda_{i,h}(\rho_1)$ have the following convergence [see 18]

$$\lambda_{i,h}(\rho_0) \to \lambda_i(\rho_0),$$
$$\lambda_{i,h}(\rho_1) \to \lambda_i(\rho_1).$$

Therefore,

$$|\lambda_{i,h}(\rho_h)| < C, \ i = 1, 2, \dots, N.$$

It implies that there exists a subsequence, also denoted by $\lambda_{i,h}(\rho_h)$, such that

$$\lim_{h\to 0} \lambda_{i,h}(\rho_h) = \lambda_i^*, \ i = 1, 2, \dots, N. \tag{85}$$

By (78), the eigenfunction $u_{i,h}(\rho_h) \in M_h(\lambda_{i,h}(\rho_h))$ satisfies

$$\int_\Omega |\nabla u_{i,h}(\rho_h)|^2 dx = 1, \tag{86}$$

and by Poincar$\acute{e}$ inequality, we have

$$\|u_{i,h}(\rho_h)\|_{H^1(\Omega)} < C, \tag{87}$$

where C is a constant independent of h. Thus, there exists a subsequence, also denoted by $u_{i,h}(\rho_h)$, such that

$$u_{i,h}(\rho_h) \rightharpoonup u_i^* \text{ in } H^1(\Omega) \text{ and } u_{i,h}(\rho_h) \to u_i^* \text{ in } L^2(\Omega), \ i = 1, 2, \dots, N, \tag{88}$$

as $h \to 0$.

We need to prove that (λ_i^*, u_i^*) is the eigenpair of (4) corresponding to ρ^* for $i = 1, 2, \dots, N$. Notice that the eigenpair $(\lambda_{i,h}(\rho_h), \ u_{i,h}(\rho_h))$ corresponding to ρ_h satisfies the weak formula (73). Substituting v_h in (73) by $\mathcal{R}_h v$, $\forall v \in H_0^1(\Omega)$, we obtain

$$\int_\Omega \nabla u_{i,h}(\rho_h) \cdot \nabla \mathcal{R}_h v dx = \lambda_{i,h}(\rho_h) \int_\Omega \rho_h u_{i,h}(\rho_h) \mathcal{R}_h v dx, \ i = 1, 2, \dots, m. \tag{89}$$

The RHS item of equation of (89) could be rewritten by

$$\begin{aligned}\lambda_{i,h}(\rho_h) \int_\Omega \rho_h u_{i,h}(\rho_h) \mathcal{R}_h v dx &= (\lambda_{i,h}(\rho_h) - \lambda_i^*) \int_\Omega \rho_h u_{i,h}(\rho_h) \mathcal{R}_h v dx \\ &+ \lambda_i^* \int_\Omega \rho_h (u_{i,h}(\rho_h) - u_i^*) \mathcal{R}_h v dx + \lambda_i^* \int_\Omega (\rho_h - \rho^*) u_i^* \mathcal{R}_h v dx \\ &+ \lambda_i^* \int_\Omega \rho^* u_i^* (\mathcal{R}_h v - v) dx + \lambda_i^* \int_\Omega \rho^* u_i^* v dx.\end{aligned} \tag{90}$$

By (11), (71), (87), (85), (88), (84) and $\rho_h \overset{*}{\rightharpoonup} \rho^*$ in $L^\infty(\Omega)$ as $h \to 0$, we have

$$\lim_{h\to 0} \lambda_{i,h}(\rho_h) \int_\Omega \rho_h u_{i,h}(\rho_h) \mathcal{R}_h v dx = \lambda_i^* \int_\Omega \rho^* u_i^* v dx. \tag{91}$$

By (88) and (84), we have

$$\lim_{h\to 0}\int_\Omega \nabla u_{i,h}(\rho_h)\cdot\nabla\mathcal{R}_h v dx = \int_\Omega \nabla u_i^*\cdot\nabla v dx \tag{92}$$

Taking the limitation of equation (89) as $h\to 0$, combined with (91) and (92), we have

$$\int_\Omega \nabla u_i^*\cdot\nabla v dx = \lambda_i^*\int_\Omega \rho^* u_i^* v dx. \tag{93}$$

Therefore, it could conclude that (λ_i^*, u_i^*) $(i=1,2,\ldots,N)$ are eigenpairs of (4) corresponding to ρ^*. Replacing $\tilde{\rho},\ \tilde{u}_i,\ \tilde{\lambda}_i$ by $\rho^*,\ u_i^*,\ \lambda_i^*$ in Lemma 4.3, we have

$$\begin{aligned}
&\int_\Omega |\nabla(u_{i,h}(\rho_h)-u_i^*)|^2 dx\\
&=\lambda_i^*\int_\Omega \rho^*(u_{i,h}(\rho_h)-u_i^*)^2dx + (\lambda_{i,h}(\rho_h)-\lambda_i^*)\int_\Omega \rho_h u_{i,h}^2(\rho_h)dx\\
&\quad+\lambda_i^*\int_\Omega(\rho_h-\rho^*)u_{i,h}^2(\rho_h)dx\\
&\to 0,\ \text{as } h\to 0
\end{aligned} \tag{94}$$

with (11), (71), (88), (87), (85) and $\rho_h \overset{*}{\rightharpoonup} \rho^*\ in\ L^\infty(\Omega)$. Combining (88) and (94), we have

$$u_{i,h}(\rho_h)\to u_i^* \text{ in } H^1(\Omega),\ i=1,2,\ldots,N, \tag{95}$$

as $h\to 0$.

Taking the limitation of (78) as $h\to 0$, by (71), (87), (95), (85) and $\rho_h \overset{*}{\rightharpoonup} \rho^*\ in\ L^\infty(\Omega)$ as $h\to 0$, the orthogonality of u_i^* is obtained by

$$\int_\Omega \nabla u_i^*\cdot\nabla u_j^* dx = \lambda_i^*\int_\Omega \rho^* u_i^* u_j^* dx = \delta_{ij},\ i,j=1,\ 2,\ \ldots,\ m. \tag{96}$$

Finally, we prove (λ_i^*, u_i^*) $(i=1,2,\ldots,N)$ is the i-th eigenpair of (4) corresponding to ρ^* by induction. For $i=1$, by (7) (replacing ρ by ρ^*), we have

$$\begin{aligned}
\lambda_1(\rho^*) &= \frac{\int_\Omega |\nabla u_1(\rho^*)|^2\,dx}{\int_\Omega \rho^* u_1^2(\rho^*)\,dx}\\
&\le \frac{\int_\Omega |\nabla u_1^*|^2\,dx}{\int_\Omega \rho^* u_1^{*2}\,dx} = \lambda_1^*.
\end{aligned} \tag{97}$$

On the other hand, by (85), (74), (71), (84) and $\rho_h \overset{*}{\rightharpoonup} \rho^*\ in\ L^\infty(\Omega)$ as $h\to 0$, we have

$$\begin{aligned}
\lambda_1^* &= \lim_{h\to 0}\lambda_{1,h}(\rho_h)\\
&= \lim_{h\to 0}\frac{\int_\Omega |\nabla u_{1,h}(\rho_h)|^2\,dx}{\int_\Omega \rho_h u_{1,h}^2(\rho_h)\,dx}\\
&\le \lim_{h\to 0}\frac{\int_\Omega |\nabla \mathcal{R}_h u_1(\rho^*)|^2\,dx}{\int_\Omega \rho_h(\mathcal{R}_h u_1(\rho^*))^2\,dx}\\
&= \frac{\int_\Omega |\nabla u_1(\rho^*)|^2\,dx}{\int_\Omega \rho^* u_1^2(\rho^*)\,dx} = \lambda_1(\rho^*).
\end{aligned} \tag{98}$$

Combining (97) and (98), we obtain $\lambda_1^* = \lambda_1(\rho^*)$ and $u_1^* \in M(\lambda_1(\rho^*))$. We set $u_1(\rho^*) \in M(\lambda_1(\rho^*))$ to satisfy $u_1^* = u_1(\rho^*)$.

For $i = 1, 2, \ldots, k$, assuming that

$$\lambda_i^* = \lambda_i(\rho^*), \tag{99}$$

$$u_i^* = u_i(\rho^*), \tag{100}$$

we need to prove that $\lambda_{k+1}^* = \lambda_{k+1}(\rho^*)$ and $u_{k+1}^* = u_{k+1}(\rho^*)$. On one hand, similar to the proof of (41), we have

$$\lambda_{k+1}(\rho^*) \le \lambda_{k+1}^*. \tag{101}$$

On the other hand, we can conclude that

$$\dim span \left\{u_{1,h}(\rho_h), u_{2,h}(\rho_h), ..., u_{k,h}(\rho_h), \mathcal{R}_h u_{k+1}(\rho^*)\right\} = k+1. \tag{102}$$

If the claim is negated to assume that

$$\dim span \left\{u_{1,h}(\rho_h), u_{2,h}(\rho_h), ..., u_{k,h}(\rho_h), \mathcal{R}_h u_{k+1}(\rho^*)\right\} = k,$$

since $u_{1,h}(\rho_h), u_{2,h}(\rho_h), ..., u_{k,h}(\rho_h)$ are orthogonal by (78), then

$$\mathcal{R}_h u_{k+1}(\rho^*) = \gamma_1 u_{1,h}(\rho_h) + \gamma_2 u_{2,h}(\rho_h) + ... + \gamma_k u_{k,h}(\rho_h). \tag{103}$$

By the orthogonality (78), we obtain

$$\begin{aligned}\int_\Omega \nabla \mathcal{R}_h u_{k+1}(\rho^*) \nabla u_{i,h}(\rho_h) dx &= \sum_{j=1}^{k} \gamma_j \int_\Omega \nabla u_{j,h}(\rho_h) \nabla u_{i,h}(\rho_h) dx \\ &= \gamma_i, \ i = 1, 2, \ldots, k.\end{aligned} \tag{104}$$

Taking the limitation of (104) as $h \to 0$, with (84), (95), (100) and (9) (replacing ρ by ρ^*), we obtain

$$0 = \int_\Omega \nabla u_{k+1}(\rho^*) \nabla u_i(\rho^*) dx = \gamma_i, \ i = 1, 2, \ldots, k. \tag{105}$$

Therefore, $\mathcal{R}_h u_{k+1}(\rho^*) \equiv 0$. By (84), it implies that $u_{k+1}(\rho^*) \equiv 0$, which contradicts the definition of a nontrivial eigenfunction.

By (74) and (102), we have

$$\begin{aligned}\lambda_{k+1,h}(\rho_h) &= \frac{\int_\Omega |\nabla u_{k+1,h}(\rho_h)|^2 \, dx}{\int_\Omega \rho_h u_{k+1,h}^2(\rho_h) \, dx} \\ &= \min_{V_{k+1} \subset V_h, \ \dim(V_{k+1}) = k+1} \max_{u_h \in V_{k+1}} \frac{\int_\Omega |\nabla u_h|^2 dx}{\int_\Omega \rho_h u_h^2 dx} \\ &\le \max_{u_h \in span\{u_{1,h}(\rho_h), u_{2,h}(\rho_h), \ldots, u_{k,h}(\rho_h), \mathcal{R}_h u_{k+1}(\rho^*)\}} \frac{\int_\Omega |\nabla u_h|^2 dx}{\int_\Omega \rho_h u_h^2 dx} \\ &= \frac{\int_\Omega |\nabla \tilde{u}_h|^2 dx}{\int_\Omega \rho_h \tilde{u}_h^2 dx}.\end{aligned} \tag{106}$$

Let $\tilde{u}_h = \beta_1 u_{1,h}(\rho_h) + \beta_2 u_{2,h}(\rho_h) +, ..., \beta_k u_{k,h}(\rho_h) + \beta_{k+1}\mathcal{R}_h u_{k+1}(\rho^*)$. By (78), (95), (100), (84), $\rho_h \overset{*}{\rightharpoonup} \rho^*\ in\ L^\infty(\Omega)$ as $h \to 0$ and (9) (replacing ρ by ρ^*), we have

$$\int_\Omega |\nabla \tilde{u}_h|^2\,dx = \sum_{j=1}^k \beta_j^2 \int_\Omega |\nabla u_{j,h}(\rho_h)|^2\,dx + \beta_{k+1}^2 \int_\Omega |\nabla \mathcal{R}_h u_{k+1}(\rho^*)|^2\,dx$$
$$+ 2\sum_{j=1}^k \beta_j\beta_{k+1} \int_\Omega \nabla u_{j,h}(\rho_h)\nabla \mathcal{R}_h u_{k+1}(\rho^*)\,dx$$
$$\to \sum_{j=1}^k \beta_j^2 \int_\Omega |\nabla u_j(\rho^*)|^2\,dx + \beta_{k+1}^2 \int_\Omega |\nabla u_{k+1}(\rho^*)|^2\,dx \quad \text{as } h \to 0 \quad (107)$$

and

$$\int_\Omega \rho_h \tilde{u}_h^2\,dx = \sum_{j=1}^k \beta_j^2 \int_\Omega \rho_h u_{j,h}^2(\rho_h)dx + \beta_{k+1}^2 \int_\Omega \rho_h(\mathcal{R}_h u_{k+1}(\rho^*))^2 dx$$
$$+ 2\sum_{j=1}^k \beta_j\beta_{k+1} \int_\Omega \rho_h u_{j,h}(\rho_h)\mathcal{R}_h u_{k+1}(\rho^*)dx$$
$$\to \sum_{j=1}^k \beta_j^2 \int_\Omega \rho^* u_j^2(\rho^*)dx + \beta_{k+1}^2 \int_\Omega \rho^* u_{k+1}^2(\rho^*)dx \quad \text{as } h \to 0. \quad (108)$$

By (5) and (9) (replacing ρ by ρ^*), (107) could be further evaluated by

$$\sum_{j=1}^k \beta_j^2 \int_\Omega |\nabla u_j(\rho^*)|^2\,dx + \beta_{k+1}^2 \int_\Omega |\nabla u_{k+1}(\rho^*)|^2\,dx$$
$$= \sum_{j=1}^k \beta_j^2 \lambda_j(\rho^*) \int_\Omega \rho^* u_j^2(\rho^*)\,dx + \beta_{k+1}^2 \lambda_{k+1}(\rho^*) \int_\Omega \rho^* u_{k+1}^2(\rho^*)\,dx$$
$$\le \lambda_{k+1}(\rho^*)\Big[\sum_{j=1}^k \beta_j^2 \int_\Omega \rho^* u_j^2(\rho^*)\,dx + \beta_{k+1}^2 \int_\Omega \rho^* u_{k+1}^2(\rho^*)\,dx\Big]. \quad (109)$$

Combining (85), (106), (107), (108) and (109), it implies that

$$\lambda_{k+1}^* = \lim_{h\to 0} \lambda_{k+1,h}(\rho_h) \le \lambda_{k+1}(\rho^*). \quad (110)$$

With (101) and (110), we obtain $\lambda_{k+1}^* = \lambda_{k+1}(\rho^*)$. The eigenfunction u_{k+1}^* corresponding to λ_{k+1}^* is also the eigenfunction corresponding to $\lambda_{k+1}(\rho^*)$. That is, $u_{k+1}^* \in M(\lambda_{k+1}(\rho^*))$. We set $u_{k+1}(\rho^*) \in M(\lambda_{k+1}(\rho^*))$ to satisfy $u_{k+1}^* = u_{k+1}(\rho^*)$. By induction, we conclude that $\lambda_i^* = \lambda_i(\rho^*),\ u_i^* = u_i(\rho^*),\ i = 1, 2, \ldots, N$. Combined with (85) and (95), the proof is complete. □

Lemma 4.5 (See [21, LEMMA 4.3]) *$C^\infty(\Omega)$ is weak-* dense in $L^\infty(\Omega)$.*

Theorem 4.2 *Let $\{\rho_h^*\}_{h>0}$ be a sequence of minimizers to the discrete minimization problem* (72) *and* (73)*. Then each subsequence of $\{\rho_h^*\}_{h>0}$ has a subsequence converging to a minimizer of the continuous optimization problem* (10) *and* (4)*.*

Proof The uniform boundedness of the sequence ρ_h^* in $L^\infty(\Omega)$ implies that there exists a subsequence, also denoted by ρ_h^*, and some ρ^* such that

$$\rho_h^* \overset{*}{\rightharpoonup} \rho^* \ \text{ in } L^\infty(\Omega) \text{ as } h \to 0. \tag{111}$$

Lemma 4.4 implies that

$$\lambda_{i,h}(\rho_h^*) \to \lambda_i(\rho^*) \text{ as } h \to 0. \tag{112}$$

By Lemma 4.5, for any $\rho \in \mathcal{A}$ and fixed $\delta > 0$, there exists a $\rho^\delta \in C^\infty(\Omega)$ such that

$$\rho^\delta \overset{*}{\rightharpoonup} \rho \ \text{ in } L^\infty(\Omega) \text{ as } \delta \to 0. \tag{113}$$

By its construction, $\rho^\delta \in \mathcal{A}$. Let $\rho_h^\delta = \mathcal{I}_h \rho^\delta \in \mathcal{A}_h$. By (83) and Lemma 4.4, we have

$$\lambda_{i,h}(\rho_h^\delta) \to \lambda_i(\rho^\delta) \ as \ h \to 0. \tag{114}$$

Noting that ρ_h^* is the minimizer of $F_h(\cdot)$ over $\mathcal{A}_h$, we have

$$\begin{aligned} F_h(\rho_h^*) &= \frac{1}{2}\sum_{i=1}^{N}(\lambda_{i,h}(\rho_h^*) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho_h^*)^2 dx \\ &\le F_h(\rho_h^\delta) = F_h(\mathcal{I}_h \rho^\delta). \end{aligned} \tag{115}$$

Thus, by (112), (115), (114) and the lower semicontinuity of a norm, we have

$$\begin{aligned} F(\rho^*) &= \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^*) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^*)^2 dx \\ &\le \lim_{h\to 0} \frac{1}{2}\sum_{i=1}^{N}(\lambda_{i,h}(\rho_h^*) - \widehat{\lambda}_i)^2 + \liminf_{h\to 0} \frac{\varepsilon}{2}\int_\Omega (\rho_h^*)^2 dx \\ &\le \liminf_{h\to 0} F_h(\rho_h^*) \le \liminf_{h\to 0} F_h(\mathcal{I}_h \rho^\delta) \\ &= \liminf_{h\to 0} \Big[\frac{1}{2}\sum_{i=1}^{N}(\lambda_{i,h}(\rho_h^\delta) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho_h^\delta)^2 dx\Big] \\ &\le \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho^\delta) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega (\rho^\delta)^2 dx \end{aligned}$$

Letting δ tend to zero, we obtain that

$$F(\rho^*) \le \frac{1}{2}\sum_{i=1}^{N}(\lambda_i(\rho) - \widehat{\lambda}_i)^2 + \frac{\varepsilon}{2}\int_\Omega \rho^2 dx.$$

Therefore, ρ^* is a minimizer of $F(\rho)$. □

5 Algorithm

For $\rho \in \mathcal{A}$ and $i = 1, 2, \ldots, N$, if eigenvalues $\lambda_i(\rho)$ of (4) are simple, we solve the optimization problem (10) and (4) by the Fletcher–Reeves conjugate gradient algorithm

[22]. That is, with the Fr$\acute{e}$chet derivative (59), the decent direction is set by

$$d_k = -F'(\rho^{(k)}) + \beta_{k-1} d_{k-1}. \tag{116}$$

With the Fletcher–Reeves scheme [22], the conjugate coefficient β_{k-1} is given by

$$\beta_{k-1} = \frac{\|F'(\rho^{(k)})\|^2_{L^2(\Omega)}}{\|F'(\rho^{(k-1)})\|^2_{L^2(\Omega)}} \tag{117}$$

and $\beta_{-1} = 0$. The step length α_k in the conjugate direction d_k is determined by

$$\alpha_k = \arg\min_{\alpha} F(\rho^{(k)} + \alpha d_k).$$

In our numerical algorithm, by taking the derivative of $F(\rho^{(k)} + \alpha d_k)$ with respect to α and setting it to be zero, the step length α_k is approximated by

$$\alpha_k = -\frac{F'(\rho^{(k)})[d_k]}{\sum_{i=1}^{N} (\lambda'(\rho^{(k)})[d_k])^2 + \varepsilon \|d_k\|^2_{L^2(\Omega)}}. \tag{118}$$

Now, we summarize the Fletcher–Reeves conjugate gradient algorithm as follows:

Algorithm 5.1 *STEP 1. Initialize the density function $\rho^{(0)}$, and set the regularity coefficient ε and the input data $\widehat{\lambda}_i, i = 1, 2, \ldots, N$.*

STEP 2. Solve the eigenpairs $(\lambda_i(\rho^{(k)}), u_i(\rho^k)), i = 1, 2, \ldots, N$ of (4).

STEP 3. Determine the gradient $F'(\rho^{(k)})$ by (59) and calculate the descent direction d_k by (116) and (117).

STEP 4. Calculate the step length α_k by (118).

STEP 5. Update the density function $\rho^{(k)}$ by

$$\rho^{(k+1)} = \rho^{(k)} + \alpha_k d_k.$$

STEP 6. Set $k = k + 1$ and go to STEP 2. Repeat the procedure until a stopping criterion is satisfied.

Remark 5.1 In the numerical experiments, the stopping criterion is set that the norm of the decent direction is small enough. That is, if $||d_k||_{L^2(\Omega)} < \delta$, where δ is a given number, the iteration of the Fletcher–Reeves conjugate gradient algorithm is stopped.

6 Numerical Results

In this section, we present numerical results of the optimization problem (10) and (4) in $1D$ and $2D$ cases. The domain is partitioned into uniform meshes. The piecewise linear (bilinear) basis functions are used. The noisy data is generated by using the following formula:

$$\widehat{\lambda}_i(x) = \lambda_i(x)(1 + \sigma\zeta) \text{ in } \Omega, i = 1, 2, \ldots, N$$

where ζ is a uniformly distributed random variable in $[-1, 1]$ and σ dictates the level of noise. The eigenpairs of (4) are solved by *eigs* found in MATLAB. The Algorithm 5.1 is conducted in MATLAB codes.

Table 1 The first 5 least eigenvalues for $1D$ case ($\sigma = 0.001$)

Eigenvalues	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$\widehat{\lambda}_i$	0.91535490	3.93915202	8.46917234	15.34538138	23.96039121
Initial λ_i	1.00000021	4.00000329	9.00001665	16.00005264	25.00012851
Final λ_i	0.91535602	3.93915164	8.46917274	15.34538126	23.96039094

Table 2 The first 10 least eigenvalues for $1D$ case with continuous density ($\sigma = 0.001$)

Eigenvalues	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$\widehat{\lambda}_i$	0.91426066	3.93801755	8.47125715	15.34100987	23.98835641
Initial λ_i	1.00000021	4.00000329	9.00001665	16.00005264	25.00012851
Final λ_i	0.91426135	3.93801751	8.47125730	15.34100986	23.98835641
Eigenvalues	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$
$\widehat{\lambda}_i$	34.24481084	46.87913637	61.16809688	77.22502908	95.68667178
Initial λ_i	36.00026648	49.00049369	64.00084221	81.00134906	100.00205618
Final λ_i	34.24481083	46.87913638	61.16809687	77.22502906	95.68667179

6.1 1*D* Case

Set the domain $\Omega = (0, \pi)$, which is uniformly partitioned into 2000 elements. The exact density function is set by

$$\rho(x) = \begin{cases} 1 + e^{-\frac{128}{9\pi^2 - 256(x-\frac{\pi}{2})^2}}, & \frac{5\pi}{16} < x < \frac{11\pi}{16}, \\ 1, & \text{otherwise.} \end{cases} \tag{119}$$

The first 5 least eigenvalues with noise level $\sigma = 0.001$ are listed in the first row of Table 1 and used as the input data. Set $\varepsilon = 10^{-6}$ and $\delta = 10^{-4}$. With the initialization of density function $\rho^{(0)} = 1$ in the whole domain (see the dash line in Fig. 1a), the evolutions of density function are illustrated in Fig. 1, after $k = 10,\ 20,\ 30,\ 50,\ 156$ iterations. The solid curve represents the exact density function, while the dash curve represents the recovered density function. After 156 iterations, the curve of the recovered density function fits well with the one of the exact density function, but it has some oscillations. The initial and the finial first 5 least eigenvalues are listed in the second row and the third row of Table 1, respectively.

With more input data, the recovery of the density function is improved. Using the first 10 least eigenvalues as the input data (see Table 2), with the same setting parameters, the evolutions of density function are illustrated in Fig. 2, after $k = 10,\ 30,\ 50,\ 100,\ 483$ iterations. The solid curve represents the exact density function, while the dash curve represents the recovered density function. After 483 iterations, the recovered density function fits the exact one much better, compared to the numerical results shown in Fig. 1 using first 5 least eigenvalues as the input data. The initial and the finial first 10 least eigenvalues are listed in Table 2.

Various levels of noise are considered. Set $\sigma = 0.001, 0.005, 0.01, 0.02,$ respectively. Set $\varepsilon = 10^{-6}$ and $\delta = 10^{-4}$. With the first 5 small least eigenvalues as the input data, we initialize the density function $\rho^{(0)} = 1$ in the whole domain. The final recovered density function are illustrated in Fig. 3, with $\sigma = 0.001, 0.005, 0.01, 0.02$, respectively. The solid

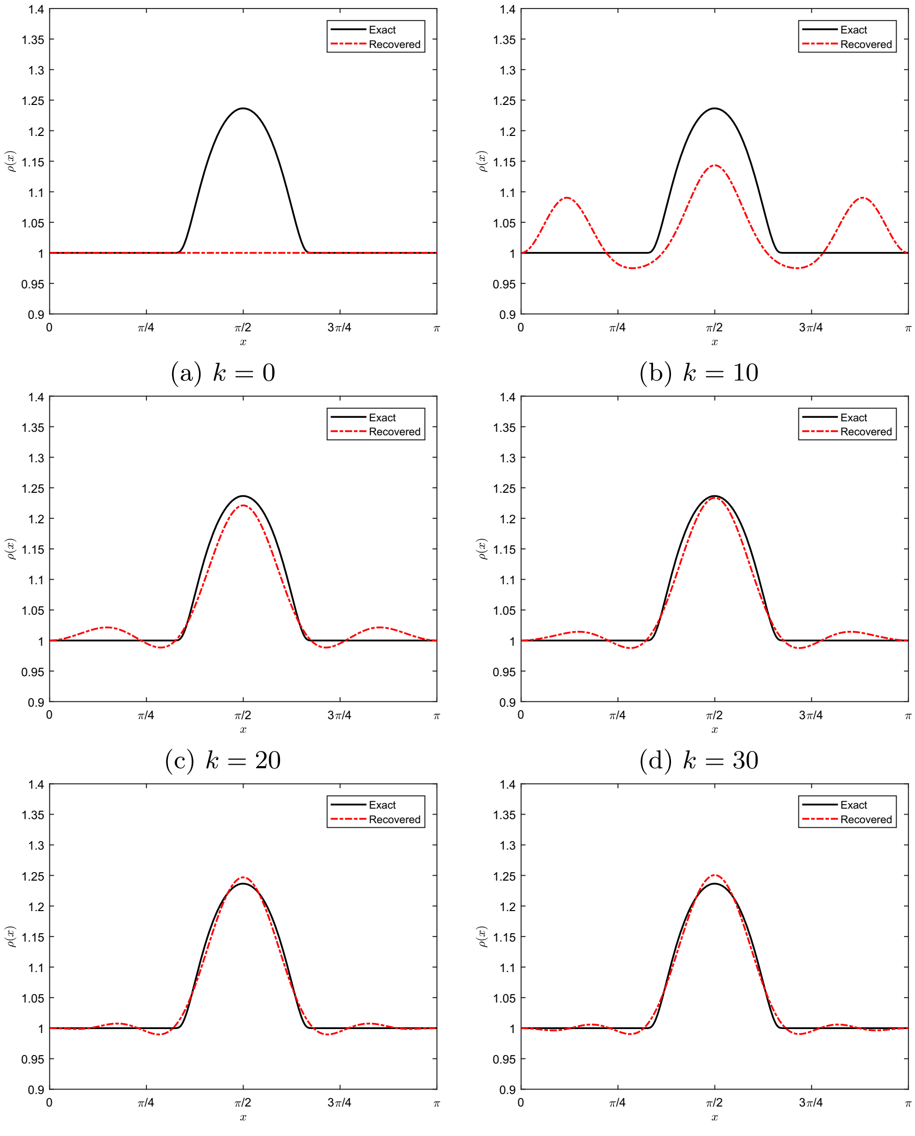


Fig. 1 The evolutions of the density function with the first 5 eigenvalues as the input data. The noise level is $\sigma = 0.001$

curve represents the exact density function, while the dash curve represents the recovered density function. Note that the higher level of noise incurs the inaccuracy of the recovery of density function. It may be caused by the sensitivity of the eigenvalue to the variation of the density function.

The measured spectral data with gaps are considered. As listed in the first row of Table 2, the eigenvalues $\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3, \widehat{\lambda}_6, \widehat{\lambda}_9$ with noise level $\sigma = 0.001$ are chosen as the input data. Setting $\varepsilon = 10^{-6}$, $\delta = 10^{-4}$ and $\rho^{(0)} = 1$ in the whole domain, the evolutions of density

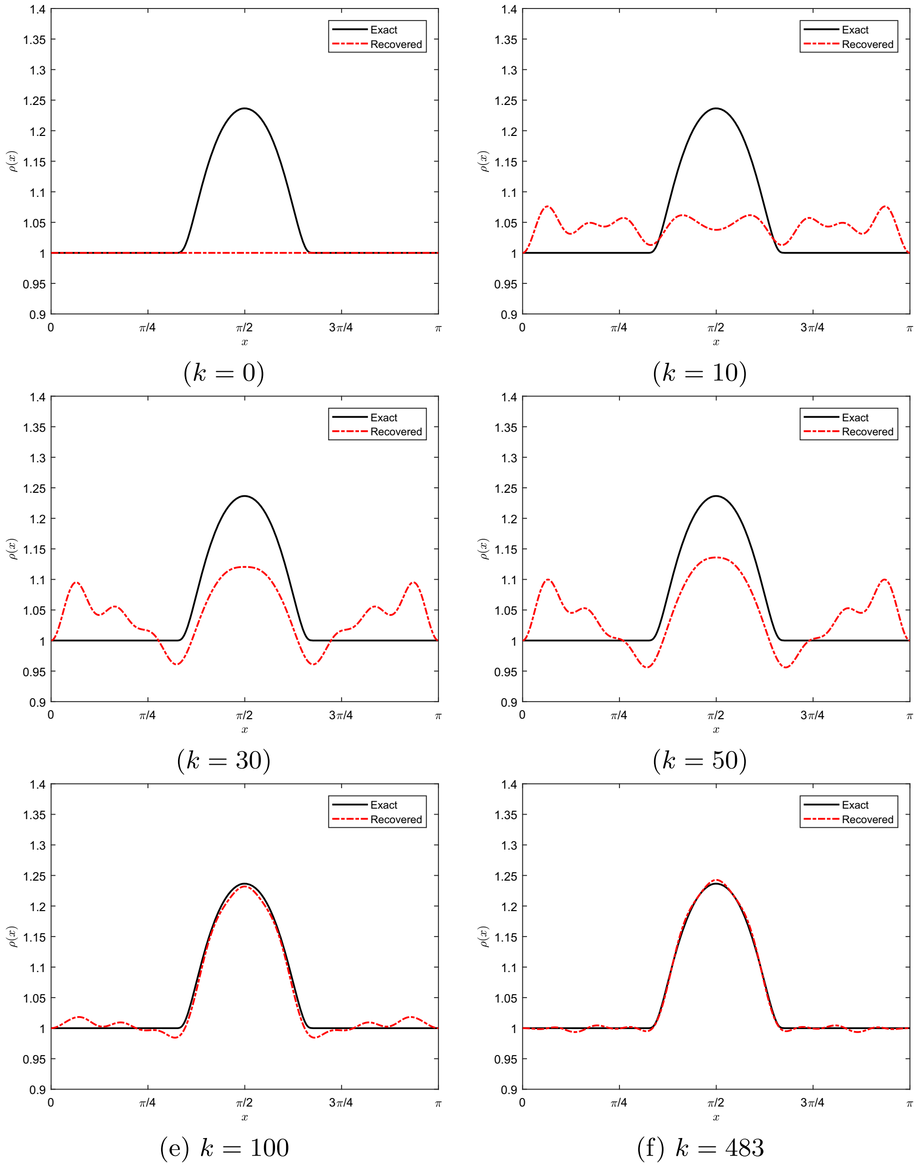
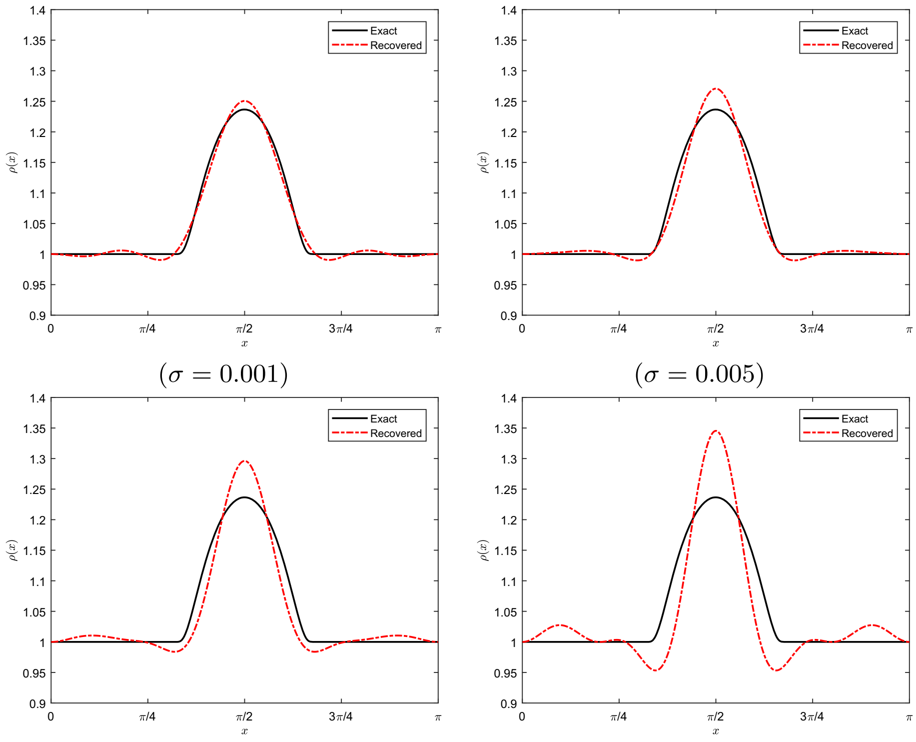


Fig. 2 The evolutions of the density function with the first 10 eigenvalues as the input data. The noise level is $\sigma = 0.001$

function are illustrated in Fig. 4, after $k = 10,\ 20,\ 30,\ 50,\ 94$ iterations. The solid curve represents the exact density function, while the dash curve represents the recovered density function. Compared to the numerical results shown in Fig. 1, it is observed that the first 5 least eigenvalues as the measured spectral data could recover the density function better than the 5 eigenvalues with gaps. After 94 iterations, the finial eigenvalues are $\lambda_1 = 0.91426600$, $\lambda_2 = 3.93801788$, $\lambda_3 = 8.47125761$, $\lambda_6 = 34.24481075$, $\lambda_9 = 77.22502904$, respectively.



$(\sigma = 0.001)$ $(\sigma = 0.005)$ $(\sigma = 0.01)$ $(\sigma = 0.02)$

Fig. 3 The recovered density functions with different noise levels of input data

The discontinuous density case is also considered, which is defined by

$$\rho(x) = \begin{cases} 1 + e^{-\frac{128}{9\pi^2 - 128(x-\frac{\pi}{2})^2}}, & \frac{5\pi}{16} < x < \frac{11\pi}{16}, \\ 1, & \text{otherwise}. \end{cases} \tag{120}$$

The first 10 least eigenvalues with noise level $\sigma = 0.001$ are listed in the first row of Table 3 and used as the input data. Set $\varepsilon = 10^{-6}$ and $\delta = 10^{-4}$. With the initialization of density function $\rho^{(0)} = 1$ in the whole domain (see the dash line in Fig. 5 (a)), the evolutions of density function are illustrated in Fig. 5, after $k = 10,\ 30,\ 50,\ 80,\ 368$ iterations. The solid curve represents the exact density function with discontinuity at $x = \frac{5\pi}{16}$ and $\frac{11\pi}{16}$, while the dash curve represents the recovered density function. After 368 iterations, the curve of the recovered density function fits well with the one of the exact density function at most part of the region $\Omega = (0, \pi)$. It has some deviations in the vicinity of $x = \frac{5\pi}{16}$ and $\frac{11\pi}{16}$. The initial and the finial first 10 least eigenvalues are listed in Table 3.

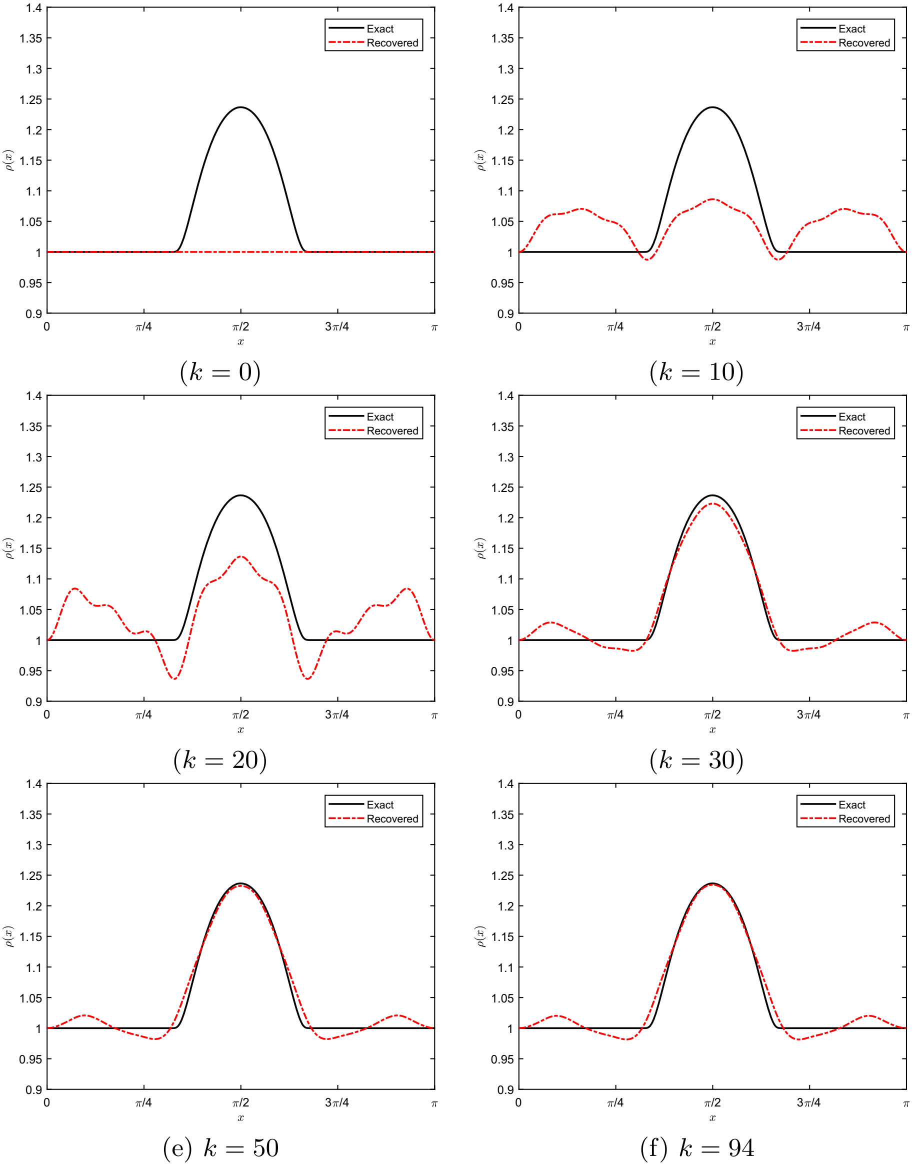


Fig. 4 The evolutions of the density function with 5 eigenvalues with gaps as the input data. The noise level is $\sigma = 0.001$

Table 3 The first 10 least eigenvalues for $1D$ case with discontinuous density ($\sigma = 0.001$)

Eigenvalues	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$\widehat{\lambda}_i$	0.89001245	3.86192032	8.43324097	14.93928067	23.60787650
Initial λ_i	1.00000021	4.00000329	9.00001665	16.00005264	25.00012851
Final λ_i	0.89001292	3.86192088	8.43324117	14.93928077	23.60787651
Eigenvalues	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$
$\widehat{\lambda}_i$	33.87947366	45.88524670	60.12939069	76.22760711	93.92399887
Initial λ_i	36.00026648	49.00049369	64.00084221	81.00134906	100.00205618
Final λ_i	33.87947364	45.88524677	60.12939067	76.22760714	93.92399888

Table 4 The first 10 least eigenvalues for $2D$ case ($\sigma = 0.001$)

Eigenvalues	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$\widehat{\lambda}_i$	1.79963276	4.49335781	4.85849079	7.57561646	8.76898607
Initial λ_i	1.90015627	4.60126675	4.90139014	7.60250063	9.10607981
Final λ_i	1.79963354	4.49335767	4.85848986	7.57561686	8.76898595
Eigenvalues	$i = 6$	$i = 7$	$i = 8$	$i = 9$	$i = 10$
$\widehat{\lambda}_i$	9.59767168	12.04452035	12.34272158	14.92739013	16.52536169
Initial λ_i	9.90673798	12.10731368	12.60784846	15.41904185	16.92114025
Final λ_i	9.59767184	12.04452034	12.34272144	14.92739008	16.52536182

6.2 2*D* Case

Set the domain $\Omega = (0, \pi/a) \times (0, \pi)$ with $a = \sqrt{0.9}$. It is uniformly partitioned into 100×100 rectangular elements. The exact density function is set by

$$\rho(x) = \begin{cases} 1 + e^{-\frac{128}{9\pi^2 - 64(x_1 - \frac{\pi}{2a})^2 - 256(x_2 - \frac{\pi}{2})^2}}, & (x_1 - \frac{\pi}{2a})^2 + 4(x_2 - \frac{\pi}{2})^2 < \frac{9\pi^2}{64}, \\ 1, & \text{otherwise.} \end{cases} \tag{121}$$

The first 10 least eigenvalues with noise level $\sigma = 0.001$ are listed in the Table 4 and used as the input data. Set $\varepsilon = 10^{-8}$ and $\delta = 10^{-4}$. The exact density function is illustrated in Fig. 6a. With the initialization of density function $\rho^{(0)} = 1$ in the whole domain (see Fig. 6b), the evolutions of density function are illustrated in Fig. 6, after $k = 10,\ 30,\ 50,\ 165$ iterations. After 156 iterations, the recovered density function fits the exact one well. The initial and the finial first 10 least eigenvalues are listed in the Table 4.

Various levels of noise are considered. Set $\sigma = 0.001, 0.005, 0.01, 0.02$, respectively. Set $\varepsilon = 10^{-8}$ and $\delta = 10^{-4}$. With the first 10 small least eigenvalues as the input data and with the initialization of the density function $\rho^{(0)} = 1$ in the whole domain, the final recovered density functions are illustrated in Fig. 7.

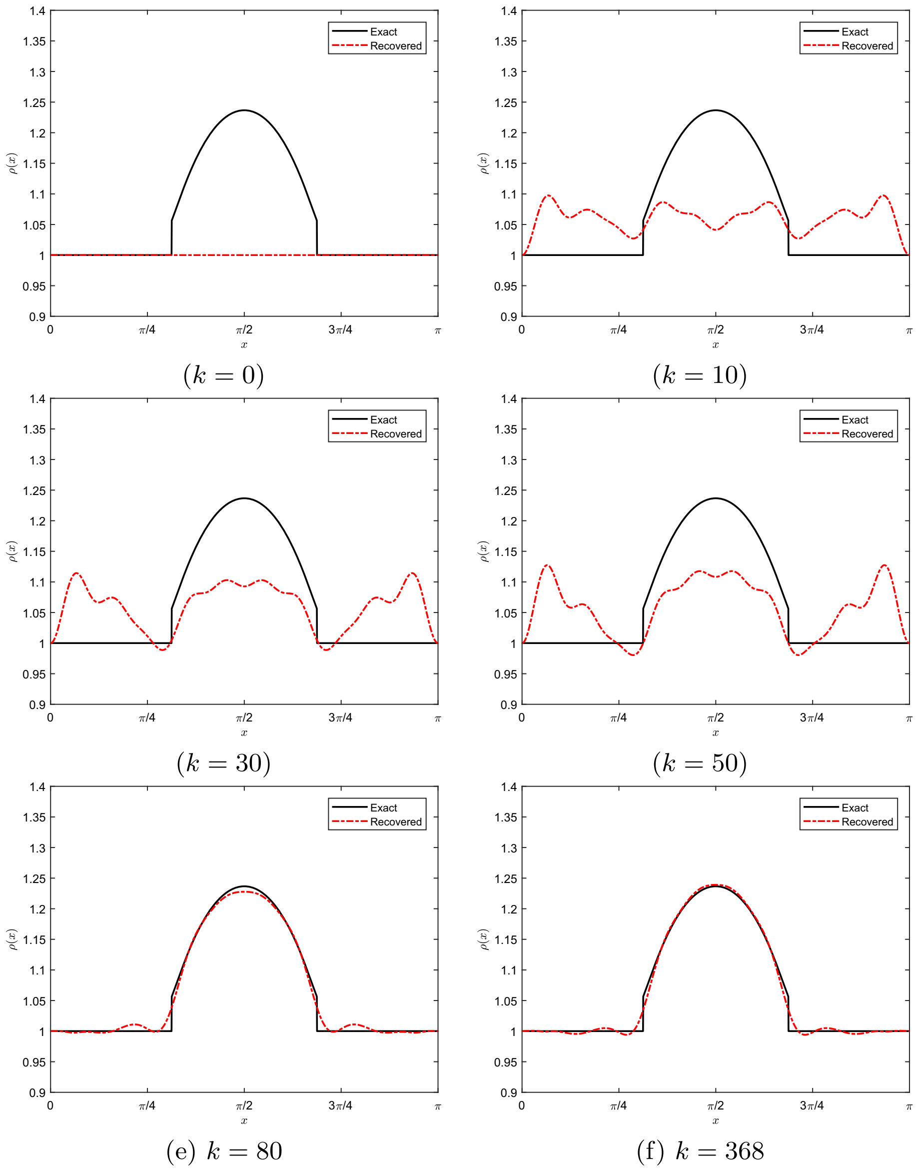
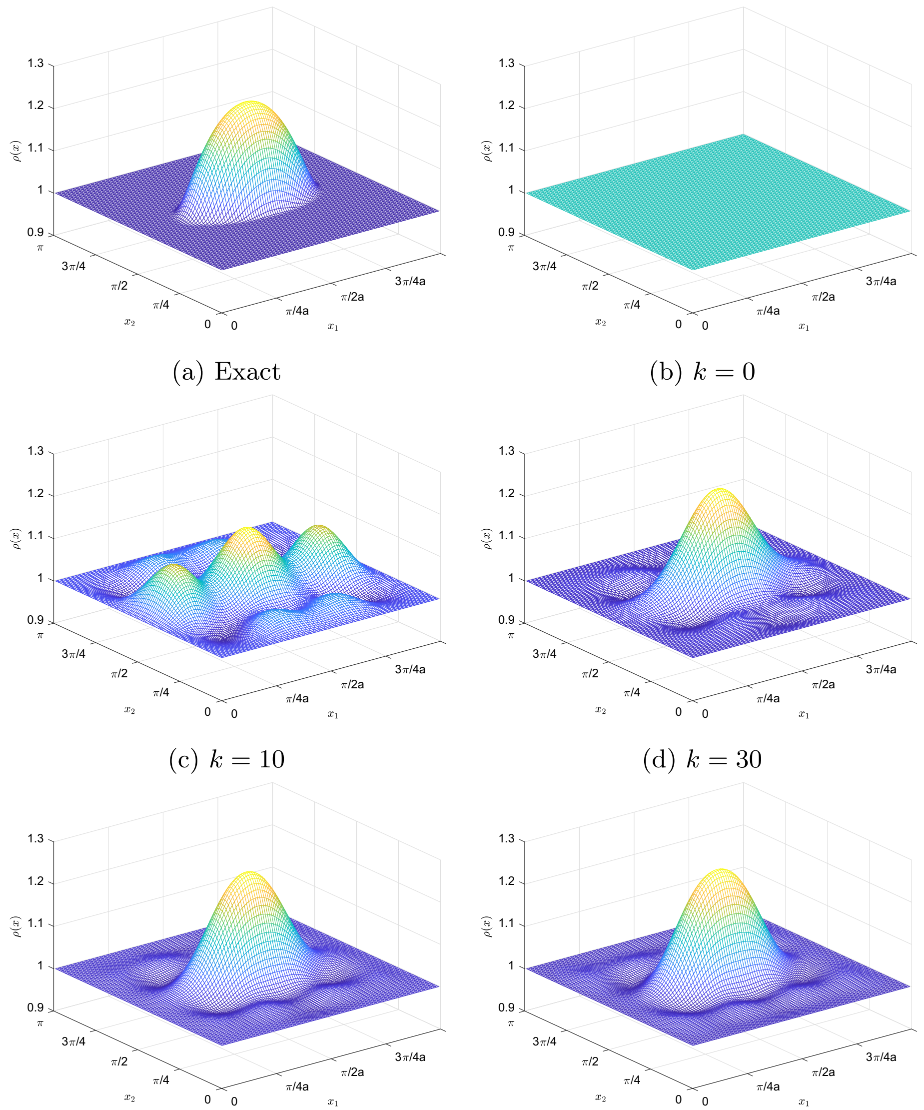


Fig. 5 The evolutions of the density function with the first 10 eigenvalues as the input data. The noise level is $\sigma = 0.001$. The original density function is discontinuous

7 Conclusions

The inverse eigenvalue problem for a weighted Helmholtz equation is investigated. The continuity of the eigenvalue and the eigenfunction with respect to the density function is proved by induction. Then the properties of existence, stability and Fr*é*chet derivative of the continuous optimization problems are established. The finite element method is applied to



(a) Exact (b) $k = 0$ (c) $k = 10$ (d) $k = 30$ (e) $k = 50$ (f) $k = 165$

Fig. 6 The exact and the recovered density function after $k = 0,\ 10,\ 30,\ 50,\ 165$ iterations with the first 10 eigenvalues as the input data. The noise level is $\sigma = 0.001$

solve the weighted Helmholtz equation. The convergence of the discrete i-th eigenpair to the continuous i-th eigenpair is proved. The properties of existence and the convergence of the discrete optimization problems are derived. A conjugate gradient algorithm is proposed. In the numerical experiments, reconstructions of continuous density function from different input eigenvalue data are discussed, including the first 5 least eigenvalues, the first 10 least eigenvalues and 5 eigenvalues with gaps. Also, the reconstruction of a discontinuous density function from the first 10 least eigenvalues as the input data is investigated. The proposed

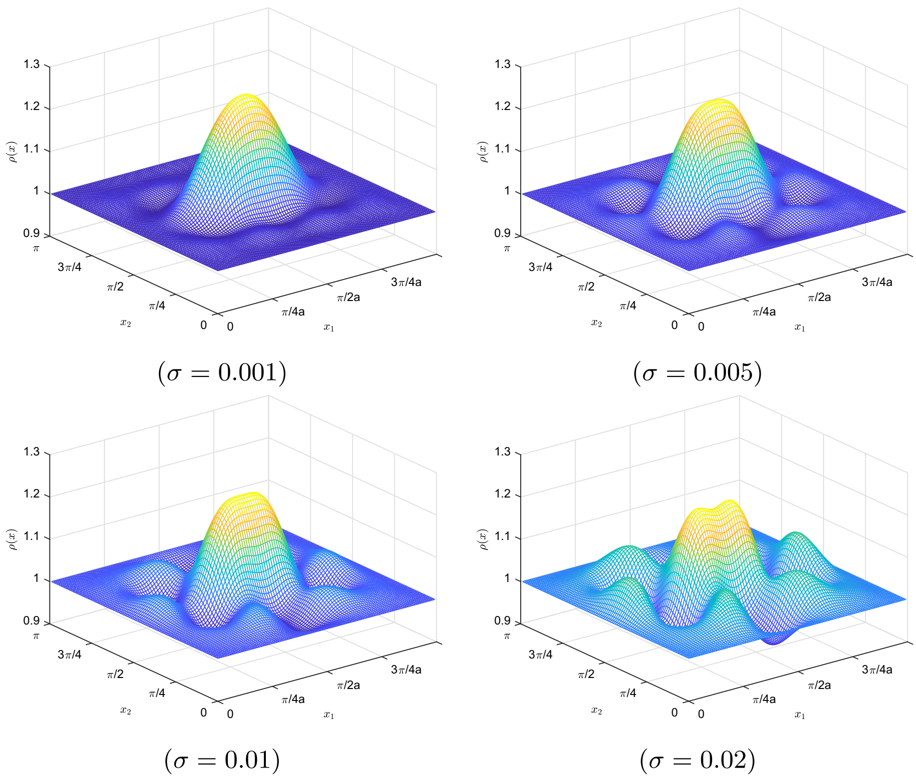


Fig. 7 The recovered density functions with different noise levels of input data

algorithm has the capacity to reconstruct the density function efficiently, especially for the cases that the density function is continuous and the input eigenvalue data are the first few least ones without gaps.

Author Contributions All authors contributed to the study conception and algorithm design. The theoretical part is derived mainly by ZZ and XC. The numerical part is mainly performed by XG. The first draft of the manuscript was written by XG and all authors commented on previous versions of the manuscript. All authors read and approved the final manuscript.

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Data Availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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