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Global-in-Time H^1-Stability of L2-1_σ Method on General Nonuniform Meshes for Subdiffusion Equation

Chaoyu Quan[1] · **Xu Wu**[2,3]

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Abstract
In this work the L2-1_σ method on general nonuniform meshes is studied for the subdiffusion equation. When the time step ratio is no less than 0.475329, a bilinear form associated with the L2-1_σ fractional-derivative operator is proved to be positive semidefinite and a new global-in-time H^1-stability of L2-1_σ schemes is then derived under simple assumptions on the initial condition and the source term. In addition, the sharp L^2-norm convergence is proved under the constraint that the time step ratio is no less than 0.475329.

Keywords Subdiffusion equation · L2-1_σ method · Nonuniform meshes · H^1-stability · Convergence

1 Introduction

In the past decade, many numerical methods have been proposed to solve the time-fractional diffusion equations [6, 21]. If the solution is sufficiently smooth (which requires the initial value to be smooth and satisfying some compatibility conditions), it has been proved that the L2-1_σ scheme has second order accuracy [2] and the L2-type methods can achieve $(3-\alpha)$-order accuracy [5, 20].

However, simple examples show that for given smooth data, the solutions to time-fractional problems typically have weak singularities. Some works start to focus on the numerical solution of more typical fractional problems whose solutions exhibit weak singularities. In particular, the L1, L2-1_σ, and L2 methods on the graded meshes have been developed. Stynes-Riordan-Gracia [25] prove the sharp error analysis of L1 scheme on graded meshes. Kopteva

✉ Xu Wu
11849596@mail.sustech.edu.cn

Chaoyu Quan
quancy@sustech.edu.cn

1 SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, China

2 Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

3 Department of Mathematics, Southern University of Science and Technology, Shenzhen, China

provides a different analysis framework of the L1 scheme on graded meshes in two and three spatial dimensions in [10]. Chen-Stynes [3] prove the second-order convergence of the L2-1_σ scheme on fitted meshes combining the graded meshes and quasiuniform meshes. Kopteva-Meng [12] provide sharp pointwise-in-time error bounds for quasi-graded termporal meshes with arbitrary degree of grading for the L1 and L2-1_σ schemes. Later Kopteva generalize this sharp pointwise error analysis to an L2-type scheme on quasi-graded meshes [11]. Liao-Li-Zhang establish the sharp error analysis for the L1 scheme of subdiffusion equation on general nonuniform meshes in [13] and then Liao-Mclean-Zhang study the L2-1_σ scheme in [14, 15], where a discrete Grönwall inequality is introduced. This analysis for general nonuniform meshes can be used to design adaptive strategies of time steps.

Taking into account the singularity of exact solution, Mustapha-Abdallah-Furati [22] analyze the global high-order convergence of the discontinuous Galerkin method for subdiffusion equation on graded mesh. Jin-Li-Zhou [7, 8] combine BDF (backward differentiation formula) CQ methods with corrections to achieve higher (more than two) order convergence which can also overcome the weak singularity problem for time-fractional diffusion equation.

In this work, we first study the H^1-stability of the L2-1_σ method proposed initially in [2] on general nonuniform meshes for subdiffusion equation with homogeneous Dirichlet boundary condition:

$$\partial_t^\alpha u(t,x) = \varDelta u(t,x) + f(t,x), \quad (t,x) \in (0,\infty) \times \varOmega, \tag{1.1}$$

where $\varOmega$ is a bounded Lipschitz domain in $\mathbb{R}^d$. For the L2-1_σ fractional-derivative operator denoted by $L_k^{\alpha,*}$, we prove that the following bilinear form

$$\mathscr{B}_n(v,w) = \sum_{k=1}^{n} \langle L_k^{\alpha,*} v, \delta_k w \rangle, \quad \delta_k w := w^k - w^{k-1},\ n \geq 1, \tag{1.2}$$

is positive semidefinite under the restrictions (3.2) on time step ratios $\rho_k := \tau_k/\tau_{k-1}$ with τ_k the kth time step and $k \geq 2$. In fact, the positive semidefiniteness of $\mathscr{B}_n$ on general nonuniform meshes is an open problem as stated in the conclusion of [16], where the maximum principle and convergence analysis are provided for L2-1_σ scheme of the time-fractional Allen–Cahn equation but not the positive definiteness of L2-1_σ operator. On the positive definiteness, Karaa presents in [1, 9] a general criteria ensuring the positivity of quadratic forms that can be applied to the time-fractional operators such as the L1 formula. In [17], Liao-Tang-Zhou proves the positive definiteness of a new L1-type operator.

Based on the positive semidefiniteness of $\mathscr{B}_n$ associated with L2-1_σ operator, we propose a new *global-in-time* H^1-stability result in Theorem 2 for the L2-1_σ scheme. In particular, when $\rho_k \geq 0.475329$ for $k \geq 2$, the restrictions (3.2) hold and the H^1-stability can be ensured for all time.

Besides the global-in-time H^1-stability of the L2-1_σ scheme in Theorem 2, we revisit the sharp convergence analysis in [15] by Liao-Mclean-Zhang. We provide a proof of sharp L^2-norm convergence based on new properties of the L2-1_σ coefficients, where the restriction on time step ratios is relaxed from $\rho_k \geq 4/7$ in [15] to $\rho_k \geq 0.475329$.

In the numerical implementations, we compare the L2-1_σ schemes on the standard graded meshes [25] and the r-variable graded meshes (with varying grading parameter). According to our stability analysis, these methods are all H^1-stable. In our example, it can be observed that choosing proper r-variable graded meshes can lead to better numerical performance.

This work is organized as follows. In Sect. 2, the derivation, explicit expression and reformulation of L2-1_σ fractional-derivative operator are provided. In Sect. 3, we prove the positive semidefiniteness of the bilinear form $\mathscr{B}_n$ under some mild restrictions on the time step ratios.

In Sect. 4, we establish a new global-in-time H^1-stability of the L2-1$_\sigma$ scheme for the subdiffusion equation, based on the positive semidefiniteness result. Moreover we show the global error estimate when $\rho_k \geq 0.475329$ under low regularity assumptions on the exact solution. In Sect. 5, we do some first numerical tests.

2 Discrete Fractional-Derivative Operator

In this part we show the derivation, explicit expression and reformulation of L2-1$_\sigma$ operator on an arbitrary nonuniform mesh.

We consider the L2-1$_\sigma$ approximation of the fractional-derivative operator defined by

$$\partial_t^\alpha u = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u'(s)}{(t-s)^\alpha}\, \mathrm{d}s.$$

Take a nonuniform time mesh $0 = t_0 < t_1 < \ldots < t_{k-1} < t_k < \ldots$ with $k \geq 1$. Let $\tau_j = t_j - t_{j-1}$ and $\sigma = 1-\alpha/2$ (c.f. [2] for this setting of σ). The fractional derivative $\partial_t^\alpha u(t)$ at $t = t_k^* := t_{k-1} + \sigma\tau_k$ could be approximated by the following L2-1$_\sigma$ fractional-derivative operator

$$\begin{aligned} L_k^{\alpha,*}u &= \frac{1}{\Gamma(1-\alpha)} \left(\sum_{j=1}^{k-1} \int_{t_{j-1}}^{t_j} \frac{\partial_s H_2^j(s)}{(t_k^*-s)^\alpha}\, \mathrm{d}s + \int_{t_{k-1}}^{t_k^*} \frac{\partial_s H_1^k(s)}{(t_k^*-s)^\alpha}\, \mathrm{d}s \right) \\ &= \frac{1}{\Gamma(1-\alpha)} \left(\sum_{j=1}^{k-1} (a_j^{(k)} u^{j-1} + b_j^{(k)} u^j + c_j^{(k)} u^{j+1}) \right) + \frac{\sigma^{1-\alpha}(u^k - u^{k-1})}{\Gamma(2-\alpha)\tau_k^\alpha}, \end{aligned} \tag{2.1}$$

where for $1 \leq j \leq k-1$,

$$\begin{aligned} H_2^j(t) =& \frac{(t-t_j)(t-t_{j+1})}{(t_{j-1}-t_j)(t_{j-1}-t_{j+1})} u^{j-1} + \frac{(t-t_{j-1})(t-t_{j+1})}{(t_j-t_{j-1})(t_j-t_{j+1})} u^j \\ &+ \frac{(t-t_{j-1})(t-t_j)}{(t_{j+1}-t_{j-1})(t_{j+1}-t_j)} u^{j+1}, \\ H_1^k(t) =& \frac{t-t_k}{t_{k-1}-t_k} u^{k-1} + \frac{t-t_{k-1}}{t_k-t_{k-1}} u^k, \end{aligned}$$

and

$$\begin{aligned} a_j^{(k)} &= \int_{t_{j-1}}^{t_j} \frac{2s - t_j - t_{j+1}}{\tau_j(\tau_j+\tau_{j+1})} \frac{1}{(t_k^*-s)^\alpha}\, \mathrm{d}s = \int_0^1 \frac{-2\tau_j(1-\theta) - \tau_{j+1}}{(\tau_j+\tau_{j+1})(t_k^* - (t_{j-1}+\theta\tau_j))^\alpha}\, \mathrm{d}\theta, \\ b_j^{(k)} &= -\int_{t_{j-1}}^{t_j} \frac{2s - t_{j-1} - t_{j+1}}{\tau_j\tau_{j+1}} \frac{1}{(t_k^*-s)^\alpha}\, \mathrm{d}s = -\int_0^1 \frac{2\tau_j\theta - \tau_j - \tau_{j+1}}{\tau_{j+1}(t_k^* - (t_{j-1}+\theta\tau_j))^\alpha}\, \mathrm{d}\theta, \\ c_j^{(k)} &= \int_{t_{j-1}}^{t_j} \frac{2s - t_{j-1} - t_j}{\tau_{j+1}(\tau_j+\tau_{j+1})} \frac{1}{(t_k^*-s)^\alpha}\, \mathrm{d}s = \int_0^1 \frac{\tau_j^2(2\theta-1)}{\tau_{j+1}(\tau_j+\tau_{j+1})(t_k^* - (t_{j-1}+\theta\tau_j))^\alpha}\, \mathrm{d}\theta. \end{aligned} \tag{2.2}$$

It can be verified that $a_j^{(k)} < 0, b_j^{(k)} > 0, c_j^{(k)} > 0$, and $a_j^{(k)}+b_j^{(k)}+c_j^{(k)} = 0$ for $1 \leq j \leq k-1$.

Specifically speaking, we can figure out the explicit expressions of $a_j^{(k)}$ and $c_j^{(k)}$ as follows (note that $b_j^{(k)} = -a_j^{(k)} - c_j^{(k)}$): for $1 \leq j \leq k-1$,

$$\begin{aligned} a_j^{(k)} &= \frac{\tau_{j+1}}{(1-\alpha)\tau_j(\tau_j+\tau_{j+1})}(t_k^* - t_j)^{1-\alpha} - \frac{2\tau_j+\tau_{j+1}}{(1-\alpha)\tau_j(\tau_j+\tau_{j+1})}(t_k^* - t_{j-1})^{1-\alpha} \\ &\quad + \frac{2}{(2-\alpha)(1-\alpha)\tau_j(\tau_j+\tau_{j+1})}\left[(t_k^* - t_{j-1})^{2-\alpha} - (t_k^* - t_j)^{2-\alpha}\right], \\ c_j^{(k)} &= \frac{1}{(1-\alpha)\tau_{j+1}(\tau_j+\tau_{j+1})}\Big[-\tau_j((t_k^* - t_{j-1})^{1-\alpha} + (t_k^* - t_j)^{1-\alpha}) \\ &\quad + 2(2-\alpha)^{-1}((t_k^* - t_{j-1})^{2-\alpha} - (t_k^* - t_j)^{2-\alpha})\Big]. \end{aligned}$$

We reformulate the discrete fractional derivative $L_k^{\alpha,*}$ in (2.1) as

$$L_k^{\alpha,*}u = \frac{1}{\Gamma(1-\alpha)}\left(c_{k-1}^{(k)}\delta_k u - a_1^k \delta_1 u + \sum_{j=2}^{k-1} d_j^{(k)}\delta_j u\right) + \frac{\sigma^{1-\alpha}}{\Gamma(2-\alpha)\tau_k^\alpha}\delta_k u, \tag{2.3}$$

where $\delta_j u = u^j - u^{j-1}$, $d_j^{(k)} := c_{j-1}^{(k)} - a_j^{(k)}$. Here we make a convention that $a_1^1 = 0$ and $c_0^1 = 0$.

To establish the global-in-time H^1-stability of L2-1_σ method for fractional-order parabolic problem, we shall prove the positive semidefiniteness of $\mathscr{B}_n$ defined in (1.2).

3 Positive Semidefiniteness of Bilinear Form $\mathscr{B}_n$

In this section, we first propose some properties of the L2-1_σ coefficients $a_j^{(k)}$, $c_j^{(k)}$ and $d_j^{(k)}$ in (2.3), which will be useful to establish the positive semidefiniteness of bilinear form $\mathscr{B}_n$. Then we prove rigorously the positive semidefiniteness of bilinear form $\mathscr{B}_n$ under some constraints of ρ_k, $k \geq 2$.

Lemma 1 (Properties of $a_j^{(k)}$, $c_j^{(k)}$ and $d_j^{(k)}$) For the L2-1_σ coefficients given in (2.3), given a nonuniform mesh $\{\tau_j\}_{j\geq 1}$, the following properties hold:

(P1) $a_j^{(k)} < 0,\ 1 \leq j \leq k-1,\ k \geq 2;$
(P2) $a_j^{(k+1)} - a_j^{(k)} > 0,\ 1 \leq j \leq k-1,\ k \geq 2;$
(P3) $a_{j+1}^{(k)} - a_j^{(k)} < 0,\ 1 \leq j \leq k-2,\ k \geq 3;$
(P4) $a_{j+1}^{(k)} - a_j^{(k)} < a_{j+1}^{(k+1)} - a_j^{(k+1)},\ 1 \leq j \leq k-2,\ k \geq 3;$
(P5) $c_j^{(k)} > 0,\ 1 \leq j \leq k-1,\ k \geq 2;$
(P6) $c_j^{(k+1)} - c_j^{(k)} < 0,\ 1 \leq j \leq k-1,\ k \geq 2;$
(P7) $d_j^{(k)} > 0,\ 2 \leq j \leq k-1,\ k \geq 3;$
(P8) $d_j^{(k+1)} - d_j^{(k)} < 0,\ 2 \leq j \leq k-1,\ k \geq 3.$

Furthermore, if the nonuniform mesh $\{\tau_j\}_{j\geq 1}$, with $\rho_j := \tau_j/\tau_{j-1}$ satisfies

$$\frac{1}{\rho_{j+1}} \geq \frac{1}{\rho_j^2(1+\rho_j)} - 3, \quad \forall j \geq 2, \tag{3.1}$$

then the following properties of $d_j^{(k)}$ hold:

(P9) $d_{j+1}^{(k)} - d_j^{(k)} > 0,\ 2 \le j \le k-2,\ k \ge 4;$
(P10) $d_{j+1}^{(k)} - d_j^{(k)} > d_{j+1}^{(k+1)} - d_j^{(k+1)},\ 2 \le j \le k-2,\ k \ge 4.$

Proof The proof is the same as the proof of [24, Lemma 3.1] except replacing t_k with t_k^*. We omit it here. □

Theorem 1 *Consider a nonuniform mesh* $\{\tau_k\}_{k\ge1}$ *satisfying that* $k \ge 2$,

$$\begin{cases} \rho_* < \rho_{k+1} \le \dfrac{\rho_k^2(1+\rho_k)}{1-3\rho_k^2(1+\rho_k)}, & \rho_* < \rho_k < \eta, \\ \rho_* < \rho_{k+1}, & \eta \le \rho_k, \end{cases} \tag{3.2}$$

where $\rho_* \approx 0.356341$, *and* $\eta \approx 0.475329$. *Then the for any function u defined on* $[0,\infty)\times\Omega$ *and* $n \ge 1$,

$$\mathscr{B}_n(u,u) = \sum_{k=1}^{n} \langle L_k^{\alpha,*} u, \delta_k u\rangle \ge \sum_{k=1}^{n} \frac{g_k(\alpha)}{2\Gamma(2-\alpha)} \|\delta_k u\|_{L^2(\Omega)}^2 \ge 0, \tag{3.3}$$

where

$$g_k(\alpha) = \begin{cases} \frac{1}{(\sigma\tau_1)^\alpha}\left(2\sigma - \frac{1-\alpha}{\rho_2^\alpha}\right), & k=1, \\ (1-\alpha)c_{k-1}^{(k)} + \frac{1}{(\sigma\tau_k)^\alpha}\left(2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha}\int_0^1 \frac{s(\rho_{k+1}+s)}{\sigma\rho_{k+1}+s}\,\mathrm{d}s\right), & 2\le k\le n-1, \\ (1-\alpha)c_{n-1}^{(n)} + \frac{1}{(\sigma\tau_n)^\alpha}(2\sigma-(1-\alpha)), & k=n\ne 2, \end{cases} \tag{3.4}$$

are always positive for $\alpha \in (0,1)$.

Proof According to (2.3), we can rewrite $\mathscr{B}_n(u,u)$ in the following matrix form

$$\mathscr{B}_n(u,u) = \sum_{k=1}^{n} \langle L_k^{\alpha,*} u, \delta_k u\rangle = \frac{1}{\Gamma(1-\alpha)} \int_\Omega \psi \mathbf{M} \psi^{\mathrm{T}} \mathrm{d}x,$$

where $\psi = [\delta_1 u, \delta_2 u, \cdots, \delta_n u]$, and

$$\mathbf{M} = \begin{pmatrix} \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_1^\alpha} & & & & \\ -a_1^{(2)} & c_1^{(2)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} & & & \\ -a_1^{(3)} & d_2^{(3)} & c_2^{(3)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_3^\alpha} & & \\ \vdots & \vdots & \ddots & \ddots & \\ -a_1^{(n)} & d_2^{(n)} & \cdots & d_{n-1}^{(n)} & c_{n-1}^{(n)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_n^\alpha} \end{pmatrix}, \tag{3.5}$$

We split $\mathbf{M}$ as $\mathbf{M} = \mathbf{A} + \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} \beta_1 & & & & \\ -a_1^{(2)} & \beta_2 & & & \\ -a_1^{(3)} & d_2^{(3)} & \beta_3 & & \\ \vdots & \vdots & \ddots & \ddots & \\ -a_1^{(n)} & d_2^{(n)} & \cdots & d_{n-1}^{(n)} & \beta_n \end{pmatrix},$$

and

$$\mathbf{B} = \text{diag}\left(\frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_1^\alpha} - \beta_1,\ c_1^{(2)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} - \beta_2,\ \cdots,\ c_{n-1}^{(n)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_n^\alpha} - \beta_n\right),$$

with

$$\begin{aligned} &2\beta_1 = -a_1^{(2)}, \quad 2\beta_2 - d_2^{(3)} = a_1^{(3)} - a_1^{(2)}, \\ &2\beta_k - d_k^{(k+1)} = d_{k-1}^{(k)} - d_{k-1}^{(k+1)}, \quad 3 \le k \le n-1, \\ &2\beta_n = d_{n-1}^{(n)}, \quad n \ge 3. \end{aligned} \tag{3.6}$$

Consider the following symmetric matrix $\mathbf{S} = \mathbf{A} + \mathbf{A}^{\mathrm{T}} + \varepsilon \mathbf{e}_n^{\mathrm{T}} \mathbf{e}_n$ with small constant $\varepsilon > 0$ and $\mathbf{e}_n = (0, \cdots, 0, 1) \in \mathbb{R}^{1\times n}$. According to Lemma 1, if the condition (3.1) holds, $\mathbf{S}$ satisfies the following three properties:

(1) $\forall\, 1 \le j < i \le n$, $[\mathbf{S}]_{i-1,j} \ge [\mathbf{S}]_{i,j}$;
(2) $\forall\, 1 < j \le i \le n$, $[\mathbf{S}]_{i,j-1} < [\mathbf{S}]_{i,j}$;
(3) $\forall\, 1 < j < i \le n$, $[\mathbf{S}]_{i-1,j-1} - [\mathbf{S}]_{i,j-1} \le [\mathbf{S}]_{i-1,j} - [\mathbf{S}]_{i,j}$.

From [23, Lemma 2.1], $\mathbf{S}$ is positive definite. Let $\varepsilon \to 0$. We can claim that $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$ is positive semidefinite.

In the following we will prove $[\mathbf{B}]_{kk} \ge 0$, $k \ge 1$, under some constraints on ρ_k. We first provide two equivalent forms of $a_j^{(k)}$ according to (2.2): $\forall 1 \le j \le k-1$,

$$\begin{aligned} a_j^{(k)} &= \int_0^1 \frac{-2\tau_j(1-s) - \tau_{j+1}}{(\tau_j + \tau_{j+1})(t_k^* - (t_{j-1} + s\tau_j))^\alpha}\,\mathrm{d}s \\ &= \frac{1}{\tau_j + \tau_{j+1}} \int_0^1 (t_k^* - (t_{j-1} + s\tau_j))^{-\alpha}\,\mathrm{d}(\tau_j s^2 - (2\tau_j + \tau_{j+1})s) \\ &= -(t_k^* - t_j)^{-\alpha} + \frac{\alpha\tau_j}{\tau_j + \tau_{j+1}} \int_0^1 (\tau_j + \tau_{j+1} + s\tau_j)(1-s)(t_k^* - t_j + s\tau_j)^{-\alpha-1}\,\mathrm{d}s \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} a_j^{(k)} &= \int_0^1 \frac{-2\tau_j(1-s) - \tau_{j+1}}{(\tau_j + \tau_{j+1})(t_k^* - (t_{j-1} + s\tau_j))^\alpha}\,\mathrm{d}s = \int_0^1 \frac{-2\tau_j s - \tau_{j+1}}{(\tau_j + \tau_{j+1})(t_k^* - t_j + s\tau_j)^\alpha}\,\mathrm{d}s \\ &= \frac{1}{\tau_j + \tau_{j+1}} \int_0^1 (t_k^* - t_j + s\tau_j)^{-\alpha}\,\mathrm{d}(-\tau_j s^2 - \tau_{j+1}s) \\ &= -(t_k^* - t_{j-1})^{-\alpha} - \frac{\alpha\tau_j}{\tau_j + \tau_{j+1}} \int_0^1 (\tau_j + \tau_{j+1} - s\tau_j)(1-s)(t_k^* - t_{j-1} - s\tau_j)^{-\alpha-1}\,\mathrm{d}s. \end{aligned} \tag{3.8}$$

Furthermore, we also reformulate $c_j^{(k)}$ in (2.2) as: $\forall 1 \le j \le k-1$,

$$\begin{aligned} c_j^{(k)} &= \int_0^1 \frac{\tau_j^2(2s-1)}{\tau_{j+1}(\tau_j + \tau_{j+1})(t_k^* - (t_{j-1} + s\tau_j))^\alpha}\,\mathrm{d}s \\ &= \frac{\tau_j^2}{\tau_{j+1}(\tau_j + \tau_{j+1})} \int_0^1 (t_k^* - (t_{j-1} + s\tau_j))^{-\alpha}\,\mathrm{d}(s^2 - s) \\ &= \frac{\alpha\tau_j^3}{\tau_{j+1}(\tau_j + \tau_{j+1})} \int_0^1 s(1-s)(t_k^* - t_j + s\tau_j)^{-\alpha-1}\,\mathrm{d}s. \end{aligned} \tag{3.9}$$

In the following content, we consider four cases: $k = 1$, $k = 2$, $3 \le k \le n-1$, and $k = n$.

Case 1 When $k = 1$, from (2.2) and $2\beta_1 = -a_1^{(2)}$ in (3.6), we have

$$[\mathbf{B}]_{11} = \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_1^\alpha} - \frac{1}{2}\int_0^1 \frac{2\tau_1(1-\theta)+\tau_2}{(\tau_1+\tau_2)(t_2^* - (t_0+\theta\tau_1))^\alpha}\,\mathrm{d}\theta$$
$$= \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_1^\alpha} - \frac{1}{2\tau_1^\alpha}\int_0^1 \frac{2s+\rho_2}{(1+\rho_2)(\sigma\rho_2+s)^\alpha}\mathrm{d}s$$
$$> \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_1^\alpha} - \frac{1}{2\tau_1^\alpha(\sigma\rho_2)^\alpha}\int_0^1 \frac{2s+\rho_2}{(1+\rho_2)}\mathrm{d}s = \frac{1}{2(1-\alpha)(\sigma\tau_1)^\alpha}\left(2\sigma - \frac{1-\alpha}{\rho_2^\alpha}\right).$$

To ensure $[\mathbf{B}]_{11} \ge 0$, we impose

$$2\sigma - \frac{1-\alpha}{\rho_2^\alpha} \ge 0. \tag{3.10}$$

Case 2 When $k = 2$, combining $2\beta_2 - d_2^{(3)} = a_1^{(3)} - a_1^{(2)}$ in (3.6) and the property (P6) in Lemma (1) gives

$$\mathbf{B}t_{22} = c_1^{(2)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} - \frac{1}{2}(d_2^{(3)} + a_1^{(3)} - a_1^{(2)}) \tag{3.11}$$
$$= \frac{1}{2}c_1^{(2)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} + \frac{1}{2}(a_1^{(2)} - a_1^{(3)} + a_2^{(3)}) + \frac{1}{2}(c_1^{(2)} - c_1^{(3)})$$
$$\ge \frac{1}{2}c_1^{(2)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} + \frac{1}{2}(a_1^{(2)} - a_1^{(3)} + a_2^{(3)}).$$

Using the forms (3.7) for $a_1^{(2)}$, $a_1^{(3)}$ and (3.8) for $a_2^{(3)}$, we can derive

$$a_1^{(2)} - a_1^{(3)} + a_2^{(3)} = -(\sigma\tau_2)^{-\alpha} + \frac{\alpha\tau_1}{\tau_1+\tau_2}\int_0^1 (\tau_1+\tau_2+s\tau_1)(1-s)(t_2^* - t_1 + s\tau_1)^{-\alpha-1}\,\mathrm{d}s$$
$$- \frac{\alpha\tau_1}{\tau_1+\tau_2}\int_0^1 (\tau_1+\tau_2+s\tau_1)(1-s)(t_3^* - t_1 + s\tau_1)^{-\alpha-1}\,\mathrm{d}s$$
$$- \frac{\alpha\tau_2}{\tau_2+\tau_3}\int_0^1 (\tau_2+\tau_3-s\tau_2)(1-s)(t_3^* - t_1 - s\tau_2)^{-\alpha-1}\,\mathrm{d}s >$$
$$- (\sigma\tau_2)^{-\alpha} - \frac{\alpha\tau_2}{\tau_2+\tau_3}\int_0^1 (\tau_2+\tau_3-s\tau_2)(1-s)(\tau_2+\sigma\tau_3 - s\tau_2)^{-\alpha-1}\,\mathrm{d}s$$
$$= -(\sigma\tau_2)^{-\alpha} - \frac{\alpha}{(1+\rho_3)\tau_2^\alpha}\int_0^1 s(\rho_3+s)(\sigma\rho_3+s)^{-\alpha-1}\,\mathrm{d}s$$
$$- (\sigma\tau_2)^{-\alpha} - \frac{\alpha}{(1+\rho_3)(\sigma\tau_2)^\alpha\rho_3^\alpha}\int_0^1 \frac{s(\rho_3+s)}{\sigma\rho_3+s}\,\mathrm{d}s. \tag{3.12}$$

Substituting (3.12) into (3.11) yields

$$\mathbf{B}t_{22} \ge \frac{1}{2}c_1^{(2)} + \frac{1}{2(1-\alpha)(\sigma\tau_2)^\alpha}\left(2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_3)\rho_3^\alpha}\int_0^1 \frac{s(\rho_3+s)}{\sigma\rho_3+s}\,\mathrm{d}s\right).$$

To make sure $[\mathbf{B}]_{22} \ge 0$, we impose

$$2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_3)\rho_3^\alpha}\int_0^1 \frac{s(\rho_3+s)}{\sigma\rho_3+s}\,\mathrm{d}s \ge 0. \tag{3.13}$$

Case 3 When $3 \le k \le n-1$, using $2\beta_k = d_k^{(k+1)} + d_{k-1}^{(k)} - d_{k-1}^{(k+1)}$ in (3.6) and $d_j^{(k)} = c_{j-1}^{(k)} - a_j^{(k)}$, we have

$$\begin{aligned}
[\mathbf{B}]_{kk} &= \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} + \frac{1}{2}c_{k-1}^{(k)} + \frac{1}{2}(c_{k-1}^{(k)} - d_k^{(k+1)} - d_{k-1}^{(k)} + d_{k-1}^{(k+1)}) \\
&= \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} + \frac{1}{2}c_{k-1}^{(k)} + \frac{1}{2}[(c_{k-1}^{(k)} - c_{k-1}^{(k+1)}) - (c_{k-2}^{(k)} - c_{k-2}^{(k+1)}) \\
&\quad + (-a_{k-1}^{(k+1)} + a_k^{(k+1)} + a_{k-1}^{(k)})].
\end{aligned} \tag{3.14}$$

From (3.7) – (3.9), if (3.1) holds for $j = k-1$, we have

$$\begin{aligned}
&(c_{k-1}^{(k)} - c_{k-1}^{(k+1)}) - (c_{k-2}^{(k)} - c_{k-2}^{(k+1)}) + (-a_{k-1}^{(k+1)} + a_k^{(k+1)} + a_{k-1}^{(k)}) \\
&= \frac{\alpha\tau_{k-1}^3}{\tau_k(\tau_{k-1}+\tau_k)} \int_0^1 s(1-s)\Big[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} \\
&\quad - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\Big]\,\mathrm{d}s - \frac{\alpha\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})}\int_0^1 s(1-s) \\
&\quad \Big[(t_k^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1} - (t_{k+1}^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1}\Big]\,\mathrm{d}s \\
&\quad + \frac{\alpha\tau_{k-1}}{\tau_{k-1}+\tau_k}\int_0^1 (\tau_{k-1} + \tau_k + s\tau_{k-1})(1-s)\Big[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} \\
&\quad - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\Big]\,\mathrm{d}s \\
&\quad - (\sigma\tau_k)^{-\alpha} - \frac{\alpha\tau_k}{\tau_k+\tau_{k+1}}\int_0^1 (\tau_k + \tau_{k+1} - s\tau_k)(1-s)(t_{k+1}^* - t_{k-1} - s\tau_k)^{-\alpha-1}\,\mathrm{d}s \\
&> -(\sigma\tau_k)^{-\alpha} - \frac{\alpha\tau_k}{\tau_k+\tau_{k+1}}\int_0^1 s(\tau_{k+1} + s\tau_k)(\sigma\tau_{k+1} + s\tau_k)^{-\alpha-1}\,\mathrm{d}s \\
&= -(\sigma\tau_k)^{-\alpha} - \frac{\alpha}{(1+\rho_{k+1})\tau_k^\alpha}\int_0^1 s(\rho_{k+1} + s)(\sigma\rho_{k+1} + s)^{-\alpha-1}\,\mathrm{d}s > \\
&\quad - (\sigma\tau_k)^{-\alpha} - \frac{\alpha}{(1+\rho_{k+1})(\sigma\tau_k)^\alpha\rho_{k+1}^\alpha}\int_0^1 \frac{s(\rho_{k+1}+s)}{\sigma\rho_{k+1}+s}\,\mathrm{d}s,
\end{aligned} \tag{3.15}$$

where we use the forms (3.7) for $a_{k-1}^{(k)}$, $a_{k-1}^{(k+1)}$ and (3.8) for $a_k^{(k+1)}$. The first inequality in (3.15) can be derived as follows. For fixed j, it is easy to see that

$$(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} > 0$$

decreases w.r.t. s and $\int_0^1 (1-3s)(1-s)\,\mathrm{d}s = 0$, thus

$$\begin{aligned}
&\int_0^1 (\tau_{k-1} + \tau_k + s\tau_{k-1})(1-s)[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}]\,\mathrm{d}s \\
&\quad \ge \int_0^1 (4\tau_{k-1} + 3\tau_k)s(1-s)[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}]\,\mathrm{d}s.
\end{aligned}$$

Moreover the convexity of the function $t^{-1-\alpha}$ gives

$$
\begin{aligned}
&(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} \\
&\quad > (t_k^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1} - (t_{k+1}^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1},
\end{aligned}
$$

Then we can get the following result:

$$
\begin{aligned}
&\frac{\alpha\tau_{k-1}^3}{\tau_k(\tau_{k-1}+\tau_k)} \int_0^1 s(1-s)\Big[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} \\
&\quad - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\Big] \mathrm{d}s - \frac{\alpha\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})} \int_0^1 s(1-s) \\
&\quad \Big[(t_k^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1} - (t_{k+1}^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1}\Big] \mathrm{d}s \\
&\quad + \frac{\alpha\tau_{k-1}}{\tau_{k-1}+\tau_k} \int_0^1 (\tau_{k-1}+\tau_k+s\tau_{k-1})(1-s)\Big[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} \\
&\quad - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\Big] \mathrm{d}s \\
&\quad > \alpha\left(\frac{\tau_{k-1}^3}{\tau_k(\tau_{k-1}+\tau_k)} - \frac{\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})} + \frac{(4\tau_{k-1}+3\tau_k)\tau_{k-1}}{\tau_{k-1}+\tau_k}\right) \int_0^1 s(1-s) \\
&\quad \Big[(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1} - (t_{k+1}^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\Big] \mathrm{d}s \ge 0,
\end{aligned}
$$

as (3.1) for $j = k-1$ gives

$$
\frac{\tau_{k-1}^3}{\tau_k(\tau_{k-1}+\tau_k)} - \frac{\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})} + \frac{(4\tau_{k-1}+3\tau_k)\tau_{k-1}}{\tau_{k-1}+\tau_k} \ge 0.
$$

Combining (3.15) with (3.14) yields

$$
\mathbf{B}t_{kk} \ge \frac{1}{2}c_{k-1}^{(k)} + \frac{1}{2(1-\alpha)(\sigma\tau_k)^\alpha}\left(2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha}\int_0^1 \frac{s(\rho_{k+1}+s)}{\sigma\rho_{k+1}+s}\,\mathrm{d}s\right).
$$

Thus, to ensure $[\mathbf{B}]_{kk} \ge 0$ for $3 \le k \le n-1$, it is sufficient to impose

$$
\begin{aligned}
&\frac{1}{\rho_k} \ge \frac{1}{\rho_{k-1}^2(1+\rho_{k-1})} - 3, \\
&2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha}\int_0^1 \frac{s(\rho_{k+1}+s)}{\sigma\rho_{k+1}+s}\,\mathrm{d}s \ge 0.
\end{aligned}
\tag{3.16}
$$

Case 4 When $k = n$, we show $[\mathbf{B}]_{nn} \ge 0$ under some constraints on ρ_n. From (3.6), (3.7) and (3.9), we can derive

$$
\begin{aligned}
[\mathbf{B}]_{nn} &= c_{n-1}^{(n)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_n^\alpha} - \frac{1}{2}(c_{n-2}^{(n)} - a_{n-1}^{(n)}) \\
&= \frac{1}{2}c_{n-1}^{(n)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_n^\alpha} + \frac{1}{2}(c_{n-1}^{(n)} - c_{n-2}^{(n)} + a_{n-1}^{(n)}) \\
&= \frac{1}{2}c_{n-1}^{(n)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_n^\alpha} + \frac{1}{2}\Bigg(\frac{\alpha\tau_{n-1}^3}{\tau_n(\tau_{n-1}+\tau_n)}\int_0^1 s(1-s)(t_n^* - t_{n-1} + s\tau_{n-1})^{-\alpha-1}\,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{\alpha \tau_{n-2}^3}{\tau_{n-1}(\tau_{n-2}+\tau_{n-1})} \int_0^1 s(1-s)(t_n^* - t_{n-2} + s\tau_{n-2})^{-\alpha-1}\, \mathrm{d}s \\
&- (\sigma\tau_n)^{-\alpha} + \frac{\alpha \tau_{n-1}}{\tau_{n-1}+\tau_n} \int_0^1 (\tau_{n-1}+\tau_n+s\tau_{n-1})(1-s)(t_n^* - t_{n-1} + s\tau_{n-1})^{-\alpha-1}\, \mathrm{d}s\Big) \\
&> \frac{1}{2} c_{n-1}^{(n)} + \frac{1}{2(1-\alpha)(\sigma\tau_n)^\alpha} \left(2\sigma - (1-\alpha)\right), \qquad (3.17)
\end{aligned}
$$

if (3.1) holds for $j = n-1$. The proof of the last inequality in (3.17) is similar to the previous proof of (3.15), where we use the facts

$$
\begin{aligned}
&\int_0^1 (\tau_{n-1}+\tau_n+s\tau_{n-1})(1-s)(t_n^* - t_{n-1} + s\tau_{n-1})^{-\alpha-1}\, \mathrm{d}s \\
&\quad \geq \int_0^1 (4\tau_{n-1}+3\tau_n)s(1-s)(t_n^* - t_{n-1} + s\tau_{n-1})^{-\alpha-1}\, \mathrm{d}s,
\end{aligned}
$$

and

$$
(t_n^* - t_{n-1} + s\tau_{n-1})^{-\alpha-1} > (t_n^* - t_{n-2} + s\tau_{n-2})^{-\alpha-1}.
$$

We omit the details here. To ensure $[\mathbf{B}]_{nn} \geq 0$, it is sufficient to impose

$$
\frac{1}{\rho_n} \geq \frac{1}{\rho_{n-1}^2(1+\rho_{n-1})} - 3, \quad 2\sigma - (1-\alpha) \geq 0. \qquad (3.18)
$$

Combining (3.10), (3.13), (3.16) and (3.18), we can conclude that if the condition (3.1) holds for $2 \leq k \leq n-1$ and

$$
\begin{aligned}
&2\sigma - \frac{1-\alpha}{\rho_2^\alpha} \geq 0, \\
&2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha} \int_0^1 \frac{s(\rho_{k+1}+s)}{\sigma\rho_{k+1}+s}\, \mathrm{d}s \geq 0, \quad 2 \leq k \leq n-1, \qquad (3.19) \\
&2\sigma - (1-\alpha) \geq 0,
\end{aligned}
$$

then $[\mathbf{B}]_{kk} \geq 0$, $k \geq 1$. We have proved the following results:

- Positive semidefiniteness of $\mathbf{A} + \mathbf{A}^{\mathrm{T}}$: (3.1) holds;
- Positive definiteness of $\mathbf{B}$: (3.19) holds and (3.1) holds for $2 \leq k \leq n-1$;

which ensure

$$
\mathbf{M} + \mathbf{M}^{\mathrm{T}} = (\mathbf{A} + \mathbf{A}^{\mathrm{T}}) + 2\mathbf{B} \geq 2\mathbf{B} \geq (1-\alpha)^{-1} \mathrm{diag}\left(g_1(\alpha), g_2(\alpha), \ldots, g_n(\alpha)\right) \geq 0,
$$

where $g_k(\alpha)$ is given in (3.4). In the following content, we just simplify the above constraints for the positive semidefiniteness of $\mathbf{M} + \mathbf{M}^{\mathrm{T}}$.

The condition (3.1) actually says that (ρ_j, ρ_{j+1}) lies on the right-hand side of the blue solid curve in Fig. 1. Let $\rho_* \approx 0.356341$ be the root of $\rho(1+\rho) = 1 - 3\rho^2(1+\rho)$. It can be found that if $\rho_j \leq \rho_*$ for some j, then $\rho_* \geq \rho_j \geq \rho_{j+1} \geq \rho_{j+2} \geq \ldots$ and τ_j will shrink to 0 quickly as j increases. This doesn't make sense in practice. We shall impose $\rho_j > \rho_*$, $\forall j \geq 2$. As a consequence, we have the following constraints: for $j \geq 2$,

$$
\begin{cases}
\rho_* < \rho_{j+1} \leq \dfrac{\rho_j^2(1+\rho_j)}{1 - 3\rho_j^2(1+\rho_j)}, & \rho_* < \rho_j < \eta, \\
\rho_* < \rho_{j+1}, & \eta \leq \rho_j,
\end{cases} \qquad (3.20)
$$

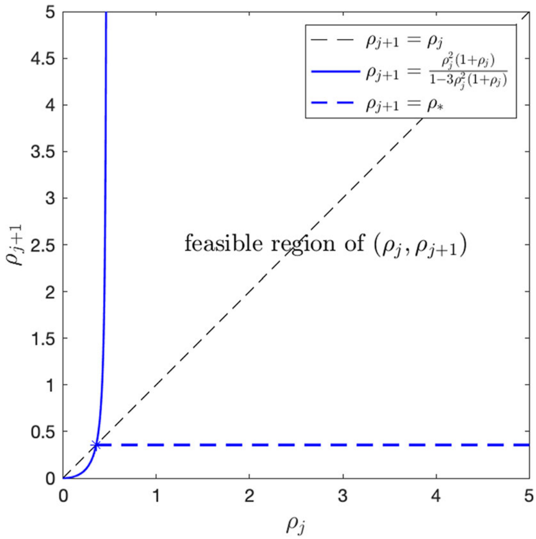

Fig. 1 Feasible region of (ρ_j, ρ_{j+1}), on the right-hand side of the blue solid curve and above the blue dashed line, obtained from the constraint (3.20) for $j \geq 2$. The blue star marker denotes (ρ_*, ρ_*)

where $\eta \approx 0.475329$ be the unique positive root of $1 - 3\rho^2(1+\rho) = 0$.

We now prove that (3.20) leads to (3.19) when $\sigma = 1 - \alpha/2 \geq 1/2$. In fact, it is easy to check that

$$2\sigma - \frac{1-\alpha}{\rho_2^\alpha} \geq 2 - \alpha - \frac{1-\alpha}{\rho_*^\alpha} \geq 0, \quad 2\sigma - (1-\alpha) = 1,$$

and for $2 \leq k \leq n-1$, we have

$$\begin{aligned}
&2\sigma - (1-\alpha) - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha} \int_0^1 \frac{s(\rho_{k+1}+s)}{\sigma\rho_{k+1}+s}\,\mathrm{d}s \\
&\geq 1 - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha} \int_0^1 \frac{s(\rho_{k+1}+s)}{\frac{1}{2}\rho_{k+1}+s}\,\mathrm{d}s \\
&\geq 1 - \frac{\alpha(1-\alpha)}{(1+\rho_{k+1})\rho_{k+1}^\alpha} \geq 1 - \frac{\alpha(1-\alpha)}{(1+\rho_*)\rho_*^\alpha} \geq 1 - \frac{1}{4(1+\rho_*)\rho_*} \geq 0.
\end{aligned}$$

In summary, if (3.20) holds, then

$$\mathscr{B}_n(u,u) = \sum_{k=1}^n \langle L_k^{\alpha,*}u, \delta_k u\rangle \geq \sum_{k=1}^n \frac{g_k(\alpha)}{2\Gamma(2-\alpha)} \|\delta_k u\|_{L^2(\Omega)}^2 \geq 0,$$

with $g_k(\alpha)$ given in (3.4). □

Remark 1 If $\rho_k \geq \eta \approx 0.475329$ for all $k \geq 2$, then the condition (3.2) holds, for which the positive semidefiniteness of bilinear form $\mathscr{B}_n(u,u)$ (3.3) can be guaranteed.

4 Stability and Convergence of L2-1$_\sigma$ Method for Subdiffusion Equation

We consider the following subdiffusion equation:

$$\begin{aligned} \partial_t^\alpha u(t,x) &= \Delta u(t,x) + f(t,x), && (t,x)\in(0,\infty)\times\Omega,\\ u(t,x) &= 0, && (t,x)\in(0,\infty)\times\partial\Omega,\\ u(0,x) &= u^0(x), && x\in\Omega, \end{aligned} \tag{4.1}$$

where Ω is a bounded Lipschitz domain in $\mathbb{R}^d$. Given an arbitrary nonuniform mesh $\{\tau_k\}_{k\geq 1}$, the L2-1$_\sigma$ scheme of this subdiffusion equation is written as

$$\begin{aligned} L_k^{\alpha,*}u &= (1-\alpha/2)\Delta u^k + \alpha/2\Delta u^{k-1} + f^k, && \text{in } \Omega,\\ u^k &= 0, && \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

where $f^k = f(t_k^*, \cdot)$.

4.1 Global-in-Time H^1-Stability of L2-1$_\sigma$ Scheme for Subdiffusion Equation

Theorem 2 *Assume that $f(t,x)\in L^\infty([0,\infty);L^2(\Omega))\cap BV([0,\infty);L^2(\Omega))$ is a bounded variation function in time and $u^0\in H_0^1(\Omega)$. If the nonuniform mesh $\{\tau_k\}_{k\geq 1}$ satisfies (3.2) (for example $\rho_k\geq\eta\approx 0.475329$ for $k\geq 2$), then the numerical solution u^n of the L2-1$_\sigma$ scheme (4.2) satisfies the following global-in-time H^1-stability*

$$\|\nabla u^n\|_{L^2(\Omega)} \leq \|\nabla u^0\|_{L^2(\Omega)} + 2C_f C_\Omega,$$

where $C_f = 2\|f\|_{L^\infty([0,\infty);L^2(\Omega))} + \|f\|_{BV([0,\infty);L^2(\Omega))}$, C_Ω is the Sobolev embedding constant depending on Ω and the spatial dimension d.

Proof Multiplying (4.2) with $\delta_k u$, integrating over Ω, and summing up the derived equations over k yield

$$\begin{aligned} \sum_{k=1}^n \langle L_k^{\alpha,*}u, \delta_k u\rangle &= \sum_{k=1}^n \langle (1-\alpha/2)\Delta u^k + \alpha/2\Delta u^{k-1}, \delta_k u\rangle + \sum_{k=1}^n \langle f^k, \delta_k u\rangle\\ &= -\frac{1}{2}\|\nabla u^n\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\nabla u^0\|_{L^2(\Omega)}^2 - \frac{1-\alpha}{2}\sum_{k=1}^n \|\nabla\delta_k u\|_{L^2(\Omega)}^2\\ &\quad + \langle f^n, u^n\rangle - \langle f^1, u^0\rangle - \sum_{k=2}^n \langle \delta_k f, u^{k-1}\rangle. \end{aligned}$$

Applying the Cauchy–Schwarz inequality yields

$$\begin{aligned} &\langle f^n, u^n\rangle - \langle f^1, u^0\rangle + \sum_{k=2}^n \langle \delta_k f, u^{k-1}\rangle\\ &\leq \left(2\|f\|_{L^\infty([0,\infty);L^2(\Omega))} + \|f\|_{BV([0,\infty);L^2(\Omega))}\right)\max_{0\leq k\leq n}\|u^k\|_{L^2(\Omega)}\\ &\leq C_f C_\Omega \max_{0\leq k\leq n}\|\nabla u^k\|_{L^2(\Omega)}, \end{aligned}$$

where $C_f = 2\|f\|_{L^\infty([0,\infty);L^2(\Omega))} + \|f\|_{BV([0,\infty);L^2(\Omega))}$, and C_Ω is the Sobolev embedding constant depending on Ω and the spatial dimension. From Theorem 1, we then have for

$n \geq 1$,

$$\|\nabla u^n\|^2_{L^2(\Omega)} \leq \|\nabla u^0\|^2_{L^2(\Omega)} - (1-\alpha)\sum_{k=1}^{n} \|\nabla \delta_k u\|^2_{L^2(\Omega)} - \sum_{k=1}^{n} \frac{g_k(\alpha)}{\Gamma(2-\alpha)} \|\delta_k u\|^2_{L^2(\Omega)} + 2C_f C_\Omega \max_{0\leq k\leq n} \|\nabla u^k\|_{L^2(\Omega)} \leq \|\nabla u^0\|^2_{L^2(\Omega)} + 2C_f C_\Omega \max_{0\leq k\leq n} \|\nabla u^k\|_{L^2(\Omega)}. \tag{4.3}$$

For any $N \geq 1$, we take $\max_{0\leq n\leq N}$ on both sides of (4.3), to obtain

$$\max_{0\leq n\leq N} \|\nabla u^n\|^2_{L^2(\Omega)} \leq \|\nabla u^0\|^2_{L^2(\Omega)} + 2C_f C_\Omega \max_{0\leq n\leq N} \|\nabla u^n\|_{L^2(\Omega)},$$

which indicates

$$\max_{0\leq n\leq N} \|\nabla u^n\|_{L^2(\Omega)} \leq \|\nabla u^0\|_{L^2(\Omega)} + 2C_f C_\Omega.$$

The proof is completed. □

Remark 2 Assume that the solution of subdiffusion equation satisfies $u(t,x) \in C([0,\infty); H_0^1(\Omega) \cap C^1((0,\infty); H_0^1(\Omega))$ and the source term satisfies $f(t,x) \in C([0,\infty); L^2(\Omega))$, $\partial_t f(t,x) \in L^1([0,\infty); L^2(\Omega))$. For any fixed $T > 0$, multiplying the first equation of (4.1) with $\partial_t u(t,x)$ and integrating over $(0,T)\times\Omega$ yield

$$\int_0^T \int_\Omega \partial_t^\alpha u(t,x)\partial_t u(t,x)\,\mathrm{d}x\mathrm{d}t = \frac{1}{2}\int_0^T \int_\Omega \partial_t |\nabla u(t,x)|^2\,\mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega f(t,x)\partial_t u(t,x)\,\mathrm{d}x\mathrm{d}t.$$

According to [26],

$$\int_0^T \int_\Omega \partial_t^\alpha u(t,x)\partial_t u(t,x)\,\mathrm{d}x\mathrm{d}t \geq 0,$$

and moreover,

$$\begin{aligned} &\int_0^T \int_\Omega f(t,x)\partial_t u(t,x)\,\mathrm{d}x\mathrm{d}t \\ &= \left(\int_\Omega f(t,x)u(t,x)\,\mathrm{d}x\right)\Big|_0^T - \int_0^T \int_\Omega \partial_t f(t,x)u(t,x)\,\mathrm{d}x\mathrm{d}t \\ &\leq \left(2\|f\|_{L^\infty([0,\infty);L^2(\Omega))} + \int_0^\infty \|\partial_t f(t,x)\|_{L^2(\Omega)}\,\mathrm{d}t\right) C_\Omega \|\nabla u\|_{L^\infty([0,T];L^2(\Omega))} \\ &=: C_f^{\mathrm{cont}} C_\Omega \|\nabla u\|_{L^\infty([0,T];L^2(\Omega))}. \end{aligned}$$

Thus we derive the H^1-stability at the continuous level

$$\|\nabla u(T,x)\|_{L^2(\Omega)} \leq \|\nabla u(0,x)\|_{L^2(\Omega)} + 2C_f^{\mathrm{cont}} C_\Omega, \quad \forall\, T > 0,$$

which corresponds to our H^1-stability result in Theorem 2 for the L2-1_σ scheme of the subdiffusion equation (4.1).

Remark 3 In the case of $\alpha = 1$, i.e., the standard diffusion equation, the energy stability (or H^1-stability) has been established for the second order BDF2 schemes in [19, Theorem 2.1] and for the third order BDF3 schemes in [18, Theorem 3.1] on general nonuniform meshes.

4.2 Sharp Convergence of L2-1$_\sigma$ Scheme for Subdiffusion Equation

We show the error estimate of the L2-1$_\sigma$ scheme (4.2) for the subdiffusion equation (4.1), that is different from the one in [14, 15]. To be precise we will reduce the restriction on time step ratios from $\rho_k \geq 4/7$ in [15] to $\rho_k \geq 0.475329$. We first reformulate the discrete fractional operator (2.3):

$$L_k^{\alpha,*}u = \frac{1}{\Gamma(1-\alpha)}\left([\mathbf{M}]_{k,k}u^k - \sum_{j=2}^{k}([\mathbf{M}]_{k,j} - [\mathbf{M}]_{k,j-1})u^{j-1} - [\mathbf{M}]_{k,1}u^0\right),$$

where $\mathbf{M}$ is given by (3.5). We now give some properties on $[\mathbf{M}]_{k,j}$.

Lemma 2 *Under the condition* (3.2)*, the following properties of* $[\mathbf{M}]_{k,j}$ *given by* (3.5) *hold:*

(Q1)

$$[\mathbf{M}]_{k,j} \geq \frac{\rho_*}{(1+\rho_*)\tau_j}\int_{t_{j-1}}^{\min\{t_j,t_k^*\}}(t_k^*-s)^{-\alpha}\,\mathrm{d}s, \quad 1 \leq j \leq k. \tag{4.4}$$

(Q2) *For all* $2 \leq j \leq k-1$,

$$[\mathbf{M}]_{k,j} - [\mathbf{M}]_{k,j-1} \geq \frac{\alpha\tau_j}{\tau_j+\tau_{j+1}}\int_0^1(\tau_j+\tau_{j+1}-s\tau_j)(1-s)(t_k^*-t_{j-1}-s\tau_j)^{-\alpha-1}\,\mathrm{d}s,$$

and

$$[\mathbf{M}]_{k,k} - [\mathbf{M}]_{k,k-1} \geq \frac{\alpha}{2(1-\alpha)(\sigma\tau_k)^\alpha}.$$

(Q3) *Moreover, if* $\rho_k \geq \eta \approx 0.475329$ *for all* $k \geq 2$*, then*

$$\frac{1-\alpha}{\sigma}[\mathbf{M}]_{k,k} - [\mathbf{M}]_{k,k-1} \geq 0.$$

Here η *is the real root of* $1-3\rho^2(1+\rho)=0$.

Proof From (3.5), for $1 \leq j \leq k-1$,

$$\begin{aligned}\mathbf{M}t_{k,j} \geq -a_j^{(k)} &= \int_0^1 \frac{2\tau_j(1-\theta)+\tau_{j+1}}{(\tau_j+\tau_{j+1})(t_k^*-(t_{j-1}+\theta\tau_j))^\alpha}\,\mathrm{d}\theta \\ &\geq \frac{\rho_{j+1}}{1+\rho_{j+1}}\int_0^1\frac{1}{(t_k^*-(t_{j-1}+\theta\tau_j))^\alpha}\,\mathrm{d}\theta \geq \frac{\rho_*}{(1+\rho_*)\tau_j}\int_{t_{j-1}}^{t_j}(t_k^*-s)^{-\alpha}\mathrm{d}s,\end{aligned} \tag{4.5}$$

and for $j=k$,

$$[\mathbf{M}]_{k,k} = c_{k-1}^{(k)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} \geq \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} = \frac{1}{\tau_k}\int_{t_{k-1}}^{t_k^*}(t_k^*-s)^{-\alpha}\mathrm{d}s.$$

The inequality (4.4) holds.

For $2 \leq j \leq k-1$, according to (3.7) – (3.9),

$$\begin{aligned}
&[\mathbf{M}]_{k,j} - [\mathbf{M}]_{k,j-1} \\
&= \frac{\alpha\tau_{j-1}^3}{\tau_j(\tau_{j-1}+\tau_j)} \int_0^1 s(1-s)(t_k^* - t_{j-1} + s\tau_{j-1})^{-\alpha-1}\,\mathrm{d}s \\
&- \frac{\alpha\tau_{j-2}^3}{\tau_{j-1}(\tau_{j-2}+\tau_{j-1})} \int_0^1 s(1-s)(t_k^* - t_{j-2} + s\tau_{j-2})^{-\alpha-1}\,\mathrm{d}s \\
&+ \frac{\alpha\tau_{j-1}}{\tau_{j-1}+\tau_j} \int_0^1 (\tau_{j-1}+\tau_j+s\tau_{j-1})(1-s)(t_k^* - t_{j-1} + s\tau_{j-1})^{-\alpha-1}\,\mathrm{d}s \\
&+ \frac{\alpha\tau_j}{\tau_j+\tau_{j+1}} \int_0^1 (\tau_j+\tau_{j+1}-s\tau_j)(1-s)(t_k^* - t_{j-1} - s\tau_j)^{-\alpha-1}\,\mathrm{d}s \\
&\geq \frac{\alpha\tau_j}{\tau_j+\tau_{j+1}} \int_0^1 (\tau_j+\tau_{j+1}-s\tau_j)(1-s)(t_k^* - t_{j-1} - s\tau_j)^{-\alpha-1}\,\mathrm{d}s,
\end{aligned}$$

under the condition (3.2) (for simplicity we make a convention that $\tau_0 = 0$). Note that (3.2) indicates the sum of first three terms is positive, using the techniques in (3.17). When $j = k = 2$, we obtain from (3.7)

$$[\mathbf{M}]_{2,2} - [\mathbf{M}]_{2,1} = c_1^{(2)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} + a_1^{(2)} \geq \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_2^\alpha} - \frac{1}{(\sigma\tau_2)^\alpha} = \frac{\alpha}{2(1-\alpha)(\sigma\tau_2)^\alpha},$$

where we use the fact $\sigma = 1 - \alpha/2$. Moreover when $j = k \geq 3$, we have

$$\begin{aligned}
&[\mathbf{M}]_{k,k} - [\mathbf{M}]_{k,k-1} \\
&= \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} + (c_{k-1}^{(k)} - c_{k-2}^{(k)} + a_{k-1}^{(k)}) \\
&= \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} + \Bigg(\frac{\alpha\tau_{k-1}^3}{\tau_k(\tau_{k-1}+\tau_k)} \int_0^1 s(1-s)(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\,\mathrm{d}s \\
&- \frac{\alpha\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})} \int_0^1 s(1-s)(t_k^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1}\,\mathrm{d}s \\
&- (\sigma\tau_k)^{-\alpha} + \frac{\alpha\tau_{k-1}}{\tau_{k-1}+\tau_k} \int_0^1 (\tau_{k-1}+\tau_k+s\tau_{k-1})(1-s)(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\,\mathrm{d}s \Bigg) > \\
&\frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_k^\alpha} - \frac{1}{(\sigma\tau_k)^\alpha} = \frac{\alpha}{2(1-\alpha)(\sigma\tau_k)^\alpha},
\end{aligned}$$

when the condition (3.2) holds. This inequality coincide with (3.17) by replacing n with k.

For the property (Q3), the case of $k = 2$ is trivial. In the case of $k \geq 3$, we have

$$\begin{aligned}
&\frac{1-\alpha}{\sigma}[\mathbf{M}]_{k,k} - [\mathbf{M}]_{k,k-1} \\
&\geq (\sigma\tau_k)^{-\alpha} - c_{k-2}^{(k)} + a_{k-1}^{(k)} \\
&= (\sigma\tau_k)^{-\alpha} - \frac{\alpha\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})} \int_0^1 s(1-s)(t_k^* - t_{k-2} + s\tau_{k-2})^{-\alpha-1}\,\mathrm{d}s \\
&- (\sigma\tau_k)^{-\alpha} + \frac{\alpha\tau_{k-1}}{\tau_{k-1}+\tau_k} \int_0^1 (\tau_{k-1}+\tau_k+s\tau_{k-1})(1-s)(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\,\mathrm{d}s > \\
&\alpha \left(\frac{\tau_{k-1}(4\tau_{k-1}+3\tau_k)}{\tau_{k-1}+\tau_k} - \frac{\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})} \right) \int_0^1 s(1-s)(t_k^* - t_{k-1} + s\tau_{k-1})^{-\alpha-1}\,\mathrm{d}s
\end{aligned}$$

$$\geq 0,$$

where we use the facts

$$\begin{aligned}&\int_0^1 (\tau_{k-1}+\tau_k+s\tau_{k-1})(1-s)(t_k^*-t_{k-1}+s\tau_{k-1})^{-\alpha-1}\,\mathrm{d}s\\&\quad\geq (4\tau_{k-1}+3\tau_k)\int_0^1 s(1-s)(t_k^*-t_{k-1}+s\tau_{k-1})^{-\alpha-1}\,\mathrm{d}s,\\&\quad(t_k^*-t_{k-1}+s\tau_{k-1})^{-\alpha-1}\geq (t_k^*-t_{k-2}+s\tau_{k-2})^{-\alpha-1},\end{aligned}$$

and

$$\frac{\tau_{k-1}(4\tau_{k-1}+3\tau_k)}{\tau_{k-1}+\tau_k}-\frac{\tau_{k-2}^3}{\tau_{k-1}(\tau_{k-2}+\tau_{k-1})}\geq 0,$$

when $\rho_k \geq \eta \approx 0.475329$ for all $k \geq 2$. □

Consider the following three standard Lagrange interpolation operators with the following interpolation points:

$$\Pi_{1,j}: t_{j-1}, t_j, \quad \Pi_{2,j}: t_{j-1}, t_j, t_{j+1}, \quad \Pi_{2,j}^*: t_{j-1}, t_j^*, t_j.$$

As stated in [12], when $\sigma = 1-\alpha/2$,

$$\int_{t_{k-1}}^{t_k^*} (\Pi_{1,k}v - \Pi_{2,k}^* v)'(s)(t_k^*-s)^{-\alpha}\,\mathrm{d}s = 0.$$

We now analyze the approximation error of the discrete fractional operator in the following lemma.

Lemma 3 *Given a function u satisfying $|\partial_t^m u(t)| \leq C_m(1+t^{\alpha-m})$ for $m = 1, 3$ and nonuniform mesh $\{\tau_k\}_{k\geq 1}$ satisfying condition (3.2), the approximation error is given by*

$$r_k := \frac{1}{\Gamma(1-\alpha)}\int_0^{t_k^*}(t_k^*-s)^{-\alpha}\partial_s[u(s)-I_2u(s)]\,\mathrm{d}s, \quad k\geq 1, \tag{4.6}$$

where $I_2u = \Pi_{2,j}u$ on (t_{j-1}, t_j) for $j < k$ and $I_2u = \Pi_{2,k}^*u$ on (t_{k-1}, t_k^*). Then for $k \geq 1$,

$$|r_k| \leq \frac{C}{\Gamma(1-\alpha)}\left([\mathbf{M}]_{k,1}\big(t_2^\alpha/\alpha+t_2\big)+\sum_{j=2}^{k}([\mathbf{M}]_{k,j}-[\mathbf{M}]_{k,j-1})(1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3\right), \tag{4.7}$$

where C is a constant depending on C_m for $m = 1, 3$.

Proof The case of $k = 1$ is not difficult to prove. We now consider the case of $k \geq 2$. Let $\chi(s) := u - I_2u$. Three subcases are discussed in the following content.

Subcase 1 On the interval (t_0, t_1), we have

$$\partial_s I_2u(s) = \frac{2s-t_1-t_2}{\tau_1(\tau_1+\tau_2)}u(t_0) - \frac{2s-t_2}{\tau_1\tau_2}u(t_1) + \frac{2s-t_1}{\tau_2(\tau_1+\tau_2)}u(t_2)$$

that is linear w.r.t. s. Then we have

$$|\partial_s I_2u(s)| \leq \max\{|\partial_s I_2u(t_0)|, |\partial_s I_2u(t_1)|\} \leq C_1\frac{1+\rho_2}{\tau_1\rho_2}(t_2+t_2^\alpha/\alpha),$$

where we use the facts

$$
\begin{aligned}
\partial_s I_2 u(t_0) &= -\frac{2\tau_1+\tau_2}{\tau_1(\tau_1+\tau_2)}u(t_0) + \frac{\tau_1+\tau_2}{\tau_1\tau_2}u(t_1) - \frac{\tau_1}{\tau_2(\tau_1+\tau_2)}u(t_2)\\
&= -\frac{2\tau_1+\tau_2}{\tau_1(\tau_1+\tau_2)}(u(t_0)-u(t_1)) + \frac{\tau_1}{\tau_2(\tau_1+\tau_2)}(u(t_1)-u(t_2))\\
&\le \left(\frac{2\tau_1+\tau_2}{\tau_1(\tau_1+\tau_2)} + \frac{\tau_1}{\tau_2(\tau_1+\tau_2)}\right)\max\{|u(t_0)-u(t_1)|, |u(t_1)-u(t_2)|\}\\
&= \frac{\tau_1+\tau_2}{\tau_1\tau_2}\max\{|u(t_0)-u(t_1)|, |u(t_1)-u(t_2)|\},\\
\partial_s I_2 u(t_1) &= -\frac{\tau_2}{\tau_1(\tau_1+\tau_2)}u(t_0) - \frac{\tau_1-\tau_2}{\tau_1\tau_2}u(t_1) + \frac{\tau_1}{\tau_2(\tau_1+\tau_2)}u(t_2)\\
&= -\frac{\tau_2}{\tau_1(\tau_1+\tau_2)}(u(t_0)-u(t_1)) - \frac{\tau_1}{\tau_2(\tau_1+\tau_2)}(u(t_1)-u(t_2))\\
&\le \left(\frac{\tau_2}{\tau_1(\tau_1+\tau_2)} + \frac{\tau_1}{\tau_2(\tau_1+\tau_2)}\right)\max\{|u(t_0)-u(t_1)|, |u(t_1)-u(t_2)|\}\\
&= \frac{\tau_1^2+\tau_2^2}{\tau_1\tau_2(\tau_1+\tau_2)}\max\{|u(t_0)-u(t_1)|, |u(t_1)-u(t_2)|\},\\
|u(t_0)-u(t_1)| &= |\int_0^{t_1}\partial_s u(s)\,\mathrm{d}s| \le C_1(\tau_1+\tau_1^\alpha/\alpha),\\
|u(t_1)-u(t_2)| &= |\int_{t_1}^{t_2}\partial_s u(s)\,\mathrm{d}s| \le C_1(\tau_2+(t_2^\alpha-t_1^\alpha)/\alpha).
\end{aligned}
$$

Therefore, we have

$$
|\partial_s\chi(s)| \le |\partial_s u| + |\partial_s I_2 u| \le C_1\left(s^{\alpha-1} + 1 + \frac{1+\rho_2}{\tau_1\rho_2}(t_2+t_2^\alpha/\alpha)\right),
$$

which yields

$$
\begin{aligned}
&|\frac{1}{\Gamma(1-\alpha)}\int_0^{t_1}(t_k^*-s)^{-\alpha}\partial_s\chi(s)\,\mathrm{d}s|\\
&\le \frac{C_1}{\Gamma(1-\alpha)}\left(\int_0^{t_1}s^{\alpha-1}(t_k^*-s)^{-\alpha}\,\mathrm{d}s + \frac{\tau_1+(1+\rho_2)/\rho_2(t_2+t_2^\alpha/\alpha)}{\tau_1}\int_0^{t_1}(t_k^*-s)^{-\alpha}\,\mathrm{d}s\right)\\
&\le \frac{C_1}{\Gamma(1-\alpha)}\left(\frac{\tau_1^\alpha}{\alpha(t_k^*-\tau_1)^\alpha} + \frac{\tau_1+(1+\rho_2)/\rho_2(t_2+t_2^\alpha/\alpha)}{\tau_1}\int_0^{t_1}(t_k^*-s)^{-\alpha}\,\mathrm{d}s\right)\\
&\le \frac{C(t_2^\alpha/\alpha+t_2)}{\Gamma(1-\alpha)}[\mathbf{M}]_{k,1},
\end{aligned}
\tag{4.8}
$$

where C is an absolute constant only depending on C_1. In the last inequality of (4.8), we use the fact

$$
\begin{aligned}
[\mathbf{M}]_{k,1} &\ge \frac{\rho_2}{(1+\rho_2)\tau_1}\int_0^{t_1}(t_k^*-s)^{-\alpha}\,\mathrm{d}s \ge \frac{\rho_2}{(1+\rho_2)(t_k^*)^\alpha}\\
&\ge \frac{\rho_2^{1+\alpha}}{(1+\rho_2)(2+\rho_2)^\alpha(t_k^*-\tau_1)^\alpha} \ge \frac{\rho_*^{1+\alpha}}{(1+\rho_*)(2+\rho_*)^\alpha(t_k^*-\tau_1)^\alpha}
\end{aligned}
$$

obtained from the inequality (4.5).

Subcase 2 On the interval (t_{j-1}, t_j), $2 \le j \le k-1$,

$$|\chi(s)| = |\frac{u^{(3)}(\xi)}{6}(s-t_{j-1})(s-t_j)(s-t_{j+1})| \le C_3(1+t_{j-1}^{\alpha-3})(s-t_{j-1})(s-t_j)(s-t_{j+1}),$$

where $\xi \in (t_{j-1}, t_{j+1})$. Then we have

$$\begin{aligned}
&|\frac{1}{\Gamma(1-\alpha)}\int_{t_{j-1}}^{t_j}(t_k^*-s)^{-\alpha}\partial_s\chi(s)\,\mathrm{d}s| = |\frac{-\alpha}{\Gamma(1-\alpha)}\int_{t_{j-1}}^{t_j}(t_k^*-s)^{-\alpha-1}\chi(s)\,\mathrm{d}s| \\
&\le \frac{C_3\alpha(1+t_{j-1}^{\alpha-3})}{\Gamma(1-\alpha)}\int_{t_{j-1}}^{t_j}(t_k^*-s)^{-\alpha-1}(s-t_{j-1})(s-t_j)(s-t_{j+1})\,\mathrm{d}s \\
&= \frac{C_3\alpha(1+t_{j-1}^{\alpha-3})\tau_j^3}{\Gamma(1-\alpha)}\int_0^1 s(\tau_j+\tau_{j+1}-s\tau_j)(1-s)(t_k^*-t_{j-1}-s\tau_j)^{-\alpha-1}\,\mathrm{d}s \\
&\le \frac{C_3(1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3}{\Gamma(1-\alpha)}([\mathbf{M}]_{k,j}-[\mathbf{M}]_{k,j-1}),
\end{aligned} \tag{4.9}$$

from (Q2) in Lemma 2.

Subcase 3 On the interval (t_{k-1}, t_k^*),

$$|\chi(s)| \le C_3(1+t_{k-1}^{\alpha-3})(s-t_{k-1})(t_k^*-s)(t_k-s) \le C_3(1+t_{k-1}^{\alpha-3})\tau_k^2(t_k^*-s),$$

which yields

$$\begin{aligned}
&|\frac{1}{\Gamma(1-\alpha)}\int_{t_{k-1}}^{t_k^*}(t_k^*-s)^{-\alpha}\partial_s\chi(s)\,\mathrm{d}s| = |\frac{-\alpha}{\Gamma(1-\alpha)}\int_{t_{k-1}}^{t_k^*}(t_k^*-s)^{-\alpha-1}\chi(s)\,\mathrm{d}s| \\
&\le \frac{C_3\alpha(1+t_{k-1}^{\alpha-3})\tau_k^2}{\Gamma(1-\alpha)}\int_{t_{k-1}}^{t_k^*}(t_k^*-s)^{-\alpha}\,\mathrm{d}s = \frac{2C_3\sigma(1+t_{k-1}^{\alpha-3})\tau_k^3}{\Gamma(1-\alpha)}\frac{\alpha}{2(1-\alpha)(\sigma\tau_k)^\alpha} \\
&\le \frac{2C_3\sigma(1+t_{k-1}^{\alpha-3})\tau_k^3}{\Gamma(1-\alpha)}([\mathbf{M}]_{k,k}-[\mathbf{M}]_{k,k-1})
\end{aligned} \tag{4.10}$$

from (Q2) in Lemma 2.

Combining (4.8), (4.9) and (4.10) we obtain the estimation (4.7) of approximation error. □

Theorem 3 *Assume that* $u \in C^3((0,T], H_0^1(\Omega))$ *and* $|\partial_t^m u(t)| \le C_m(1+t^{\alpha-m})$, *for* $m = 1, 2, 3$ *for* $0 < t \le T$. *If the nonuniform mesh satisfies* $\rho_k \ge \eta \approx 0.475329$, *then the numerical solutions of L2-1*$_\sigma$ *scheme* (4.2) *have the following global error estimate*

$$\begin{aligned}
&\max_{1\le k\le n}\|u(t_k)-u^k\|_{L^2(\Omega)} \\
&\le C\Bigg(t_2^\alpha/\alpha + t_2 + \frac{1}{1-\alpha}\max_{2\le k\le n}(1+\rho_{k+1})(1+t_{k-1}^{\alpha-3})(t_{k-1}^*)^\alpha\tau_k^3\tau_{k-1}^{-\alpha} \\
&\quad + (\tau_1^\alpha/\alpha+\tau_1)\tau_1^{\alpha/2} + \sqrt{\Gamma(1-\alpha)}\max_{2\le k\le n}(t_k^*)^{\alpha/2}(1+t_{k-1}^{\alpha-2})\tau_k^2\Bigg),
\end{aligned}$$

where C *is a constant depending only on* C_m, $m = 1, 2, 3$ *and* Ω.

Proof Let $e^k := u(t_k) - u^k$. We have

$$L_k^{\alpha,*}e = \Delta e_k^* - r_k + \Delta R_k^*, \tag{4.11}$$

where $e_k^* := (1-\alpha/2)e^k + \alpha/2e^{k-1}$, r_k is given in (4.6), and $R_k^* := u(t_k^*) - ((1-\alpha/2)u(t_k) + \alpha/2u(t_{k-1}))$. Multiplying (4.11) with e_k^* and integrating over Ω yield

$$\langle L_k^{\alpha,*} e, e_k^* \rangle = -\|\nabla e_k^*\|_{L^2(\Omega)}^2 - \langle r_k, e_k^* \rangle - \langle \nabla R_k^*, \nabla e_k^* \rangle. \tag{4.12}$$

According to [2, Lemma 1] as well as Lemma 2, we can derive

$$\begin{aligned}\langle L_k^{\alpha,*} e, e_k^* \rangle &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{k} [\mathbf{M}]_{k,j} \langle (e^j - e^{j-1}), (1-\alpha/2)e^k + \alpha/2e^{k-1} \rangle \\ &\geq \frac{1}{2\Gamma(1-\alpha)} \sum_{j=1}^{k} [\mathbf{M}]_{k,j} \left(\|e^j\|_{L^2(\Omega)}^2 - \|e^{j-1}\|_{L^2(\Omega)}^2 \right).\end{aligned}$$

Applying Cauchy-Schwarz inequality in (4.12) yields

$$\begin{aligned}\textstyle\sum_{j=1}^{k} [\mathbf{M}]_{k,j} \left(\|e^j\|_{L^2(\Omega)}^2 - \|e^{j-1}\|_{L^2(\Omega)}^2 \right) &\leq 2\Gamma(1-\alpha) \|r_k\|_{L^2(\Omega)} \|e_k^*\|_{L^2(\Omega)} \\ &\quad + \Gamma(1-\alpha) \|R_k^*\|_{H^1(\Omega)}^2.\end{aligned} \tag{4.13}$$

We define a lower triangular $\mathbf{P}$ matrix such that

$$\mathbf{P}\mathbf{M} = \mathbf{E}_{\mathrm{L}}$$

where

$$\mathbf{E}_{\mathrm{L}} = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & \vdots & \ddots & \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

In other words,

$$\sum_{l=j}^{k} [\mathbf{P}]_{k,l} [\mathbf{M}]_{l,j} = 1, \quad \forall 1 \leq j \leq k \leq n.$$

Here $\mathbf{P}$ is called complementary discrete convolution kernel in the work [14]. It can be easily checked that $[\mathbf{P}]_{k,l} \geq 0$ due to the monotonicity properties of $\mathbf{M}$. From (4.13) we can derive that $\forall 1 \leq k \leq n$,

$$\begin{aligned}\|e^k\|_{L^2(\Omega)}^2 &\leq 2\Gamma(1-\alpha) \sum_{l=1}^{k} [\mathbf{P}]_{k,l} \|r_l\|_{L^2(\Omega)} \|e_l^*\|_{L^2(\Omega)} + \Gamma(1-\alpha) \sum_{l=1}^{k} [\mathbf{P}]_{k,l} \|R_l^*\|_{H^1(\Omega)}^2 \\ &\leq 2\Gamma(1-\alpha) \left(\max_{1 \leq l \leq k} \|e_l^*\|_{L^2(\Omega)} \right) \sum_{l=1}^{k} [\mathbf{P}]_{k,l} \|r_l\|_{L^2(\Omega)} + \Gamma(1-\alpha) \sum_{l=1}^{k} [\mathbf{P}]_{k,l} \|R_l^*\|_{H^1(\Omega)}^2,\end{aligned} \tag{4.14}$$

where we use

$$\begin{aligned}&\sum_{l=1}^{k} [\mathbf{P}]_{k,l} \sum_{j=1}^{l} [\mathbf{M}]_{l,j} \left(\|e^j\|_{L^2(\Omega)}^2 - \|e^{j-1}\|_{L^2(\Omega)}^2 \right) \\ &= \sum_{j=1}^{k} \left(\|e^j\|_{L^2(\Omega)}^2 - \|e^{j-1}\|_{L^2(\Omega)}^2 \right) \sum_{l=j}^{k} [\mathbf{P}]_{k,l} [\mathbf{M}]_{l,j}\end{aligned}$$

$$= \sum_{j=1}^{k} \left(\|e^j\|^2_{L^2(\Omega)} - \|e^{j-1}\|^2_{L^2(\Omega)} \right) = \|e^k\|^2_{L^2(\Omega)}.$$

According to Lemma 3,

$$\begin{aligned}
&\Gamma(1-\alpha) \sum_{l=1}^{k} [\mathbf{P}]_{k,l} \|r_l\| \\
&\le C|\Omega| \sum_{l=1}^{k} [\mathbf{P}]_{k,l} \left([\mathbf{M}]_{l,1} (t_2^{\alpha}/\alpha + t_2) + \sum_{j=2}^{l} ([\mathbf{M}]_{l,j} - [\mathbf{M}]_{l,j-1})(1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3 \right) \\
&= C|\Omega| \left((t_2^{\alpha}/\alpha + t_2) + \sum_{j=2}^{k} (1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3 \sum_{l=j}^{k} [\mathbf{P}]_{k,l} ([\mathbf{M}]_{l,j} - [\mathbf{M}]_{l,j-1}) \right) \\
&= C|\Omega| \left((t_2^{\alpha}/\alpha + t_2) + \sum_{j=2}^{k} (1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3 [\mathbf{P}]_{k,j-1} [\mathbf{M}]_{j-1,j-1} \right) \\
&= C|\Omega| \left((t_2^{\alpha}/\alpha + t_2) + \sum_{j=2}^{k} [\mathbf{P}]_{k,j-1} [\mathbf{M}]_{j-1,1} \frac{[\mathbf{M}]_{j-1,j-1}}{[\mathbf{M}]_{j-1,1}} (1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3 \right) \\
&\le C|\Omega| \left((t_2^{\alpha}/\alpha + t_2) + \max_{2\le j\le k} \frac{[\mathbf{M}]_{j-1,j-1}}{[\mathbf{M}]_{j-1,1}} (1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})\tau_j^3 \right) \\
&\le C|\Omega| \left((t_2^{\alpha}/\alpha + t_2) + \frac{1}{1-\alpha} \max_{2\le j\le k} (1+\rho_{j+1})(1+t_{j-1}^{\alpha-3})(t_{j-1}^{*})^{\alpha} \tau_j^3 \tau_{j-1}^{-\alpha} \right),
\end{aligned}$$

where C is a constant only depending on C_m. The last inequality is obtained by the following upper bound of $[\mathbf{M}]_{j,j}$ and lower bound of $[\mathbf{M}]_{j,1}$:

$$\begin{aligned}
\mathbf{M}t \quad {}_{j,j} &= c_{j-1}^{(j)} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_j^{\alpha}} \qquad (4.15) \\
&= \int_0^1 \frac{\tau_{j-1}^2 (2\theta - 1)}{\tau_j(\tau_{j-1}+\tau_j)(t_j^{*} - (t_{j-2} + \theta\tau_{j-1}))^{\alpha}} \mathrm{d}\theta + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_j^{\alpha}} \\
&\le \frac{1}{\rho_j(1+\rho_j)(\sigma\tau_j)^{\alpha}} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_j^{\alpha}} \le \frac{1}{\eta(1+\eta)(\sigma\tau_j)^{\alpha}} + \frac{\sigma^{1-\alpha}}{(1-\alpha)\tau_j^{\alpha}}, \\
[\mathbf{M}]_{j,1} &\ge \frac{\eta}{(1+\eta)\tau_1} \int_0^{t_1} (t_j^{*} - s)^{-\alpha} \mathrm{d}s \ge \frac{\eta}{(1+\eta)(t_j^{*})^{\alpha}},
\end{aligned}$$

where we use (Q1) in Lemma 2 for the inequality of $[\mathbf{M}]_{j,1}$.

Using the Taylor formula with integral remainder for R_j^{*} gives

$$R_j^{*} = -\alpha/2 \int_{t_{j-1}}^{t_j^{*}} (s - t_{j-1}) u''(s)\, \mathrm{d}s - (1-\alpha/2) \int_{t_j^{*}}^{t_j} (t_j - s) u''(s)\, \mathrm{d}s, \quad 1 \le j \le k.$$

Under the regularity assumption, we have

$$\|R_1^{*}\|_{H^1(\Omega)} \le C(\tau_1^{\alpha}/\alpha + \tau_1), \quad \|R_j^{*}\|_{H^1(\Omega)} \le C(1 + t_{j-1}^{\alpha-2})\tau_j^2, \quad 2 \le j \le k.$$

Then we have

$$\begin{aligned}
&\sum_{l=1}^{k}[\mathbf{P}]_{k,l}\|R_l^*\|_{H^1(\Omega)}^2 \\
&\quad \le C\left([\mathbf{P}]_{k,1}[\mathbf{M}]_{1,1}\frac{1}{[\mathbf{M}]_{1,1}}(\tau_1^\alpha/\alpha+\tau_1)^2+\sum_{l=2}^{k}[\mathbf{P}]_{k,l}[\mathbf{M}]_{l,2}\frac{1}{[\mathbf{M}]_{l,2}}\left((1+t_{l-1}^{\alpha-2})\tau_l^2\right)^2\right) \\
&\quad \le C\left(\frac{1}{[\mathbf{M}]_{1,1}}(\tau_1^\alpha/\alpha+\tau_1)^2+\max_{2\le l\le k}\frac{1}{[\mathbf{M}]_{l,2}}\left((1+t_{l-1}^{\alpha-2})\tau_l^2\right)^2\right) \\
&\quad \le C\left((1-\alpha)\tau_1^\alpha(\tau_1^\alpha/\alpha+\tau_1)^2+\max_{2\le l\le k}(t_l^*)^\alpha((1+t_{l-1}^{\alpha-2})\tau_l^2)^2\right),
\end{aligned}$$

where we use $[\mathbf{M}]_{l,2}\ge[\mathbf{M}]_{l,1}$ and (4.15).

Taking the max for $1\le k\le n$ on both sides of (4.14), we can derive

$$\begin{aligned}
\max_{1\le k\le n}\|e_k\|_{L^2(\Omega)} \le C\Bigg(&(t_2^\alpha/\alpha+t_2)+\frac{1}{1-\alpha}\max_{2\le k\le n}(1+\rho_{k+1})(1+t_{k-1}^{\alpha-3})(t_{k-1}^*)^\alpha\tau_k^3\tau_{k-1}^{-\alpha} \\
&+(\tau_1^\alpha/\alpha+\tau_1)\tau_1^{\alpha/2}+\sqrt{\Gamma(1-\alpha)}\max_{2\le k\le n}(t_k^*)^{\alpha/2}(1+t_{k-1}^{\alpha-2})\tau_k^2\Bigg).
\end{aligned} \tag{4.16}$$

The proof is completed. □

In the case of graded mesh with grading parameter r,

$$t_j=\left(\frac{j}{K}\right)^r T,\quad \tau_j=t_j-t_{j-1}=\left[\left(\frac{j}{K}\right)^r-\left(\frac{j-1}{K}\right)^r\right]T, \tag{4.17}$$

where K is the total time step number, $1\le j\le K,\ t_K=T$. As a consequence, the two terms after max operations in (4.16) can be estimated as follows:

$$\begin{aligned}
&(1+\rho_{k+1})(1+t_{k-1}^{\alpha-3})(t_{k-1}^*)^\alpha\tau_k^3\tau_{k-1}^{-\alpha}\le Ct_{k-1}^{2\alpha-3}\tau_k^{3-\alpha} \\
&\quad = Ct_{k-1}^{2\alpha-3}(t_k-t_{k-1})^{3-\alpha}=C(t_{k-1})^\alpha(t_k/t_{k-1}-1)^{3-\alpha} \\
&\quad = Ct_{k-1}^\alpha((1+1/(k-1))^r-1)^{3-\alpha} \\
&\quad \le C\,r^{3-\alpha}T^\alpha\frac{(k-1)^{r\alpha-(3-\alpha)}}{K^{r\alpha}}=\frac{C_{T,1}}{K^{\min\{r\alpha,3-\alpha\}}}
\end{aligned} \tag{4.18}$$

and

$$\begin{aligned}
&(t_k^*)^{\alpha/2}(1+t_{k-1}^{\alpha-2})\tau_k^2\le Ct_{k-1}^{\alpha-2}\tau_k^2=Ct_{k-1}^{\alpha-2}(t_k-t_{k-1})^2=Ct_{k-1}^\alpha(t_k/t_{k-1}-1)^2 \\
&\quad =CT^\alpha\left(\frac{k-1}{K}\right)^{r\alpha}((1+1/(k-1))^r-1)^2\le C\,r^2T^\alpha\frac{(k-1)^{r\alpha-2}}{K^{r\alpha}}=\frac{C_{T,2}}{K^{\min\{r\alpha,2\}}}.
\end{aligned} \tag{4.19}$$

In (4.18) and (4.19), $C_{T,1}$ and $C_{T,2}$ only depend on T. Therefore, if u satisfies the regularity assumptions in Theorem 3, then we have the following error estimate of numerical solutions of the L2-1_σ scheme on the graded mesh with grading parameter r:

$$\max_{1\le k\le K}\|u(t_k)-u^k\|_{L^2(\Omega)}\le\frac{\tilde{C}}{K^{\min\{r\alpha,2\}}}. \tag{4.20}$$

where $\tilde{C}$ depends on C_m with $m=1,2,3,\alpha$ and Ω.

Table 1 $\max_{1\le k\le K}\|u(t_k)-u^k\|_{L^2(\Omega)}$ for the graded meshes with different grading parameters and time step numbers where $\alpha=0.3$

	$K=40$	$K=80$	$K=160$	$K=320$	$K=480$	$K=640$
$r=1$	2.3600e−2	2.2505e−2	2.0661e−2	1.8461e−2	1.7117e−2	1.6165e−2
order	–	0.0685	0.1233	0.1625	0.1863	0.1988
$r=2$	1.3254e−2	9.4767e−3	6.5872e−3	4.4967e−3	3.5761e−3	3.0338e−3
order	–	0.4841	0.5247	0.5508	0.5650	0.5716
$r=2/\alpha$	2.7182e−4	7.4873e−5	1.9983e−5	5.2316e−6	2.3816e−6	1.3655e−6
order	–	1.8601	1.9056	1.9335	1.9408	1.9334
$r=3/\alpha$	5.6542e−4	1.5847e−4	4.2808e−5	1.1281e−5	5.1370e−6	2.9371e−6
order	–	1.8351	1.8883	1.9239	1.9403	1.9432

Remark 4 When $\alpha\to 1^-$, the constant $\tilde{C}$ in (4.20) will tend to infinity. However, using the technique by Chen-Stynes in [4], one can obtain α-robust error estimate in the sense that $\tilde{C}$ won't tend to infinity when $\alpha\to 1^-$.

5 Numerical Tests

In this section, we provide some numerical tests on the L2-1_σ scheme (4.2) of the subdiffusion equation (4.1).

As in [3, 15], the discrete coefficients $a_j^{(k)}$ and $c_j^{(k)}$ in (2.2) are computed by adaptive Gauss-Kronrod quadrature, to avoid roundoff error problems.

5.1 1D Example

We first test the convergence rate of an 1D example, where $\Omega=[0,2\pi]$, $T=1$, $u^0(x)\equiv 0$, and $f(t,x)=(\Gamma(1+\alpha)+t^\alpha)\sin(x)$. It can be checked that the exact solution is $u(t,x)=t^\alpha\sin(x)$.

The graded mesh (4.17) with grading parameter r and time step number K is adopted in time. We use the central finite difference method in space with grid spacing $h=2\pi/10000$. The maximum L_2-error is computed by $\max_{1\le k\le K}\|u(t_k)-u^k\|_{L^2(\Omega)}$. Tables 1 , 2 and 3 present the maximum L_2-errors for $\alpha=0.3,\ 0.5,\ 0.7$ and $r=1,\ 2,\ 2/\alpha,\ 3/\alpha$ respectively. It can be observed that the convergence rates are consistent with (4.20) derived from Theorem 3.

In [10, 25], the authors state that the large value of r in the graded mesh increases the temporal mesh width near the final time $t=T$ which can lead to large errors. Indeed, when $r=3/\alpha$, the errors seem larger than the case of $r=2/\alpha$, as observed in Tables 1, 2 and 3. We then propose to use the graded mesh with varying grading parameter r_j (dependent on the time), called r-variable graded mesh. In particular, for this example, we use the following r-variable graded mesh

$$
\begin{aligned}
&r_j=2/\alpha+1.5-\frac{3(j-1)}{K-1},\\
&t_j=\left(\frac{j}{K}\right)^{r_j}T,\quad \tau_j=t_j-t_{j-1}=\left[\left(\frac{j}{K}\right)^{r_j}-\left(\frac{j-1}{K}\right)^{r_{j-1}}\right]T.
\end{aligned}
\tag{5.1}
$$

Table 2 $\max_{1\le k\le K} \|u(t_k) - u^k\|_{L^2(\Omega)}$ for the graded meshes with different grading parameters and time step numbers where $\alpha = 0.5$

	$K = 40$	$K = 80$	$K = 160$	$K = 320$	$K = 480$	$K = 640$
$r = 1$	1.8575e−2	1.4568e−2	1.1059e−2	8.2145e−3	6.8534e−3	6.0116e−3
order	–	0.3506	0.3976	0.4290	0.4468	0.4555
$r = 2$	3.9186e−3	2.0105e−3	1.0182e−3	5.1239e−4	3.4232e−4	2.5701e−4
order	–	0.9628	0.9815	0.9908	0.9947	0.9963
$r = 2/\alpha$	2.2728e−4	5.8725e−5	1.4830e−5	3.7186e−6	1.6536e−06	9.3037e−7
order	–	1.9524	1.9854	1.9957	1.9986	1.9993
$r = 3/\alpha$	3.5987e−4	9.9080e−5	2.6590e−5	7.0116e−6	3.2025e−6	1.8379e−6
order	–	1.8608	1.8977	1.9231	1.9327	1.9302

Table 3 $\max_{1\le k\le K} \|u(t_k) - u^k\|_{L^2(\Omega)}$ for the graded meshes with different grading parameters and time step numbers where $\alpha = 0.7$

	$K = 40$	$K = 80$	$K = 160$	$K = 320$	$K = 480$	$K = 640$
$r = 1$	8.3068e−3	5.4221e−3	3.4582e−3	2.1753e−3	1.6518e−3	1.3569e−3
order	–	0.6154	0.6488	0.6688	0.6790	0.6836
$r = 2$	7.3797e−4	2.8495e−4	1.0874e−4	4.1317e−5	2.3437e−5	1.5672e−5
order	–	1.3729	1.3898	1.3961	1.3983	1.3989
$r = 2/\alpha$	1.7758e−4	4.6703e−5	1.1903e−5	2.9940e−6	1.3323e−6	7.4975e−7
order	–	1.9269	1.9721	1.9913	1.9970	1.9985
$r = 3/\alpha$	1.5861e−4	4.3872e−5	1.1918e−5	3.1981e−6	1.4809e−6	8.6093e−7
order	–	1.8541	1.8802	1.8978	1.8987	1.8855

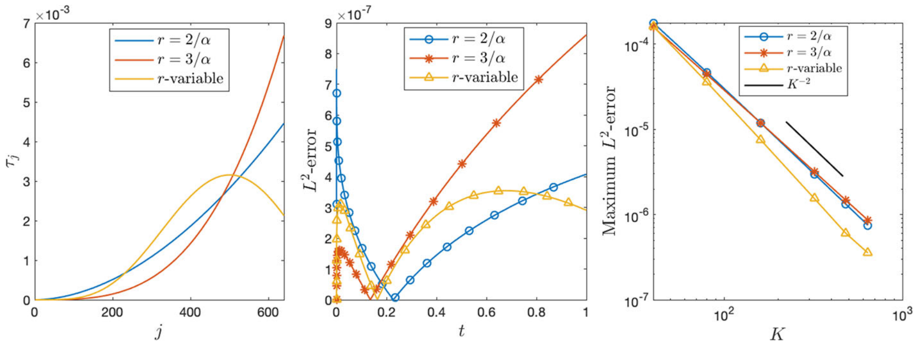

Fig. 2 Time steps (left), pointwise L^2-errors (middle), and maximum L^2-errors (right) of the L2-1_σ scheme in 1D on the r-variable graded mesh (5.1) and the graded meshes (4.17) with $r = 2/\alpha,\ 3/\alpha$ ($\alpha = 0.7$)

In Fig. 2, we compare the time steps, the pointwise L^2-errors, and the maximum L^2-errors of the r-variable graded mesh (5.1) and the standard graded meshes (4.17) with $r = 2/\alpha,\ 3/\alpha$. Here we set $\alpha = 0.7$ and for the left and middle subfigures $K = 640$. From the middle of Fig. 2, the maximum L^2-error for the r-variable graded mesh is smaller than the standard graded meshes with $r = 2/\alpha,\ 3/\alpha$.

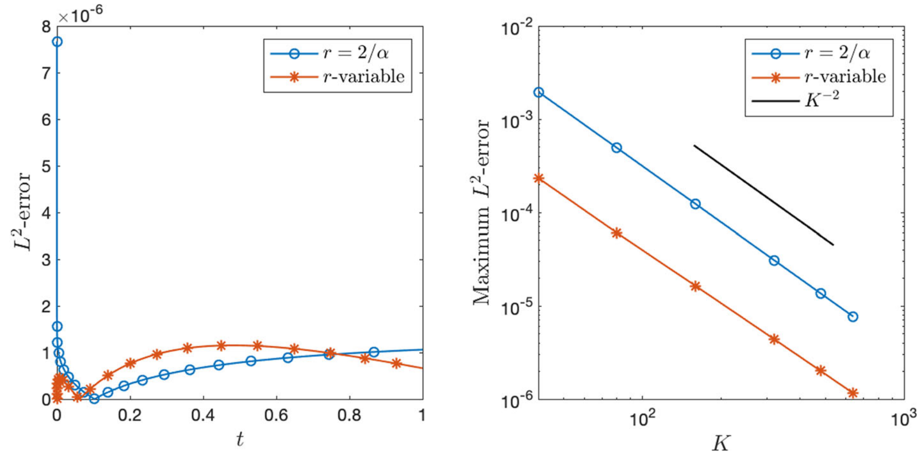

Fig. 3 Pointwise L^2-errors (left) with $K = 640$ and maximum L^2-errors (right) of L2-1_σ scheme in 2D on the r-variable graded mesh (5.1) and the graded mesh (4.17) with $r = 2/\alpha$ ($\alpha = 0.7$)

5.2 2D Example

In the 2D case, we set $f(t, x, y) = (\Gamma(1 + \alpha) + 2t^\alpha)\sin(x)\sin(y)$ and then the exact solution $u(t, x, y) = t^\alpha \sin(x)\sin(y)$. In this example, we set periodic boundary condition for the subdiffusion equation. We take $T = 1$ and $\alpha = 0.7$. Here we use Fourier spectral method in the domain $\Omega = [0, 2\pi]^2$ with 256×256 Fourier modes. In Fig. 3, we show the pointwise L^2-errors (with $K = 640$) and the maximum L^2-errors of the L2-1_σ schemes on the standard graded meshes (4.17) with $r = 2/\alpha$ and the r-variable graded mesh (5.1). One can observe that the r-variable graded mesh performs better than the graded mesh for this example.

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Data Availability Enquiries about data availability should be directed to the authors.

Declarations

Conflict of interest The authors have not disclosed any competing interests.

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