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# Convergence and Error Estimates of a Mixed Discontinuous Galerkin-Finite Element Method for the Semi-stationary Compressible Stokes System

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**Abstract**
In this paper, we study a mixed discontinuous Galerkin-finite element method (DG-FEM) for solving the semi-stationary compressible Stokes system in a bounded domain. The approximation of continuity equation is obtained by a piecewise constant discontinuous Galerkin method. The discretization of momentum equation is obtained by conforming Bernardi–Raugel finite elements. The convergence of mixed DG-FEM for nonlinear, isentropic stokes problem is rigorously established by compactness arguments and the existence analysis of Lions on the discrete level. Employing the continuous relative energy functional method and a detailed consistency analysis, we derive two error estimates for the numerical solution of the semi-stationary isentropic stokes system. In particular, we establish the $L^2$ error estimates for the pressure. All convergence results do not require the boundedness of numerical solutions.

**Keywords** Compressible Stokes system · Discontinuous Galerkin method · Bernardi–Raugel finite element · Convergence · Error estimates

## 1 Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain, we consider the following semi-stationary compressible Stokes problem:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0, \text{ in } (0, T) \times \Omega, \\ -\mu \Delta \boldsymbol{u} - (\lambda + \mu) \nabla \operatorname{div} \boldsymbol{u} + \nabla p(\rho) = \mathbf{0}, \text{ in } (0, T) \times \Omega, \end{cases} \tag{1.1}$$

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where $\rho$ is the fluid density and $\boldsymbol{u}$ is the velocity. The parameters coefficients $\mu$ and $\lambda$ are assumed to be constant and satisfy $\mu > 0$, $d\lambda + 2\mu > 0$. The pressure $p(\rho)$ is governed by the isentropic equation (or Boyle's law):

$$p(\rho) = a\rho^{\gamma}, \quad a > 0, \tag{1.2}$$

where $\gamma > 1$ is the adiabatic exponent. The internal energy $\mathcal{H}$ is given by $\mathcal{H}(\rho) = \frac{p(\rho)}{\gamma-1}$. The system (1.1)–(1.2) is supplemented with initial conditions for the density

$$\rho(0, x) = \rho_0, \text{ in } \Omega \tag{1.3}$$

Together with the following no-slip boundary condition for the velocity

$$\boldsymbol{u} = \mathbf{0}, \text{ on } (0, t) \times \partial\Omega. \tag{1.4}$$

In recent years, numerical methods for compressible Stokes equations have received some attention. In the pioneering work of [20], the authors proposed a low order mixed finite element-finite volume (FE-FV) scheme based on nonconforming $P_1$ (also called Crouzeix–Raviart) finite element for solving the stationary compressible isothermal Stokes problem and analyzed its convergence to a weak solution of the continuous problem. After that, the convergence of mixed FE-FV scheme to weak solution of the isentropic case under the assumption of $\gamma > 1$ has been established by Eymard et al. [10]. Meanwhile, they generalized the results to the well known Marker-and-Cell (MAC) scheme in [9]. Later, the convergence of mixed FE-FV scheme to weak solution of the general compressible Stokes problem ($p = \varphi(\rho)$, where $\varphi$ is a superlinear nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$) under the hypothesis $\gamma > 1$ was proved by Fettah and Gallouët in [18]. The models studied in the above mentioned literature are all steady state compressible stokes models. The semi-steady compressible Stokes model is known as a reasonable approximation of the isentropic Navier–Stokes equations when the convective effects can be neglected. The convergence of mixed DG-FEM based on nonconforming $P_1$ finite element for the semi-steady compressible Stokes flow with a Navier boundary condition was shown by Karlsen and Karper in [26]. Meanwhile, they proposed and analyzed the convergence of a new mixed DG-FEM (here the velocity and vorticity were approximated by the div-conforming and curl-conforming Nédélec finite element spaces) to the semi-stationary compressible Stokes systems in [27]. We also mention that the convergence of the MAC scheme for the semi-stationary compressible Stokes flow with Dirichlet boundary conditions was proved in [21]. Very recently, a mixed FE-FV scheme based on Bernardi–Raugel finite element scheme for the stationary compressible isothermal Stokes system was proposed in [2]. The authors gave a convergence proof for the isothermal Stokes equations and investigated the convergence of numerical solutions to its incompressible limit. The convergence analysis is restricted to the isothermal Stokes equations (the pressure of the form (1.2) with $\gamma = 1$) and the extension to the case $\gamma > 1$ remains open.

The aim of this paper is to show the convergence and error estimates of a mixed DG-FEM based on Bernardi–Raugel finite element for the semi-stationary (isentropic) compressible Stokes equations. This work consists of two major parts. The first part of this paper is to show the convergence of a mixed DG-FEM to a weak solution of the system (1.1) for any $\gamma > 1$. The convergence result of this paper is nontrivial compared to the existing literature. On the one hand, we see that the function $\boldsymbol{v}_h = \Pi_h^{\mathbb{V}} \nabla \Delta^{-1}[\rho_h]$ is not a solution to the div-curl problem

$$\operatorname{div} \boldsymbol{v}_h = \rho_h, \quad \mathbf{curl}\, \boldsymbol{v}_h = 0,$$

where $\Pi_h^{\mathbb{V}}$ is the reconstruction interpolation operator of Bernardi–Raugel finite element space $\mathbb{V}_h$. Therefore, it is more difficult to obtain the discrete version of the effective viscous flux compared to [26], which will complicate the convergence analysis in this paper. On the other hand, the convergence analysis of this paper is valid for the semi-stationary (isentropic) compressible Stokes equations for any $\gamma > 1$. Of course, it is also valid for the stationary compressible Stokes equations with a slight modification, which fills the gap in the convergence analysis of [2] for the case $\gamma > 1$. We also want to remark that the $\boldsymbol{H}^1$-conforming Bernardi–Raugel finite element has several advantages compared to the nonconforming Crouzeix–Raviart element used in the references [10, 18, 20, 23, 26]. Firstly, the conforming finite element method has less number of degrees of freedom which results in a cheaper computational cost. Secondly, the Korn's inequality is admissible for the conforming method employed to approximate the velocity unknown. It is well known that the Korn's inequality does not hold for the nonconforming Crouzeix–Raviart finite element space. Therefore, the conforming setting of this paper is easier to generalize to other viscous stress tensor compared to the nonconforming method in the references. Third, the convergence proofs of the conforming setting is less "structure dependent" than the nonconforming method. In other words, the methodology of the convergence proofs in this paper can be easily generalized to other numerical schemes.

The second part of this paper is to derive an error estimate between the mixed DG-FEM solution of the semi-stationary compressible Stokes system and its strong solution. By a detailed consistency analysis and the relative energy functional method introduced in reference [13], two error estimates for the numerical solutions of problem (1.1) under the hypothesis $\gamma > \frac{6}{5}$ are proved in this paper. All the error results are unconditional in the sense that we do not require the boundedness of numerical solutions and the CFL like condition on the temporal mesh size. The relative energy method was originally designed to analyze the weak-strong uniqueness property of the compressible Navier–Stokes equations. Recently, this idea has been used to analyze the error estimate of numerical schemes of compressible Navier–Stokes system under the hypothesis $\gamma > \frac{3}{2}$, such as the mixed DG-FEM based on nonconforming Crouzeix–Raviart finite element [12, 23], the implicit MAC scheme [24] and the finite difference method [31]. The error analysis of this paper uses similar analytical techniques but with some modifications. Firstly, our analysis is based on a detailed consistency analysis and the continuous relative energy functional method, rather than the discrete version used in the above literatures. Secondly, our numerical scheme is different from the above work and it requires to deal with some different technical estimates. Thirdly and more importantly, we derive the unconditional $L^2$ error estimate of pressure under the assumption of $\gamma > \frac{6}{5}$. To the best of our knowledge, this is the first unconditional error estimate of pressure for the compressible flows.

A brief overview of this work is provided as follows. In the next section, we introduce some notations and preliminary knowledge for this paper. In Sect. 3, we consider a mixed DG-FEM based on Bernardi–Raugel finite element for the semi-stationary compressible Stokes equations. After that, we deduce the discrete energy law, a priori estimate of pressure, the existence of numerical solutions and some uniform bounds. In Sect. 4, we establish the consistency formulation for the continuity equations. In Sect. 5, we show the boundedness of discrete time derivative and an important priori estimates for the density. The convergence of mixed DG-FEM for the nonlinear, isentropic Stokes equations is proved by compactness arguments and the existence analysis of Lions on the discrete level in Sect. 6. In Sect. 7, an unconditional error estimate for mixed DG-FEM solution of the problem (1.1) under the hypothesis $\gamma > \frac{6}{5}$ is proved by the relative energy functional method.

## 2 Notation and Preliminaries

In this section, we introduce some notations and preliminary results used in this paper. For any $1 \le q \le \infty$, $L^q(\Omega)$ denotes the usual Lebesgue space on $\Omega$. For all non-negative integers $k$ and $r$, $W^{k,r}(\Omega)$ stands for the standard Sobolev spaces. We write $H^k(\Omega) = W^{k,2}(\Omega)$. We define $H_0^1(\Omega)$ as the subspace of $H^1(\Omega)$, which is zero on $\partial\Omega$. The vector-valued quantities will be denoted in boldface notations, such as $\boldsymbol{u} = (u_i)_{i=1}^d$ and $\boldsymbol{L}^2(\Omega) = (L^2(\Omega))^d$.

**Hypothesis 2.1** *The initial data $\rho_0$ satisfies the following properties:*

$$\rho_0 \in L^\gamma(\Omega), \quad \rho_0 > 0.$$

**Definition 2.1** We say that $(\rho, \boldsymbol{u})$ is a weak solution of the problem (1.1) if it satisfies the following properties:

(i) The solution $(\rho, \boldsymbol{u})$ satisfied the regularity requirements

$$\rho \in L^\infty(0, T; L^\gamma(\Omega)) \cap L^{2\gamma}((0, T) \times \Omega), \quad \boldsymbol{u} \in L^2(0, T; \boldsymbol{H}_0^1(\Omega)).$$

(ii) For any test fuctions $(\varphi, \boldsymbol{v}) \in C_0^\infty((0, T) \times \Omega) \times \boldsymbol{C}_0^\infty((0, T) \times \Omega)$ and $t_F \in [0, T]$, there holds the weak formulation

$$\int_0^{t_F} \int_\Omega [\rho \partial_t \varphi + \rho \boldsymbol{u} \cdot \nabla \varphi] dx dt = \left[ \int_\Omega \rho \varphi dx \right]_{t=0}^{t=t_F}, \tag{2.1}$$

$$\int_0^{t_F} \int_\Omega [\mu \nabla \boldsymbol{u} : \nabla \boldsymbol{v} + (\mu + \lambda) \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} - p(\rho) \operatorname{div} \boldsymbol{v}] dx dt = 0. \tag{2.2}$$

(iii) The solution $(\rho, \boldsymbol{u})$ satisfies the energy inequality

$$\left[ \int_\Omega \mathcal{H}(\rho) dx \right]_{t=0}^{t=t_F} + \int_0^{t_F} \int_\Omega [\mu |\nabla \boldsymbol{u}|^2 + (\lambda + \mu) |\operatorname{div} \boldsymbol{u}|^2] dx dt \le 0. \tag{2.3}$$

Next, we recall the following renormalized solution argument introduced by DiPerna and Lions (see e.g., [6]).

**Definition 2.2** We say that $(\rho, \boldsymbol{u}) \in L^\infty(0, T; L^\gamma(\Omega)) \times L^2(0, T; \boldsymbol{H}_0^1(\Omega))$ is a renormalized solution of the continuity equation $\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0$ if the identity

$$\partial_t \Phi(\rho) + \operatorname{div}(\Phi(\rho) \boldsymbol{u}) + \Psi(\rho) \operatorname{div} \boldsymbol{u} = 0,$$

in $\mathcal{D}'((0, T) \times \Omega)$ holds for any $\Phi \in C[0, \infty) \cap C^1(0, \infty)$ with $\Phi(0) = 0$, $\Psi(\rho) = \Phi'(\rho)\rho - \Phi(\rho)$ and $\Phi(\rho), \boldsymbol{u}\Phi(\rho) \in L^1((0, T) \times \Omega)$.

Finally, we recall the following well-known lemma [30] which says that the weak solution $\rho$ is a renormalized solution.

**Lemma 2.1** *Suppose that couple $(\rho, \boldsymbol{u}) \in L^2((0, T) \times \Omega) \times L^2(0, T; \boldsymbol{H}_0^1(\Omega))$ satisfies the continuity equation in the weak sense (2.1). Then $(\rho, \boldsymbol{u})$ is also renormalized solution according to Definition 2.2.*

## 3 Numerical Method

In this section, we consider a mixed DG-FEM based on Bernardi–Raugel finite element for solving the compressible stokes problem (1.1).

### 3.1 Finite Dimensional Function Spaces

In order to introduce the mixed DG-FEM scheme, the mesh and some discrete function spaces are defined. Let $\mathcal{T}_h$ be a quasi-uniform tetrahedral partition of $\Omega$ with $\overline{\Omega} = \cup_{K\in\mathcal{T}_h}\overline{K}$, $K_i \cap K_j = \emptyset$ for $K_i, K_j \in \mathcal{T}_h$, $i \neq j$. The mesh size is defined by $h = \max_{K\in\mathcal{T}_h} h_K$, where $h_K$ is the mesh size of $K$. We write $\mathcal{F}_h$ as the set of faces in $\mathcal{T}_h$, while $F$ is the face. Furthermore, $\mathcal{F}_{h,ext}$ is the set of faces $F \in \partial\Omega$, while $\mathcal{F}_{h,int} = \mathcal{F}_h \setminus \mathcal{F}_{h,ext}$.

In order to discretize the problem, we introduce two families of finite-dimensional spaces. Before proceeding further, we write $\mathbb{P}_n(K)$ as the space of polynomials of degree $n$, while $\mathbb{P}_n^d(K) = [\mathbb{P}_n(K)]^d$. We define the space of piecewise constant functions

$$\mathbb{Q}_h := \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_0(K),\ \forall\, K \in \mathcal{T}_h\}$$

for the approximation of the density. In addition, we introduce the associated projection operator

$$\Pi_h^{\mathbb{Q}} : L^2(\Omega) \to \mathbb{Q}_h, \quad \Pi_h^{\mathbb{Q}}[v]|_K = \frac{1}{|K|}\int_K v dx, \quad \forall\, K \in \mathcal{T}_h.$$

By recalling the standard Poincaré and Jensen's inequalities, we have the following interpolation error estimates

$$\begin{cases} \|\Pi_h^{\mathbb{Q}}[\varphi]\|_{L^q(K)} \le \|\varphi\|_{\boldsymbol{L}^q(K)}, & \|\varphi - \Pi_h^{\mathbb{Q}}[\varphi]\|_{L^q(K)} \le Ch\|\nabla\varphi\|_{L^q(K)}, \\ \|\Pi_h^{\mathbb{Q}}[\varphi]\|_{L^q(\Omega)} \le \|\varphi\|_{\boldsymbol{L}^q(\Omega)}, & \|\varphi - \Pi_h^{\mathbb{Q}}[\varphi]\|_{L^q(\Omega)} \le Ch\|\nabla\varphi\|_{L^q(\Omega)}, \end{cases} \tag{3.1}$$

for any $K \in \mathcal{T}_h$ and $1 \le q \le \infty$. We define the trace

$$v^+ := \lim_{\delta\to 0^+} v(\boldsymbol{x} + \delta\boldsymbol{n}_F), \quad v^- := \lim_{\delta\to 0^+} v(\boldsymbol{x} - \delta\boldsymbol{n}_F),$$

where $\boldsymbol{n}_F$ is the outer normal vector to the face $F$. Moreover, we define the jumps $[\![v]\!] := v^+ - v^-$ for any $F \in \mathcal{F}_{h,int}$. Finally, we introduce the semi-norm of the space $\mathbb{Q}_h$

$$\|v\|_{\mathbb{Q}_h}^2 := \sum_{F\in\mathcal{F}_{h,int}} \int_F \frac{[\![v]\!]^2}{h} dS, \quad \forall\, v \in \mathbb{Q}_h.$$

We employ the Bernardi–Raugel finite element space (see, e.g., [3, 25])

$$\mathbb{V}_h := \{\boldsymbol{v} \in \boldsymbol{C}^0(\overline{\Omega}) : \boldsymbol{v}|_K \in \mathbb{BR}(K), \forall\, K \in \mathcal{T}_h\} \cap \boldsymbol{H}_0^1(\Omega)$$

for the approximation of the velocity. The local Bernardi–Raugel finite element space $\mathbb{BR}(K)$ is given by

$$\mathbb{BR}(K) := \mathbb{P}_1^d(K) \oplus \mathrm{Span}\{\boldsymbol{p}_i,\ 1 \le i \le d+1\}, \quad \boldsymbol{p}_i := \prod_{j=1, j\neq i}^{d+1} \lambda_j \boldsymbol{n}_i,$$

where $\lambda_j$ is the barycentric coordinate of $K$ and $\boldsymbol{n}_i$ is the unit outward normal to $F_i \subset \partial K$. We introduce the reconstruction interpolation operator (see, e.g., [25, Chapter II])

$$\Pi_h^{\mathbb{V}} : \boldsymbol{H}_0^1(\Omega) \to \mathbb{V}_h, \quad \int_\Omega \operatorname{div} \Pi_h^{\mathbb{V}}[\boldsymbol{v}]\varphi_h dS = \int_\Omega \operatorname{div} \boldsymbol{v}\varphi_h dS, \quad \forall\, \varphi_h \in \mathbb{Q}_h.$$

The interpolation operator $\Pi_h^{\mathbb{V}}$ has the following error estimates (see, e.g., [25, Chapter II, Lemma 2.2 and 2.8]):

$$|\Pi_h^{\mathbb{V}}\boldsymbol{v} - \boldsymbol{v}|_{m,\Omega} \le Ch^{k-m}|\boldsymbol{v}|_{k,\Omega}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^k(\Omega), \tag{3.2}$$

where $|\cdot|_{m,\Omega}$ is the semi-norm of $\boldsymbol{H}^m(\Omega)$ and $m = 0, 1$, $k = 1, 2$. Obviously, taking $k = m = 1$ in (3.2), the interpolation operators $\Pi_h^{\mathbb{V}}$ have the following $\boldsymbol{H}^1$-stable

$$\|\Pi_h^{\mathbb{V}} \boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega). \tag{3.3}$$

Finally, we introduce some basic estimate for finite dimensional function spaces. By recalling the following inverse estimate from [5, Theorem 4.5.11], there holds

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{r,q_1}(\Omega)} \leq Ch^{m-r+3\min\{\frac{1}{q_1}-\frac{1}{q_2},0\}}\|\boldsymbol{v}\|_{\boldsymbol{W}^{m,q_2}(\Omega)}, \tag{3.4}$$

for any polynomial functions $\boldsymbol{v}|_K \in \mathbb{P}_n^d(K)$, $K \in \mathcal{T}_h$, where $C > 0$ is a generic constant independent of the mesh-size $h$, $m$ and $r$ are two real numbers with $0 \leq m \leq r$, $q_1$ and $q_2$ are two integers with $1 \leq q_1, q_2 \leq \infty$. By applying the scaling arguments and the trace theorem, we obtain

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\partial K)} \leq Ch^{-\frac{1}{q}}(\|\boldsymbol{v}\|_{\boldsymbol{L}^q(K)} + h\|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^q(K)}), \tag{3.5}$$

for any $K \in \mathcal{T}_h$ and $1 \leq q \leq \infty$ and $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(K)$; see, e.g., [1]. Moreover, we apply the inverse estimate (3.4) and the trace inequality (3.5) to obtain

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\partial K)} \leq Ch^{-\frac{1}{q}}\|\boldsymbol{v}\|_{\boldsymbol{L}^q(K)}, \tag{3.6}$$

for any $K \in \mathcal{T}_h$ and $1 \leq q \leq \infty$, $\boldsymbol{v} \in \mathbb{P}_n^d(K)$.

### 3.2 The Discretization of the Convection Term

Before introducing the scheme, we discuss the approximation of the convection operators in the continuity equation. To this end, we define the standard upwind operator $\mathrm{Up}[r_h, \boldsymbol{v}_h]$ on a face $F$, which is described by

$$\mathrm{Up}[r_h, \boldsymbol{v}_h] = r_{h,+}[\boldsymbol{v}_{h,F} \cdot \boldsymbol{n}]^- + r_{h,-}[\boldsymbol{v}_{h,F} \cdot \boldsymbol{n}]^+, \quad \forall\, r_h \in \mathbb{Q}_h,\ \boldsymbol{v}_h \in \mathbb{V}_h,$$

where $[\boldsymbol{v}_{h,F} \cdot \boldsymbol{n}]^+ := \max\{0, \boldsymbol{v}_{h,F} \cdot \boldsymbol{n}\}$ and $[\boldsymbol{v}_{h,F} \cdot \boldsymbol{n}]^- := \min\{0, \boldsymbol{v}_{h,F} \cdot \boldsymbol{n}\}$, $\boldsymbol{v}_{h,F} := \frac{1}{|F|}\int_F \boldsymbol{v}_h dS$. By applying the following lemma, we can show the distributional error of the convective term and its numerical analogue.

**Lemma 3.1** *For all $r_h \in \mathbb{Q}_h$ and $\boldsymbol{v}_h \in \mathbb{V}_h$, $\varphi \in H_0^1(\Omega)$, we conclude that*

$$\begin{aligned}\int_\Omega r_h \boldsymbol{v}_h \cdot \nabla\varphi dx &= \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathrm{Up}[r_h, \boldsymbol{v}_h][\![\Pi_h^{\mathbb{Q}}[\varphi]]\!]dS + \int_\Omega (\Pi_h^{\mathbb{Q}}[\varphi] - \varphi) r_h \operatorname{div} \boldsymbol{v}_h dx \\ &+ \sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K}\int_F (\Pi_h^{\mathbb{Q}}[\varphi] - \varphi)[\![r_h]\!][\boldsymbol{v}_{h,F} \cdot \boldsymbol{n}]^- dS \\ &+ \sum_{K\in\mathcal{T}_h}\int_{\partial K} r_h(\varphi - \varphi_F)(\boldsymbol{v}_h - \boldsymbol{v}_{h,F}) \cdot \boldsymbol{n} dS.\end{aligned}$$

***Proof*** By the same procedure as in [14, Section 2.3], we easily see that

$$\begin{aligned}\int_\Omega r_h \boldsymbol{v}_h \cdot \nabla\varphi dx &= \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathrm{Up}[r_h, \boldsymbol{v}_h][\![g_h]\!]dS + \int_\Omega (g_h - \varphi) r_h \operatorname{div} \boldsymbol{v}_h dx \\ &+ \sum_{K\in\mathcal{T}_h}\sum_{F\subset\partial K}\int_F (g_h - \varphi)[\![r_h]\!][\boldsymbol{v}_{h,F} \cdot \boldsymbol{n}]^- dS + \sum_{K\in\mathcal{T}_h}\int_{\partial K} r_h\varphi(\boldsymbol{v}_h - \boldsymbol{v}_{h,F}) \cdot \boldsymbol{n} dS\end{aligned}$$

for any $r_h, g_h \in \mathbb{Q}_h$, $\boldsymbol{u}_h \in \mathbb{V}_h$ and $\varphi \in H_0^1(\Omega)$. It can easily be seen that

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K} r_h\varphi(\boldsymbol{v}_h - \boldsymbol{v}_{h,F})\cdot \boldsymbol{n} dS = \sum_{K\in\mathcal{T}_h}\int_{\partial K} r_h(\varphi-\varphi_F)(\boldsymbol{v}_h - \boldsymbol{v}_{h,F})\cdot \boldsymbol{n} dS.$$

Combining the above analysis, the proof is thus complete. □

### 3.3 Numerical Scheme

For the time discretization, let $N$ be a fixed integer and $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $[0, T]$ with time-step size $\tau = T/N$. Moreover, let $t_n = n\tau$ be the discrete time points and $v^n$ is the approximation value of the function $v$ at time $t_n$ for $0 \le n \le N$. For convenience, we introduce $d_t\boldsymbol{v}^n = (\boldsymbol{v}^n - \boldsymbol{v}^{n-1})/\tau$ and $D_t\boldsymbol{v}(t) = (\boldsymbol{v}(t) - \boldsymbol{v}(t-\tau))/\tau$.

We initialize the scheme $\rho_h^0 := \Pi_h^{\mathbb{Q}}[\rho_0]$. For any $1 \le n \le N$, we compute $(\rho_h^n, \boldsymbol{u}_h^n) \in \mathbb{Q}_h \times \mathbb{V}_h$ by the following numerical scheme

**Scheme 1** *Given $\rho_h^{n-1} \in \mathbb{Q}_h$, for any $(\varphi_h, \boldsymbol{v}_h) \in \mathbb{Q}_h \times \mathbb{V}_h$, find $(\rho_h^n, \boldsymbol{u}_h^n) \in \mathbb{Q}_h \times \mathbb{V}_h$ such that*

$$\int_\Omega d_t\rho_h^n\varphi_h dx - \sum_{F\in\mathcal{F}_{h,int}}\int_F \mathrm{Up}[\rho_h^n, \boldsymbol{u}_h^n][\![\varphi_h]\!] dS + h^{\epsilon-1}\sum_{F\in\mathcal{F}_{h,int}}\int_F [\![\rho_h^n]\!][\![\varphi_h]\!] dx = 0, \tag{3.7}$$

$$\int_\Omega [\mu\nabla\boldsymbol{u}_h^n : \nabla\boldsymbol{v}_h + (\mu+\lambda)\operatorname{div}\boldsymbol{u}_h^n \operatorname{div}\boldsymbol{v}_h] dx - \int_\Omega p(\rho_h^n)\operatorname{div}\boldsymbol{v}_h dx = 0. \tag{3.8}$$

***Remark 3.1*** (i) Taking $\varphi_h = 1$ in the discrete continuity equation (3.7), we can show $\int_\Omega \rho_h^n dx = \int_\Omega \rho_h^{n-1} dx$. In other words, we have immediately the scheme satisfying the conservation of mass. (ii) The stabilization term in the discrete continuity equation is useful in the convergence analysis. More specifically, it provides control over the discrete semi-norm of $\rho_h$ by some (negative) power of the mesh size $h$. We remark that the artificial stabilization term in the convergence analysis of compressible flows is introduced by [10, 18, 20].

The renormalized continuity scheme can derived by the following lemma and the proof can be referred to [14, Section 4.1] for more details.

**Lemma 3.2** (Renormalized continuity scheme). *For any $1 \le n \le N$, let $(\rho_h^n, \boldsymbol{u}_h^n) \in \mathbb{Q}_h \times \mathbb{V}_h$ satisfy the continuity scheme (3.7). Then $(\rho_h^n, \boldsymbol{u}_h^n)$ also satisfies the following renormalized continuity scheme*

$$\begin{aligned}
&\int_\Omega d_t\mathcal{B}(\rho_h^n)\varphi_h dx - \sum_{F\in\mathcal{F}_{h,int}}\int_F \mathrm{Up}[\mathcal{B}(\rho_h^n), \boldsymbol{u}_h^n][\![\varphi_h]\!] dS \\
&+ h^{\epsilon-1}\sum_{F\in\mathcal{F}_{h,int}}\int_F \mathcal{B}'(\rho_{h,+}^n)[\![\varphi_h]\!][\![\rho_h]\!] dS + h^{\epsilon-1}\sum_{F\in\mathcal{F}_{h,int}}\int_F \mathcal{B}''(\overline{\eta}_{\rho,h}^n)[\![\rho_h]\!]^2 dS \\
&+ \int_\Omega \varphi_h(\mathcal{B}'(\rho_h^n)\rho_h^n - \mathcal{B}(\rho_h^n))\operatorname{div}\boldsymbol{u}_h^n dx = -\frac{1}{2\tau}\int_\Omega \mathcal{B}''(\xi_{\rho,h}^n)|\rho_h^n - \rho_h^{n-1}|^2\varphi_h dx \\
&- \frac{1}{2}\sum_{F\in\mathcal{F}_{h,int}}\int_F \varphi_h\mathcal{B}''(\eta_{\rho,h}^n)[\![\rho_h^n]\!]^2|\boldsymbol{u}_{h,F}^n\cdot\boldsymbol{n}| dS,
\end{aligned} \tag{3.9}$$

*for any* $\mathcal{B} \in C^2(\mathbb{R}_+)$ *and* $\varphi_h \in \mathbb{Q}_h$*, where* $\xi^n_{\rho,h} \in \text{co}\{\rho_h^{n-1}, \rho_h^n\}$ *on each element* $K \in \mathcal{T}_h$ *and* $\bar{\eta}^n_{\rho,h}, \eta^n_{\rho,h} \in \text{co}\{\rho^n_h, (\rho^n_h)^+\}$ *on each face* $F \in \mathcal{F}_h$*, where* $\text{co}\{a, b\} = [\min\{a, b\}, \max\{a, b\}]$.

In the upcoming analysis, the discrete density solution $\rho_h$ is necessary for positive. For this purpose, we recall the following lemma (see, e.g., [22, 26, 28]).

**Lemma 3.3** *For any* $1 \le n \le N$*, we assume that* $\rho_h^{n-1} > 0$ *in* $\Omega$ *and* $\boldsymbol{u}_h^n \in \mathbb{V}_h$ *holds. Then the solution* $\rho_h^n \in \mathbb{Q}_h$ *of the discontinuous Galerkin method* (3.7) *satisfies*

$$\rho_h^n \ge \frac{\min_{\boldsymbol{x}\in\Omega} \rho_h^{n-1}}{1 + \tau \|\operatorname{div} \boldsymbol{u}_h^n\|_{L^\infty(\Omega)}} > 0.$$

### 3.4 A Priori Estimates

In this subsection, we establish some a priori estimates for the discrete solutions of the scheme (3.7)–(3.8), including the energy estimate and the uniformly boundedness of pressure in $L^2((0, T) \times \Omega)$.

**Theorem 3.1** (Discrete energy law) *For any* $1 \le m \le N$*, the solution* $(\rho_h^n, \boldsymbol{u}_h^n)$ *of the scheme* (3.7)–(3.8) *satisfies the following discrete energy law*

$$\mathcal{J}_h(\rho_h^m) + \tau \sum_{i=1}^{3} \sum_{n=1}^{m} \mathcal{D}^n_{i,h} + \tau \sum_{n=1}^{m} \mathcal{D}_h(\boldsymbol{u}_h^n) = \mathcal{J}_h(\rho_h^0), \tag{3.10}$$

*where the discerte energy* $\mathcal{J}_h$ *and the discrete dissipation* $\mathcal{D}_h$ *are defined by*

$$\mathcal{J}_h(\rho_h^n) := \int_\Omega \mathcal{H}(\rho_h^n)dx, \quad \mathcal{D}_h(\boldsymbol{u}_h^n) := \mu\|\nabla \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)} + (\lambda + \mu)\|\operatorname{div} \boldsymbol{u}_h^n\|^2_{L^2(\Omega)},$$

*and the numerical diffusion terms* $\mathcal{D}^n_{i,h}$ *are given by*

$$\begin{aligned}
\mathcal{D}^n_{1,h} &:= \frac{1}{2\tau} \int_\Omega \mathcal{H}''(\xi^n_{\rho,h}) \left|\rho_h^n - \rho_h^{n-1}\right|^2 dx,\\
\mathcal{D}^n_{2,h} &:= h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\bar{\eta}^n_{\rho,h})[\![\rho_h^n]\!]^2 dS,\\
\mathcal{D}^n_{3,h} &:= \frac{1}{2} \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\eta^n_{\rho,h}) \left|\boldsymbol{u}^n_{h,F} \cdot \boldsymbol{n}\right| [\![\rho_h^n]\!]^2 dS.
\end{aligned}$$

***Proof*** Taking $(\mathcal{B}, \varphi_h) = (\mathcal{H}, 1)$ in the renormalized continuity scheme (3.9) and by applying $\mathcal{H}'(\rho)\rho - \mathcal{H}(\rho) = p(\rho)$, we can show

$$\begin{aligned}
\int_\Omega p(\rho_h^n) \operatorname{div} \boldsymbol{u}_h^n dx = &- d_t \int_\Omega \mathcal{H}(\rho_h^n)dx - \frac{\tau}{2} \int_\Omega \mathcal{H}''(\xi^n_{\rho,h})(d_t\rho_h^n)^2 dx\\
&- h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\bar{\eta}^n_{\rho,h})[\![\rho_h^n]\!]^2 dS\\
&- \frac{1}{2} \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\eta^n_{\rho,h})[\![\rho_h^n]\!]^2 \left|\boldsymbol{u}^n_{h,F} \cdot \boldsymbol{n}\right| dS.
\end{aligned}$$

Let $\boldsymbol{v}_h = \boldsymbol{u}_h$ in (3.8), we conclude that

$$\mu\|\nabla \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)} + (\lambda + \mu)\|\nabla \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)} - \int_\Omega p(\rho_h^n) \operatorname{div} \boldsymbol{u}_h^n dx = 0.$$

Combining the above analysis implies

$$d_t \int_\Omega \mathcal{H}(\rho_h^n)dx + \mathcal{D}_h(\boldsymbol{u}_h^n) + \sum_{i=1}^{3} \mathcal{D}_{i,h}^n = 0, \tag{3.11}$$

for any $1 \le n \le N$. Summing (3.11) with respect to $n$ from $n = 1$ to $n = m$, we obtain (3.10). The proof is thus complete. □

In order to show the $L^2(\Omega)$ estimate of pressure, we introduce an inverse of the divergence operator $\mathbf{B}$, which satisfies the following result (see [4] and [19, Chapter 3]).

**Lemma 3.4** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain. There exists a linear operator $\mathbf{B}$ enjoying the properties*

$$\mathbf{B}[r] \in \boldsymbol{W}_0^{1,q}(\Omega), \quad \operatorname{div} \mathbf{B}[r] = r \quad \forall\, r \in L^q(\Omega), \quad \int_\Omega r dx = 0, \quad \forall\, 1 < q < \infty.$$

*Moverever, the linear operator $\mathbf{B}$ satisfies the following estimate*

$$\|\mathbf{B}[r]\|_{\boldsymbol{W}_0^{1,q}(\Omega)} \le C\|r\|_{L^q(\Omega)}, \quad \forall\, 1 < q < \infty. \tag{3.12}$$

Next, we prove the stability estimate for the discrete pressure.

**Theorem 3.2** *Suppose that Hypothesis 2.1 is satisfied. For any $1 \le m \le N$, then the pressure $p(\rho_h)$ satisfies the following estimate*

$$\tau \sum_{n=1}^{m} \|p(\rho_h^n)\|_{L^2(\Omega)}^2 \le C. \tag{3.13}$$

***Proof*** Let $r_h^n := p(\rho_h^n) - \frac{1}{|\Omega|}\int_\Omega p(\rho_h^n)dx$ for $1 \le n \le N$. Taking $\boldsymbol{v}_h^n = \Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]$ in (3.8) and by the definition of $\Pi_h^{\mathbb{V}}$ and $\mathbf{B}$, we can show

$$\begin{aligned}\|p(\rho_h^n)\|_{L^2(\Omega)}^2 =& \frac{1}{|\Omega|}\|p(\rho_h^n)\|_{L^1(\Omega)}^2 + \mu\int_\Omega \nabla\boldsymbol{u}_h^n : \nabla\Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]dx \\ &+ (\lambda+\mu)\int_\Omega \operatorname{div}\boldsymbol{u}_h^n \operatorname{div}\Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]dx.\end{aligned}$$

By applying Hölder inequality, the estimates (3.3) and (3.12), we obtain

$$\begin{aligned}\left|\int_\Omega \nabla\boldsymbol{u}_h^n : \nabla\Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]dx\right| \le& \|\nabla\boldsymbol{u}_h^n\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]\|_{\boldsymbol{L}^2(\Omega)} \\ \le& C\|\nabla\boldsymbol{u}_h^n\|_{\boldsymbol{L}^2(\Omega)}\|p(\rho_h^n)\|_{L^2(\Omega)}, \\ \left|\int_\Omega \operatorname{div}\boldsymbol{u}_h^n \operatorname{div}\Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]dx\right| \le& \|\operatorname{div}\boldsymbol{u}_h^n\|_{L^2(\Omega)}\|\operatorname{div}\Pi_h^{\mathbb{V}}\mathbf{B}[r_h^n]\|_{L^2(\Omega)} \\ \le& C\|\operatorname{div}\boldsymbol{u}_h^n\|_{L^2(\Omega)}\|p(\rho_h^n)\|_{L^2(\Omega)}.\end{aligned}$$

Combining the above analysis, by applying Young inequality, we have

$$\begin{aligned}\|p(\rho_h^n)\|_{L^2(\Omega)}^2 \le& \frac{1}{|\Omega|}\|p(\rho_h^n)\|_{L^1(\Omega)}^2 + C\|\nabla\boldsymbol{u}_h^n\|_{\boldsymbol{L}^2(\Omega)}^2 \\ &+ C\|\operatorname{div}\boldsymbol{u}_h^n\|_{\boldsymbol{L}^2(\Omega)}^2 + \frac{1}{2}\|p(\rho_h^n)\|_{L^2(\Omega)}^2,\end{aligned} \tag{3.14}$$

for any $1 \le n \le N$.

Summing (3.14) with respect to $n$ from $n = 1$ to $n = m$ and applying the discrete energy estimate (3.10) implies

$$\begin{aligned}\tau \sum_{n=1}^{m} \|p(\rho_h^n)\|_{L^2(\Omega)}^2 \le & C\tau \sum_{n=1}^{m} \|p(\rho_h^n)\|_{L^1(\Omega)}^2 + C\tau \sum_{n=1}^{m} \|\nabla \boldsymbol{u}_h^n\|_{\boldsymbol{L}^2(\Omega)}^2 \\ \le & C\mathcal{J}_h^2(\rho_h^0) + C\mathcal{J}_h(\rho_h^0) \le C\|\rho_0\|_{L^\gamma(\Omega)}^{2\gamma} + C\|\rho_0\|_{L^\gamma(\Omega)}^{\gamma}.\end{aligned} \tag{3.15}$$

By applying Hypothesis 2.1 for the inequality (3.15), we have (3.13). The proof is thus complete. □

### 3.5 Existence of Numerical Solution

By applying Schaeffer's fixed point theorem, we can show the existence of numerical solutions for the scheme (3.7)–(3.8) in this subsection. Firstly, we recall Schaeffer's fixed point theory (see, e.g., [8, Theorem 9.2.4]):

**Lemma 3.5** *Let $\mathcal{L} : D \to D$ be a continuous mapping defined on a finite dimensional normed vector space $D$. Suppose that the set*

$$\{z \in D : z = \Lambda\mathcal{L}(z),\ \ \Lambda \in [0, 1]\}$$

*is non empty and bounded. Then there exists $z \in D$ such that $z = \mathcal{L}(z)$.*

Then we can prove an existence result of numerical solutions for the scheme (3.7)–(3.8).

**Theorem 3.3** *For any $1 \le n \le N$, let $(\rho_h^{n-1}, \boldsymbol{u}_h^{n-1}) \in \mathbb{Q}_h \times \mathbb{V}_h$ and $\rho_h^{n-1} > 0$ be given. Then, for each fixed $h, \tau > 0$, the scheme (3.7)–(3.8) has at least one solution*

$$(\rho_h^n, \boldsymbol{u}_h^n) \in \mathbb{Q}_h \times \mathbb{V}_h, \quad \rho_h^n > 0.$$

The proof of Theorem 3.3 can be found in "Appendix A.1".

### 3.6 Uniform Bounds

In this subsection, we deduce some priori estimates from the discrete energy law (3.10). To this end, we need to extend the definition of discrete solution for any $t \le T$. We define the piecewise constant interpolations of $\rho_h^n$ by

$$\rho_h(t, \cdot) := \begin{cases} \rho_h^0, & \text{for } \ t \in (-\infty, 0], \\ \rho_h^n, & \text{for } \ t \in (t_{n-1}, t_n],\ \forall\, 1 \le n \le N, \end{cases} \tag{3.16}$$

and the piecewise constant interpolations of $\boldsymbol{u}_h^n$ by

$$\boldsymbol{u}_h(t, \cdot) := \boldsymbol{u}_h^n, \quad \text{for } \ t \in (t_{n-1}, t_n],\ \forall\, 1 \le n \le N. \tag{3.17}$$

The following stable results are proved by the discrete energy law and the $L^2$ estimate of pressure, which is crucial in both error estimates and convergence analysis.

**Lemma 3.6** *Suppose that Hypothesis 2.1 is satisfied. Then the family* $(\rho_h, \boldsymbol{u}_h)$ *defined in* (3.16)–(3.17) *satisfies the following estimates:*

$$\begin{aligned} \|\rho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} &\le C, \quad \|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)} \le C, \\ \|p(\rho_h)\|_{L^2((0,T)\times\Omega)} &\le C, \quad \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))} \le C, \\ \|\operatorname{div}\boldsymbol{u}_h\|_{L^2((0,T)\times\Omega)} &\le C, \quad \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \le C. \end{aligned}$$

**Lemma 3.7** *Suppose that Hypothesis 2.1 is satisfied. Then the family* $(\rho_h, \boldsymbol{u}_h)$ *defined in* (3.16)–(3.17) *satisfies the following estimates:*

$$\begin{aligned} \int_0^T \int_\Omega \mathcal{H}''(\xi_{\rho,h})|\rho_h - \rho_h^\star|^2 dxdt &\le C\tau, \\ \int_0^T h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\overline{\eta}_{\rho,h})[\![\rho_h]\!]^2 dSdt &\le C, \\ \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\eta_{\rho,h})|\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}|[\![\rho_h]\!]^2 dSdt &\le C. \end{aligned}$$

**Lemma 3.8** *Suppose that Hypothesis 2.1 and* $\gamma \ge 2$ *are satisfied. Then the family* $(\rho_h, \boldsymbol{u}_h)$ *defined in* (3.16)–(3.17) *satisfies the following estimates:*

$$\begin{aligned} &\int_0^T h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F [\![\rho_h]\!]^2 dSdt \le C, \quad \int_0^T \int_\Omega |\rho_h - \rho_h^\star|^2 dxdt \le C\tau, \\ &\int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F |\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}|[\![\rho_h]\!]^2 dSdt \le C. \end{aligned}$$

***Proof*** Taking $(\mathcal{B}(\rho), \varphi_h) = (\rho^2, 1)$ in renormalized continuity scheme (3.9) and summing this result with respect to $n$ from $n = 1$ to $n = N$, we obtain

$$\begin{aligned} &\frac{1}{\tau}\int_0^T \int_\Omega |\rho_h - \rho_h^\star|^2 dxdt + \int_0^T h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F [\![\rho_h]\!]^2 dSdt \\ &\quad + \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F |\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}|[\![\rho_h]\!]^2 dSdt \\ &\le -\int_0^T \int_\Omega \rho_h^2 \operatorname{div}\boldsymbol{u}_h dxdt - \int_\Omega \rho_h(T,\cdot)^2 dx + \int_\Omega \rho_h(0,\cdot)^2 dx := \sum_{i=1}^3 \mathcal{U}_i. \end{aligned}$$

By applying Hölder inequality and the embedding $L^{2\gamma} \hookrightarrow L^4$ and $L^\gamma \hookrightarrow L^2$ for $\gamma \ge 2$, we conclude that

$$\begin{aligned} |\mathcal{U}_1| &\le \|\rho_h\|^2_{L^4((0,T)\times\Omega)} \|\operatorname{div}\boldsymbol{u}_h\|_{L^2((0,T)\times\Omega)} \\ &\le C\|\rho_h\|^2_{L^{2\gamma}((0,T)\times\Omega)} \|\operatorname{div}\boldsymbol{u}_h\|_{L^2((0,T)\times\Omega)}, \\ |\mathcal{U}_2| &\le C\|\rho_h\|^2_{L^\infty(0,T;L^\gamma(\Omega))}, \quad |\mathcal{U}_3| \le C\|\rho_0\|^2_{L^\gamma(\Omega)}. \end{aligned}$$

Combining the above analysis with Hypothesis 2.1 and Lemma 3.6, we have the required estimates, the proof is thus complete. □

## 4 Consistency Formulation of the Continuity Scheme

In this section, we establish the consistency formulation for the discrete solution of the numerical scheme (3.7)–(3.8). In other words, the discrete solution asymptotically satisfies the weak formulation of continuous problem.

**Lemma 4.1** *The family* $(\rho_h, \boldsymbol{u}_h)$ *defined in* (3.16)–(3.17) *satisfies the following consistency formulation*

$$\int_0^T \int_\Omega [D_t \rho_h \varphi - \rho_h \boldsymbol{u}_h \cdot \nabla\varphi] dxdt = \int_0^T \int_\Omega \mathscr{R}_h \cdot \nabla\varphi dxdt, \tag{4.1}$$

*for any* $\varphi \in L^2(0, T; H^1(\Omega))$, *where the remainder functional* $\mathscr{R}_h$ *is given by*

$$\begin{aligned}\int_0^T \int_\Omega \mathscr{R}_h \cdot \nabla\varphi dxdt &= \int_0^T \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\varphi - \Pi_h^{\mathbb{Q}}[\varphi]) [\![\rho_h]\!] [\boldsymbol{u}_{h,F} \cdot \boldsymbol{n}]^- dSdt \\ &+ \int_0^T \int_\Omega (\varphi - \Pi_h^{\mathbb{Q}}[\varphi]) \rho_h \operatorname{div} \boldsymbol{u}_h dxdt + h^{\epsilon-1} \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F [\![\rho_h]\!] [\![\varphi - \Pi_h^{\mathbb{Q}}[\varphi]]\!] dSdt \\ &+ \int_0^T \sum_{K\in\mathcal{T}_h} \int_{\partial K} \rho_h (\varphi - \varphi_F)(\boldsymbol{u}_{h,F} - \boldsymbol{u}_h) \cdot \boldsymbol{n} dSdt := \sum_{i=1}^4 \mathcal{P}_i(\varphi).\end{aligned} \tag{4.2}$$

***Proof*** Taking $\varphi_h = \Pi_h^{\mathbb{Q}}[\varphi]$ in the continuity method (3.7) and summing this identity with respect to $n$ from $n = 1$ to $n = N$, we can show

$$\begin{aligned}\int_0^T \int_\Omega D_t \rho_h \Pi_h^{\mathbb{Q}}[\varphi] dxdt - \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathrm{Up}[\rho_h, \boldsymbol{u}_h] [\![\Pi_h^{\mathbb{Q}}[\varphi]]\!] dSdt \\ + h^{\epsilon-1} \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F [\![\rho_h]\!] [\![\Pi_h^{\mathbb{Q}}[\varphi] - \varphi]\!] dSdt = 0.\end{aligned}$$

It is easy to check that

$$\int_0^T \int_\Omega D_t \rho_h \Pi_h^{\mathbb{Q}}[\varphi] dxdt = \int_0^T \int_\Omega D_t \rho_h \varphi dxdt.$$

By taking $(r_h, \boldsymbol{v}_h) = (\rho_h^n, \boldsymbol{u}_h^n)$ in Lemma 3.1 and summing this idenity with respect to $n$ from $n = 1$ to $n = N$, we conclude that

$$\begin{aligned}\int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathrm{Up}\,[\rho_h, \boldsymbol{u}_h]\, [\![\Pi_h^{\mathbb{Q}}[\varphi]]\!] dSdt &= \int_0^T \int_\Omega (\varphi - \Pi_h^{\mathbb{Q}}[\varphi]) \rho_h \operatorname{div} \boldsymbol{u}_h dxdt \\ &+ \int_0^T \int_\Omega \rho_h \boldsymbol{u}_h \cdot \nabla\varphi dxdt + \int_0^T \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\varphi - \Pi_h^{\mathbb{Q}}[\varphi]) [\![\rho_h]\!] [\boldsymbol{u}_{h,F} \cdot \boldsymbol{n}]^- dSdt \\ &+ \int_0^T \sum_{K\in\mathcal{T}_h} \int_{\partial K} \rho_h (\varphi - \varphi_F)(\boldsymbol{u}_{h,F} - \boldsymbol{u}_h) \cdot \boldsymbol{n} dSdt.\end{aligned}$$

Combining the above analysis, we obtain (4.1). The proof is thus complete. □

Next, the error estimate of the remainder term $\mathscr{R}_h$ of Lemma 4.1 is proved in the following lemma.

**Lemma 4.2** *Suppose that Hypothesis 2.1 is satisfied. There exists a constant $C > 0$ independent of $h$ and $\tau$, such that the error functional $\mathscr{R}_h$ of Lemma 4.1 satisfies the following estimates*

$$\left|\int_0^T \int_\Omega \mathscr{R}_h \cdot \nabla\varphi dxdt\right| \leq Ch^A \|\nabla\varphi\|_{L^{2m_1}(0,T;L^6(\Omega))}, \tag{4.3}$$

*where the parameters $A$ and $m_1$ are given by*

$$A := \frac{\min\{1,\epsilon\}}{2}, \quad m_1 := \frac{2\gamma}{\gamma-1}.$$

***Proof*** We show the proof of this Lemma in four steps.
**Bound on** $\mathcal{P}_1$. We estimate this term for $1 < \gamma \leq 2$ and $\gamma > 2$ separately. If $1 < \gamma \leq 2$, by applying Cauchy–Schwarz inequality, we can show

$$|\mathcal{P}_1(\varphi)| \leq \sqrt{\mathcal{P}_{1,1,1}} \times \sqrt{\mathcal{P}_{1,1,2}}, \tag{4.4}$$

where $\mathcal{P}_{1,1,1}$ and $\mathcal{P}_{1,1,2}$ are given by

$$\mathcal{P}_{1,1,1} := \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\eta_{\rho,h})|\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}|[\![\rho_h]\!]^2 dSdt,$$

$$\mathcal{P}_{1,1,2} := \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F (\mathcal{H}''(\eta_{\rho,h}))^{-1}|\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}||\varphi - \Pi_h^{\mathbb{Q}}[\varphi]|^2 dSdt.$$

It is easy to check that

$$\begin{aligned}(\mathcal{H}''(\eta_{\rho,h}))^{-1} \leq& C(\rho_{h,+}+\rho_{h,-})^{2-\gamma}\\ \leq& C(1+\rho_{h,+}+\rho_{h,-}), \quad \text{with } 1<\gamma\leq 2.\end{aligned} \tag{4.5}$$

For the term $\mathcal{P}_{1,1,2}$, by applying the inequality (4.5), we obtain

$$\mathcal{P}_{1,1,2} \leq C(\mathcal{P}_{1,1,2,1}+\mathcal{P}_{1,1,2,2}). \tag{4.6}$$

where $\mathcal{P}_{1,1,2,1}$ and $\mathcal{P}_{1,1,2,2}$ are defined by

$$\mathcal{P}_{1,1,2,1} := \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F |\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}||\rho_{h,+}+\rho_{h,-}||\varphi - \Pi_h^{\mathbb{Q}}[\varphi]|^2 dSdt,$$

$$\mathcal{P}_{1,1,2,2} := \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F |\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}||\varphi - \Pi_h^{\mathbb{Q}}[\varphi]|^2 dSdt.$$

Using the trace inequalities (3.5)–(3.6), we conclude that

$$\begin{aligned}\mathcal{P}_{1,1,2,1} \leq& \int_0^T \sum_{K\in\mathcal{T}_h} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^6(\partial K)}\|\rho_h\|_{L^2(\partial K)}\|\varphi - \Pi_h^{\mathbb{Q}}[\varphi]\|^2_{L^6(\partial K)}dt\\ \leq& Ch^{-1}\int_0^T \sum_{K\in\mathcal{T}_h} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^6(K)}\|\rho_h\|_{L^2(K)}\|\varphi - \Pi_h^{\mathbb{Q}}[\varphi]\|^2_{L^6(K)}dt\\ &+ Ch\int_0^T \sum_{K\in\mathcal{T}_h} \|\boldsymbol{u}_h\|_{\boldsymbol{L}^6(K)}\|\rho_h\|_{L^2(K)}\|\nabla\varphi\|^2_{L^6(K)}dt.\end{aligned}$$

Therefore, using the Hölder inequality and the interpolation error estimate (3.1), we get that

$$\mathcal{P}_{1,1,2,1} \le Ch\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))}\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}\|\nabla\varphi\|^2_{L^{2m_1}(0,T;L^6(\Omega))}. \tag{4.7}$$

By a similar proof to the error estimate of $\mathcal{P}_{1,1,2,2}$, we find

$$\mathcal{P}_{1,1,2,2} \le Ch\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))}\|\nabla\varphi\|^2_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}. \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6), using Lemma 3.6, we have arrived at

$$\mathcal{P}_{1,1,2} \le Ch\|\nabla\varphi\|^2_{L^{2m_1}(0,T;L^6(\Omega))}. \tag{4.9}$$

By applying Lemma 3.7 and (4.9) to (4.4) leads to the bound

$$|\mathcal{P}_1(\varphi)| \le Ch^{\frac{1}{2}}\|\nabla\varphi\|_{L^{2m_1}(0,T;L^6(\Omega))}.$$

For the case $\gamma > 2$, by using Cauchy–Schwarz inequality, we obtain

$$|\mathcal{P}_1(\varphi)| \le \sqrt{\mathcal{P}_{1,2,1}} \times \sqrt{\mathcal{P}_{1,1,2,2}},$$

where $\mathcal{P}_{1,2,1}$ is defined by

$$\mathcal{P}_{1,2,1} := \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F [\![\rho_h]\!]^2 |\boldsymbol{u}_{h,F}\cdot\boldsymbol{n}| dSdt.$$

By virtue of the inequality (4.8), Lemmas 3.6 and 3.8, we have

$$|\mathcal{P}_1(\varphi)| \le Ch^{\frac{1}{2}}\|\nabla\varphi\|_{L^4(0,T;L^{\frac{12}{5}}(\Omega))}.$$

**Bound on** $\mathcal{P}_2(\varphi)$. By applying the Hölder inequality, the inverse estimate (3.4) and the embedding $L^{2\gamma} \hookrightarrow L^2$ for $\gamma > 1$, we obtain

$$\begin{aligned}
|\mathcal{P}_2(\varphi)| \le& \|\varphi - \Pi_h^{\mathbb{Q}}[\varphi]\|_{L^{m_1}(0,T;L^6(\Omega))}\|\rho_h\|_{L^{2\gamma}(0,T;L^3(\Omega))}\|\operatorname{div}\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}\\
\le& Ch^{-\frac{1}{2}}\|\varphi - \Pi_h^{\mathbb{Q}}[\varphi]\|_{L^{m_1}(0,T;L^6(\Omega))}\|\rho_h\|_{L^{2\gamma}(0,T;L^2(\Omega))}\|\operatorname{div}\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}\\
\le& Ch^{\frac{1}{2}}\|\nabla\varphi\|_{L^{m_1}(0,T;L^6(\Omega))}\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}\|\operatorname{div}\boldsymbol{u}_h\|_{L^2(0,T;L^2(\Omega))}\\
\le& Ch^{\frac{1}{2}}\|\nabla\varphi\|_{L^{m_1}(0,T;L^6(\Omega))}.
\end{aligned}$$

**Bound on** $\mathcal{P}_3(\varphi)$. We shall treat the case $1 < \gamma \le 2$ and $\gamma > 2$ separately. If $1 < \gamma \le 2$, by applying the Cauchy Schwarz inequality, we have

$$|\mathcal{P}_3(\varphi)| \le \sqrt{\mathcal{P}_{3,1,1}} \times \sqrt{\mathcal{P}_{3,1,2}}, \tag{4.10}$$

where $\mathcal{P}_{3,1}$ and $\mathcal{P}_{3,2}$ are defined by

$$\begin{aligned}
\mathcal{P}_{3,1,1} :=& \int_0^T h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\overline{\eta}_{\rho,h})[\![\rho_h]\!]^2 dSdt,\\
\mathcal{P}_{3,1,2} :=& \int_0^T h^{\epsilon-1} \sum_{F\in\mathcal{F}_{h,int}} \int_F (\mathcal{H}''(\overline{\eta}_{\rho,h}))^{-1}[\![\Pi_h^{\mathbb{Q}}[\varphi] - \varphi]\!]^2 dSdt.
\end{aligned}$$

By employing the inequality (4.5), we can show

$$\mathcal{P}_{3,1,2} \le C(\mathcal{P}_{3,1,2,1} + \mathcal{P}_{3,1,2,2}), \tag{4.11}$$

where $\mathcal{P}_{3,1,2,1}$ and $\mathcal{P}_{3,1,2,2}$ are given by

$$\mathcal{P}_{3,1,2,1} := h^{\epsilon-1} \int_0^T \sum_{F \in \mathcal{F}_{h,int}} \int_F |\varphi - \Pi_h^{\mathbb{Q}}[\varphi]|^2 dSdt,$$

$$\mathcal{P}_{3,1,2,2} := h^{\epsilon-1} \int_0^T \sum_{F \in \mathcal{F}_{h,int}} \int_F |\rho_{h,+} + \rho_{h,-}||\varphi - \Pi_h^{\mathbb{Q}}[\varphi]|^2 dSdt.$$

By applying the trace estimate (3.5)–(3.6) and the Poincaré inequality (3.1), the embedding $L^{2\gamma} \hookrightarrow L^2$ for $\gamma > 1$, we get

$$\begin{aligned}
\mathcal{P}_{3,1,2,1} \leq & Ch^{\epsilon-2}\|\Pi_h^{\mathbb{Q}}[\varphi] - \varphi\|^2_{L^2((0,T)\times\Omega)} + Ch^{\epsilon}\|\nabla\varphi\|^2_{L^2((0,T)\times\Omega)} \\
\leq & Ch^{\epsilon}\|\nabla\varphi\|^2_{L^2((0,T)\times\Omega)}, \qquad (4.12) \\
\mathcal{P}_{3,1,2,2} \leq & Ch^{\epsilon-2}\|\rho_h\|_{L^2((0,T)\times\Omega)}\|\Pi_h^{\mathbb{Q}}[\varphi] - \varphi\|^2_{L^4((0,T)\times\Omega)} \\
& + Ch^{\epsilon}\|\rho_h\|_{L^2((0,T)\times\Omega)}\|\nabla\varphi\|^2_{L^4((0,T)\times\Omega)} \\
\leq & Ch^{\epsilon}\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}\|\nabla\varphi\|^2_{L^4((0,T)\times\Omega)}. \qquad (4.13)
\end{aligned}$$

Inserting (4.12) and (4.13) into (4.11), using Lemma 3.6, we conclude that

$$\mathcal{P}_{3,1,2} \leq Ch^{\epsilon}\|\nabla\varphi\|^2_{L^4((0,T)\times\Omega)}. \qquad (4.14)$$

For the inequality (4.10), by using (4.14) and Lemma 3.7, we easily see that

$$|\mathcal{P}_3(\varphi)| \leq Ch^{\frac{\epsilon}{2}}\|\nabla\varphi\|_{L^4((0,T)\times\Omega)}.$$

For the case $\gamma > 2$, by applying the Cauchy Schwarz inequality, we obtain

$$|\mathcal{P}_3(\varphi)| \leq \sqrt{\mathcal{P}_{3,2,1}} \times \sqrt{\mathcal{P}_{3,1,2,1}},$$

where $\mathcal{P}_{3,2,1}$ is given by

$$\mathcal{P}_{3,2,1} := \int_0^T h^{\epsilon-1} \sum_{F \in \mathcal{F}_{h,int}} \int_F [\![\rho_h]\!]^2 dSdt.$$

According to Lemma 3.8 and the estimate (4.12), we have

$$|\mathcal{P}_3(\varphi)| \leq Ch^{\frac{\epsilon}{2}}\|\nabla\varphi\|_{L^4((0,T)\times\Omega)}.$$

**Bound on** $\mathcal{P}_4(\varphi)$. By employing the Hölder inequality, the trace estimates (3.5)–(3.6), the Poincaré and inverse inequalities, the embedding $L^{2\gamma} \hookrightarrow L^2$ for $\gamma > 1$, we easily establish that

$$\begin{aligned}
|\mathcal{P}_4(\varphi)| \leq & Ch^{\frac{1}{2}}\|\nabla\varphi\|_{L^{m_1}(0,T;L^6(\Omega))}\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \\
\leq & Ch^{\frac{1}{2}}\|\nabla\varphi\|_{L^{m_1}(0,T;L^6(\Omega))}.
\end{aligned}$$

Combining the above analysis, we have the required estimate (4.3). The proof is thus complete. □

## 5 Basic Estimates

This section establishes the boundedness of discrete time derivative $D_t \rho_h$ and a priori estimate of discrete density $\rho_h$ in $L^2(0, T; \mathbb{Q}_h)$.

**Lemma 5.1** *Suppose that the conditions of Lemma 4.2 are satisfied, then the discrete time derivative $D_t \rho_h$ satisfies*

$$\|D_t \rho_h\|_{L^{m_2}(0,T;W^{-1,\frac{6}{5}}(\Omega))} \le C, \quad 1 < m_2 := \frac{4\gamma}{3\gamma+1}. \tag{5.1}$$

***Proof*** Let $\phi \in L^{2m_1}(0, T; W^{1,6}(\Omega))$ such that $\|\phi\|_{L^{2m_1}(0,T;W^{1,6}(\Omega))} = 1$. Taking $\varphi_h = \Pi_h^{\mathbb{Q}}[\phi]$ in (3.7) and summing this result with respect to $n$ from $n = 1$ to $n = N$, applying the same argument as Lemma 4.1, we infer that

$$\int_0^T \int_\Omega D_t \rho_h \phi dxdt = \int_0^T \int_\Omega \rho_h \boldsymbol{u}_h \cdot \nabla\phi dxdt + \int_0^T \int_\Omega \mathscr{R}_h \cdot \nabla\phi dxdt.$$

Using Hölder inequality, Lemmas 4.2 and 3.6, we conclude that

$$\begin{aligned}|\int_0^T \int_\Omega D_t \rho_h \phi dxdt| \le & C\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)} \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{L}^6(\Omega))} \|\nabla\phi\|_{L^{m_1}(0,T;L^3(\Omega))} \\ & + C\|\nabla\phi\|_{L^{2m_1}(0,T;L^6(\Omega))} \le C\|\phi\|_{L^{2m_1}(0,T;W^{1,6}(\Omega))}.\end{aligned}$$

This inequality immediately implies Lemma 5.1. The proof is thus complete. □

**Lemma 5.2** *Suppose that Hypothesis 2.1 and the CFL condition $\tau \approx h$ are satisfied, there exists $\epsilon_0 > 0$ and $0 < \delta < 1$ such that for any $0 < \epsilon < \epsilon_0$,*

$$\int_0^T \|\rho_h\|_{\mathbb{Q}_h}^2 dt \le Ch^{-2\delta}. \tag{5.2}$$

***Proof*** We divide our proof in two steps. Firstly, if $1 < \gamma \le 2$, by applying Cauchy–Schwarz inequality, we obtain

$$\int_0^T \|\rho_h\|_{\mathbb{Q}_h}^2 dt \le \sqrt{\mathcal{P}_5} \times \sqrt{\mathcal{P}_6}, \tag{5.3}$$

where $\mathcal{P}_5$ and $\mathcal{P}_6$ are given by

$$\begin{aligned}\mathcal{P}_5 :=& h^{\epsilon-1} \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F \mathcal{H}''(\overline{\eta}_{\rho,h}) [\![\rho_h]\!]^2 dSdt, \\ \mathcal{P}_6 :=& h^{-(\epsilon+1)} \int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F (\mathcal{H}''(\overline{\eta}_{\rho,h}))^{-1} [\![\rho_h]\!]^2 dSdt.\end{aligned}$$

According to the trace estimate (3.6) and the inequality (4.5), we infer that

$$\mathcal{P}_6 \le Ch^{-(\epsilon+2)} \|\rho_h\|_{L^{4-\gamma}((0,T)\times\Omega)}^{4-\gamma}. \tag{5.4}$$

On the one hand, for $1 < \gamma < \frac{4}{3}$, it is easy check that $4 - \gamma > 2\gamma$. Therefore, by applying the inverse estimate (3.4) and the CFL condition $\tau \approx h$, we have

$$\|\rho_h\|_{L^{4-\gamma}((0,T)\times\Omega)}^{4-\gamma} \le Ch^{6-\frac{8}{\gamma}} \|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}^{4-\gamma}. \tag{5.5}$$

On the other hand, for the case $\frac{4}{3} \le \gamma < 2$, by using the embedding result $L^{2\gamma} \hookrightarrow L^{4-\gamma}$, we conclude that

$$\|\rho_h\|_{L^{4-\gamma}((0,T)\times\Omega)}^{4-\gamma} \le C\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}^{4-\gamma}. \tag{5.6}$$

Inserting (5.5) and (5.6) into (5.4), using Lemma 3.6, we obtain

$$\mathcal{P}_6 \le \begin{cases} Ch^{-4(\frac{\epsilon}{4}+\frac{2}{\gamma}-1)}, & 1<\gamma<\frac{4}{3}, \\ Ch^{-4(\frac{\epsilon}{4}+\frac{1}{2})}, & \frac{4}{3}\le\gamma\le 2. \end{cases} \tag{5.7}$$

By substituting (5.7) into (5.3), using Lemma 3.7, we get that

$$\int_0^T \|\rho_h\|_{\mathbb{Q}_h}^2 dt \le \begin{cases} Ch^{-2(\frac{\epsilon}{4}+\frac{2}{\gamma}-1)}, & 1<\gamma<\frac{4}{3}, \\ Ch^{-2(\frac{\epsilon}{4}+\frac{1}{2})}, & \frac{4}{3}\le\gamma\le 2. \end{cases} \tag{5.8}$$

Secondly, for the case $\gamma > 2$, by using Lemma 3.8, we easily see that

$$\int_0^T \|\rho_h\|_{\mathbb{Q}_h}^2 dt = h^{-1}\int_0^T \sum_{F\in\mathcal{F}_{h,int}} \int_F [\![\rho_h]\!]^2 dSdt \le Ch^{-\epsilon}. \tag{5.9}$$

Combining the inequalities (5.8) and (5.9), we have the required estimate (5.2), where the parameter $\alpha_0$ and $\delta$ are given by

$$\epsilon_0 := \begin{cases} 8(1-\frac{1}{\gamma}), & 1<\gamma<\frac{4}{3}, \\ 2, & \frac{4}{3}\le\gamma. \end{cases}, \quad \delta := \begin{cases} \frac{\epsilon}{4}+\frac{2}{\gamma}-1, & 1<\gamma<\frac{4}{3}, \\ \frac{\epsilon}{4}+\frac{1}{2}, & \frac{4}{3}\le\gamma<2, \\ \frac{\epsilon}{2}, & \gamma\ge 2. \end{cases}$$

It is easy check that $\delta < 1$. The proof is thus complete. □

***Remark 5.1*** (i) In fact, for the case of $\gamma \ge \frac{4}{3}$, the CFL condition $\tau \approx h$ is not required for the estimate (5.2). (ii) Lemma 5.2 plays a key role in deriving the discrete version of the effective viscous flux identity. See Lemma 6.7 and Theorem 6.2 for more on why it is needed.

## 6 Convergence Analysis

In this section, we will prove the family $(\rho_h, \boldsymbol{u}_h, p(\rho_h))$ defined in (3.16)–(3.17) converges to weak solution ( see Definition 2.1). For that purpose, we first need to establish a spatial compactness estimate for Bernardi–Raugel finite element space.

**Theorem 6.1** *Let $q$ satisfies $2 \le q < 6$ and $\frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{6}$, $\theta \in [0,1]$. For any $\boldsymbol{v}_h \in \mathbb{V}_h$, there exists a constant $C > 0$ such that the following estimate holds*

$$\|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|_{\boldsymbol{L}^q(\mathbb{R}^d)} \le C|\boldsymbol{\xi}|^\theta \|\nabla \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}, \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d.$$

The proof of Theorem 6.1 can been found in "Appendix A.2".

According to Lemma 3.6 and Theorem 3.2, we can assert the existence of functions

$$\rho \in L^\infty(0,T; L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega), \quad \boldsymbol{u} \in L^2(0,T; \boldsymbol{H}_0^1(\Omega))$$

such that the family $(\rho_h, \boldsymbol{u}_h)$ defined in (3.16)–(3.17) exists suitable subsequences satisfy

$$\begin{cases} \rho_h \rightharpoonup^* \rho, \text{ in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^{2\gamma}((0,T)\times\Omega), \\ p(\rho_h) \rightharpoonup p, \text{ in } L^2((0,T)\times\Omega), \quad \boldsymbol{u}_h \rightharpoonup \boldsymbol{u}, \text{ in } \boldsymbol{L}^2((0,T)\times\Omega), \\ \operatorname{div}\boldsymbol{u}_h \rightharpoonup \operatorname{div}\boldsymbol{u}, \text{ in } L^2((0,T)\times\Omega), \quad \boldsymbol{u}_h \rightharpoonup \boldsymbol{u}, \text{ in } L^2(0,T;\boldsymbol{H}^1(\Omega)). \end{cases} \tag{6.1}$$

The following lemma can be found in [26, Lemma 2.3].

**Lemma 6.1** *Let $\{f_h\}_{h>1}^\infty$ and $\{g_h\}_{h>1}^\infty$ be two function sequences on $(0,T)\times\Omega$ such that*

(i) *$f_h$ and $g_h$ converge weakly to $f$ and $g$ respectively in $L^{p_1}(0,T;L^{q_1}(\Omega))$ and $L^{p_2}(0,T;L^{q_2}(\Omega))$, where $1 \le p_1, q_1 \le \infty$, $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{q_1}+\frac{1}{q_2}=1$.*
(ii) *Assume that $\frac{g_h(t,\boldsymbol{x})-g_h(t-h,\boldsymbol{x})}{h}$ is bounded in $L^1(0,T;W^{-m,1}(\Omega))$, for some $m \ge 0$ independent of $h$. And $\|f_h(t,\boldsymbol{x})-f_h(t,\boldsymbol{x}-\boldsymbol{\xi})\|_{L^{p_1}(0,T;L^{p_2}(\Omega))} \to 0$ as $|\boldsymbol{\xi}| \to 0$ uniformly in $h$.*

*Then, $f_h g_h$ converges to $fg$ in the sense of distributions on $(0,T)\times\Omega$.*

Next, we present a weak convergent results for $\rho_h \boldsymbol{u}_h$.

**Lemma 6.2** *Suppose that the condition of Lemma 4.2 are satisfied, then the family $(\rho_h, \boldsymbol{u}_h)$ defined in (3.16)–(3.17) satisfies*

$$\rho_h \boldsymbol{u}_h \rightharpoonup \rho\boldsymbol{u}, \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

***Proof*** From Lemma 5.1, we can show

$$D_t \rho_h \in L^1(0,T;W^{-1,1}(\Omega)). \tag{6.2}$$

By applying Theorem 6.1 and Lemma 3.6, we conclude that

$$\|\boldsymbol{u}_h(t,\boldsymbol{x}) - \boldsymbol{u}_h(t,\boldsymbol{x}-\boldsymbol{\xi})\|_{L^2(0,T;\boldsymbol{L}^2(\mathbb{R}^d))} \xrightarrow{|\boldsymbol{\xi}|\to 0} 0. \tag{6.3}$$

By substituting (6.1)–(6.3) into Lemma 6.1, the proof is thus complete. □

## 6.1 Limit in the Compressible Stokes Equations

In this subsection, we can show the limit $(\rho, \boldsymbol{u}, p)$ constructed in (6.1) is a weak solution of Definition 2.1. The remaining major difficulty is to prove the pressure $p(\rho_h) \to p(\rho)$.

**Lemma 6.3** *Suppose that the condition of Lemma 4.2 is satisfied, then the accumulation point $(\rho, \boldsymbol{u})$ constructed in (6.1) satisfies the weak formulation (2.1).*

***Proof*** We pass to the limit with $h, \tau \to 0$ in the consistency formulation (4.1). Firstly, we rewrite the discrete time derivative term

$$
\begin{aligned}
\int_0^T \int_\Omega D_t \rho_h \varphi dxdt =& -\int_0^T \int_\Omega \rho_h D_t \varphi(t+\tau, \cdot) dxdt + \frac{1}{\tau}\int_{T-\tau}^T \int_\Omega \rho_h(t,\cdot)\varphi(t+\tau,\cdot) dxdt \\
& - \frac{1}{\tau}\int_{-\tau}^0 \int_\Omega \rho_h(t,\cdot)\varphi(t+\tau,\cdot) dxdt \\
=& -\int_0^T \int_\Omega \rho_h D_t \varphi(t+\tau,\cdot) dxdt - \int_\Omega \rho_h^0 \varphi(0,\cdot) dx \\
& - \int_0^\tau \int_\Omega \rho_h^0 \frac{\varphi(t,\cdot)-\varphi(0,\cdot)}{\tau} dxdt \\
=& -\int_0^T \int_\Omega \rho_h (\partial_t \varphi(t,\cdot) + \frac{\tau}{2}\partial_{tt}\varphi(t_\dagger,\cdot)) dxdt - \int_\Omega \rho_h^0 \varphi(0,\cdot) dx \\
& - \int_0^\tau \int_\Omega \rho_h^0 \partial_t \varphi(t_\ddagger,\cdot) dxdt,
\end{aligned}
$$

where $t_\dagger \in (t, t+\tau)$ and $t_\ddagger \in (0,\tau)$. By applying Lemma 3.6 and the embedding $L^\gamma \hookrightarrow L^1$, we have

$$
\begin{aligned}
\frac{\tau}{2}\left|\int_0^T \int_\Omega \rho_h \partial_{tt}\varphi(t_\dagger,\cdot) dxdt\right| &\le C\tau \|\rho_h\|_{L^1((0,T)\times\Omega)} \|\partial_{tt}\varphi\|_{L^\infty((0,T)\times\Omega)} \\
&\le C\tau \|\rho_h\|_{L^\infty(0,T;L^\gamma(\Omega))} \|\partial_{tt}\varphi\|_{L^\infty((0,T)\times\Omega)} \overset{h,\tau\to 0}{\longrightarrow} 0, \\
\left|\int_0^\tau \int_\Omega \rho_h \partial_t \varphi(t_\ddagger,\cdot) dxdt\right| &\le C\tau \|\rho_h^0\|_{L^1(\Omega)} \|\partial_t\varphi\|_{L^\infty((0,T)\times\Omega)} \\
&\le C\tau \|\rho_0\|_{L^\gamma(\Omega)} \|\partial_t\varphi\|_{L^\infty((0,T)\times\Omega)} \overset{h,\tau\to 0}{\longrightarrow} 0.
\end{aligned}
$$

According to (6.1) and $\Pi_h^{\mathbb{Q}}[\rho_0] \rightharpoonup \rho_0$ in $L^\gamma(\Omega)$, we obtain

$$
\begin{aligned}
& -\int_0^T \int_\Omega \rho_h \partial_t \varphi(t,\cdot) dxdt - \int_\Omega \rho_h^0 \varphi(0,\cdot) dx \\
& \overset{h,\tau\to 0}{\longrightarrow} -\int_0^T \int_\Omega \rho \partial_t \varphi dxdt - \int_\Omega \rho_0 \varphi(0,\cdot) dx.
\end{aligned}
$$

Next, by applying Lemma 6.2, we can show

$$
\int_0^T \int_\Omega \rho_h \boldsymbol{u}_h \cdot \nabla\varphi dxdt \overset{h,\tau\to 0}{\longrightarrow} \int_0^T \int_\Omega \rho \boldsymbol{u} \cdot \nabla\varphi dxdt.
$$

Finally, by employing the inequality (4.3) of Lemma 4.2, we conclude that

$$
\int_0^T \int_\Omega \mathscr{R}_h \cdot \nabla\varphi dxdt \overset{h,\tau\to 0}{\longrightarrow} 0.
$$

Combining the above analysis, the proof is thus complete. □

**Lemma 6.4** *Suppose that Hypothesis 2.1 is satisfied, the accumulation limit* $(\rho, \boldsymbol{u})$ *constructed in* (6.1) *satisfies the following weak formulation:*

$$\mu \int_0^T \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} dxdt + (\lambda + \mu) \int_0^T \int_\Omega \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v} dxdt \\ - \int_0^T \int_\Omega p \operatorname{div} \boldsymbol{v} dxdt = 0, \quad \forall\, \boldsymbol{v} \in \boldsymbol{C}_0^\infty((0,T) \times \Omega). \tag{6.4}$$

***Proof*** We define $\mathcal{F}_{\mathbb{V}_h}$ as the $\boldsymbol{L}^2$-orthogonal projection operator from $\boldsymbol{L}^2(\Omega)$ into $\mathbb{V}_h$. For any $\boldsymbol{v} \in \boldsymbol{C}_0^\infty((0,T) \times \Omega)$, we can choose $\boldsymbol{v}_h = \mathcal{F}_{\mathbb{V}_h} \boldsymbol{v}$ and $\boldsymbol{v}_h^n = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} \boldsymbol{v}_h dt$ such that

$$\boldsymbol{v}_h(t, \cdot) \xrightarrow{h \to 0} \boldsymbol{v}(t, \cdot), \text{ in } \boldsymbol{H}_0^1(\Omega), \tag{6.5}$$

for any $t \in (0, T)$. Taking $\boldsymbol{v}_h^n$ in (3.8), multiplying by $\tau$ and summing the results with respect to $n$ from $n = 1$ to $n = N$, we conclude that

$$\int_0^T \int_\Omega [\mu \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h + (\lambda + \mu) \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h] dxdt \\ - \int_0^T \int_\Omega p(\rho_h) \operatorname{div} \boldsymbol{v}_h dxdt = 0.$$

Obviously, by applying (6.1) and (6.5), we have the required weak formulation (6.4). The proof is thus complete. □

### 6.2 Strong Convergence of the Density

The strong convergence of the density is proved by the discrete version of the weak continuity property of the effective viscous flux introduced on the continuous level in [30]. For this purpose, we first introduce the following notation

$$\operatorname{curl} \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \text{ if } d = 2, \ \operatorname{curl} \boldsymbol{v} = \left[ \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right], \text{ if } d = 3,$$

where $\boldsymbol{v}$ is a vector-valued function. Obviously, if $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and $\boldsymbol{w} \in \boldsymbol{H}_0^1(\Omega)$, we can show

$$\int_\Omega \nabla \boldsymbol{v} : \nabla \boldsymbol{w} dx = \int_\Omega \operatorname{div} \boldsymbol{v} \operatorname{div} \boldsymbol{w} dx + \int_\Omega \operatorname{curl} \boldsymbol{v} \operatorname{curl} \boldsymbol{w} dx. \tag{6.6}$$

Next, we report the following Lemma, which plays a key role in deriving the discrete version of the effective viscous flux.

**Lemma 6.5** *Let* $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ *be a bounded open set. For any* $1 < r < \infty$ *and* $q \in L^r(\Omega)$, *there exists* $\boldsymbol{w} \in \boldsymbol{W}^{1,r}(\Omega)$ *such that*

$$\operatorname{div} \boldsymbol{w} = q, \quad \operatorname{curl} \boldsymbol{w} = 0, \text{ a.e. in } \Omega, \quad \|\boldsymbol{w}\|_{\boldsymbol{W}^{1,r}(\Omega)} \le C \|q\|_{L^r(\Omega)},$$

*where* $C$ *only depends on* $\Omega$ *and* $r$. *Moreover, if* $q \in W^{1,r}(\Omega)$ *(or* $q \in W^{-1,r}(\Omega)$*), it is possible to have* $\boldsymbol{w} \in \boldsymbol{W}^{2,r}(\Omega)$ *(or* $\boldsymbol{w} \in \boldsymbol{L}^r(\Omega)$*) such that*

$$\|\boldsymbol{w}\|_{\boldsymbol{W}^{2,r}(\Omega)} \le C \|q\|_{W^{1,r}(\Omega)}, \ (\text{or } \|\boldsymbol{w}\|_{\boldsymbol{L}^r(\Omega)} \le C \|q\|_{W^{-1,r}(\Omega)}).$$

***Proof*** It is easy to check that $\nabla\Delta^{-1}[q]$ can be served as the desired solution, where $\Delta^{-1}$ is the inverse of the Laplacian on $\mathbb{R}^3$, and here we applied to $q$ extended by 0 outside $\Omega$. Obviously, $\nabla\Delta^{-1}$ is a continuous linear operator from $L^r(\Omega)$ to $\boldsymbol{W}^{1,r}(\Omega)$ and from $W^{1,r}(\Omega)$ to $\boldsymbol{W}^{2,r}(\Omega)$, from $W^{-1,r}(\Omega)$ to $\boldsymbol{L}^r(\Omega)$ (see e.g., [29, Lemma 8.3]). The proof is thus complete. □

In the next step, we introduce the operator $\Pi_h^{\mathbb{Y}} : \mathbb{Q}_h \mapsto \mathbb{Y}_h$ which interpolates the piecewise constant functions to the space of continuous finite element space $\mathbb{Y}_h$,

$$\forall\, q_h \in \mathbb{Q}_h, \quad \Pi_h^{\mathbb{Y}}[q_h](A) := \frac{1}{\operatorname{card}(N_A)} \sum_{K\in N_A} q_h|_K,$$

for any vertices $A$ in the discretization, where $N_A$ is the set of elements $K \in \mathcal{T}_h$ of which takes $A$ as its vertices. The operator $\Pi_h^{\mathbb{Y}}$ satisfies the following results (see e.g., [10, Lemma 5.8]).

**Lemma 6.6** *For any $q_h \in \mathbb{Q}_h$, there exists a constant $C > 0$, depending only on the shape-regularity of $\mathcal{T}_h$ such that*

$$\|\nabla\Pi_h^{\mathbb{Y}}[q_h]\|_{L^2(\Omega)} \le C\|q_h\|_{\mathbb{Q}_h}, \quad \|q_h - \Pi_h^{\mathbb{Y}}[q_h]\|_{L^2(\Omega)} \le Ch\|q_h\|_{\mathbb{Q}_h}.$$

Then we can prove the following estimates.

**Lemma 6.7** *Suppose that the condition of Lemma 5.2 is satisfied, there exists a constant $C > 0$ such that the following estimates hold*

$$\int_0^T \|\rho_h - \Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{L^2(\Omega)} dt \le Ch^{2(1-\delta)}, \quad \int_0^T \|\Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{L^2(\Omega)} dt \le C,$$
$$\int_0^T \|\Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{H^1(\Omega)} dt \le Ch^{-2\delta}.$$

***Proof*** By applying the inequality (5.2) and Lemma 6.6, we can show

$$\int_0^T \|\rho_h - \Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{L^2(\Omega)} dt \le Ch^{2(1-\delta)}, \quad \int_0^T \|\nabla\Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{L^2(\Omega)} dt \le Ch^{-2\delta}.$$

According to the embedding $L^{2\gamma} \hookrightarrow L^2$ for $\gamma > 1$, we have

$$\begin{aligned}\int_0^T \|\Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{L^2(\Omega)} dt \le& 2\int_0^T \|\rho_h - \Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{L^2(\Omega)} dt + 2\int_0^T \|\rho_h\|^2_{L^2(\Omega)} dt \\ \le& Ch^{2(1-\delta)} + C\|\rho_h\|^2_{L^{2\gamma}((0,T)\times\Omega)} \le C.\end{aligned}$$

These inequalities immediately implies

$$\int_0^T \|\Pi_h^{\mathbb{Y}}[\rho_h]\|^2_{H^1(\Omega)} dt \le Ch^{-2\delta}.$$

Combining the above analysis, the proof is thus complete. □

**Theorem 6.2** *Suppose that the condition of Lemma 5.2 is satisfied. The family $(\rho_h, \boldsymbol{u}_h)$ defined in (3.16)–(3.17) and the accumulation limit $(\rho, \boldsymbol{u})$ constructed in (6.1) satisfy the following convergence properties:*

$$\begin{aligned}&\lim_{h,\tau\to 0} \int_0^T \psi \int_\Omega ((\lambda + 2\mu)\operatorname{div}\boldsymbol{u}_h - p(\rho_h))\rho_h\varphi\, dxdt \\ &= \int_0^T \psi \int_\Omega ((\lambda + 2\mu)\operatorname{div}\boldsymbol{u} - p)\rho\varphi\, dxdt. \qquad (6.7)\end{aligned}$$

*for any* $\psi \in C_0^\infty((0,T))$ *and* $\varphi \in C_0^\infty(\Omega)$.

***Proof*** According to Lemmas 6.5, 6.7 and 3.6, the inequality (5.1), and there exists $\boldsymbol{w}_{\mathbb{Y},h} \in L^2(0,T;\boldsymbol{H}^2(\Omega))$ and $\boldsymbol{w}_h \in L^2(0,T;\boldsymbol{H}^1(\Omega))$ such that

$$\begin{cases} \operatorname{div} \boldsymbol{w}_{\mathbb{Y},h} = \Pi_h^{\mathbb{Y}}[\rho_h], \text{ in } (0,T)\times\Omega, \\ \operatorname{curl} \boldsymbol{w}_{\mathbb{Y},h} = 0, \text{ in } (0,T)\times\Omega, \end{cases} \quad \begin{cases} \operatorname{div} \boldsymbol{w}_h = \rho_h, \text{ in } (0,T)\times\Omega, \\ \operatorname{curl} \boldsymbol{w}_h = 0, \text{ in } (0,T)\times\Omega, \end{cases} \tag{6.8}$$

and a generic constant $C$ independent of $h$ and $\tau$ such that

$$\begin{cases} \|\boldsymbol{w}_{\mathbb{Y},h}\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \le C, \\ \|\boldsymbol{w}_{\mathbb{Y},h}\|_{L^2(0,T;\boldsymbol{H}^2(\Omega))} \le C h^{-\delta} \end{cases} \quad \begin{cases} \|\boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \le C, \\ \|D_t \boldsymbol{w}_h\|_{L^{m_2}(0,T;\boldsymbol{L}^{\frac{6}{5}}(\Omega))} \le C. \end{cases} \tag{6.9}$$

Subtracting the right side of (6.8) from its left side, we obtain

$$\operatorname{div}(\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h) = \Pi_h^{\mathbb{Y}}[\rho_h] - \rho_h, \quad \mathbf{curl}(\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h) = 0, \text{ in } (0,T)\times\Omega.$$

By employing Lemmas 6.5 and 6.7, we can show

$$\|\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \le C\|\rho_h - \Pi_h^{\mathbb{Y}}[\rho_h]\|_{L^2((0,T)\times\Omega)} \le Ch^{1-\delta}. \tag{6.10}$$

Using the Lemma A.4 together with the estimates (6.9), we deduce for a suitable subsequence that

$$\boldsymbol{w}_h \to \boldsymbol{w}, \text{ in } \boldsymbol{L}^2((0,T)\times\Omega). \tag{6.11}$$

In addition, the accumulation limit $\boldsymbol{w}$ satisfies the following properties

$$\operatorname{div} \boldsymbol{w} = \rho, \quad \operatorname{curl} \boldsymbol{w} = 0, \text{ in } (0,T)\times\Omega. \tag{6.12}$$

Taking $\boldsymbol{v}_h = \Pi_h^{\mathbb{V}}[\varphi \boldsymbol{w}_{\mathbb{Y},h}]$ in (3.8), multiplying by $\psi \in C_0^\infty((0,T))$ and integrating from $t=0$ to $T$, we derive

$$\begin{aligned} &\mu \int_0^T \psi \int_\Omega \nabla \boldsymbol{u}_h : \nabla(\varphi \boldsymbol{w}_{\mathbb{Y},h}) dxdt + (\lambda+\mu)\int_0^T \psi \int_\Omega \operatorname{div} \boldsymbol{u}_h \operatorname{div}(\varphi \boldsymbol{w}_{\mathbb{Y},h}) dxdt \\ &- \int_0^T \psi \int_\Omega p(\rho_h) \operatorname{div}(\varphi \boldsymbol{w}_{\mathbb{Y},h}) dxdt = \mathscr{R}_{1,h}, \end{aligned} \tag{6.13}$$

where $\mathscr{R}_{1,h}$ is given by

$$\begin{aligned} \mathscr{R}_{1,h} :=& \mu \int_0^T \psi \int_\Omega \nabla \boldsymbol{u}_h : \nabla(\varphi \boldsymbol{w}_{\mathbb{Y},h} - \Pi_h^{\mathbb{V}}[\varphi \boldsymbol{w}_{\mathbb{Y},h}]) dxdt \\ &+ (\lambda+\mu)\int_0^T \psi \int_\Omega \operatorname{div} \boldsymbol{u}_h \operatorname{div}(\varphi \boldsymbol{w}_{\mathbb{Y},h} - \Pi_h^{\mathbb{V}}[\varphi \boldsymbol{w}_{\mathbb{Y},h}]) dxdt. \end{aligned}$$

By applying the inequalities (3.2) and (6.9), we can show

$$\begin{aligned} |\mathscr{R}_{1,h}| \le& C\|\psi\|_{L^\infty((0,T))}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))}\|\varphi \boldsymbol{w}_{\mathbb{Y},h} - \Pi_h^{\mathbb{V}}[\varphi \boldsymbol{w}_{\mathbb{Y},h}]\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \\ \le& Ch\|\psi\|_{L^\infty((0,T))}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))}\|\varphi \boldsymbol{w}_{\mathbb{Y},h}\|_{L^2(0,T;\boldsymbol{H}^2(\Omega))} \\ \le& Ch^{1-\delta}\|\psi\|_{L^\infty((0,T))}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))}\|\varphi\|_{W^{2,\infty}(\Omega)}. \end{aligned} \tag{6.14}$$

Obviously, we have $\operatorname{div}(\varphi \boldsymbol{w}_{\mathbb{Y},h}) = \Pi_h^{\mathbb{Y}}[\rho_h]\varphi + \boldsymbol{w}_{\mathbb{Y},h} \cdot \nabla\varphi$ and $\operatorname{curl}(\varphi \boldsymbol{w}_{\mathbb{Y},h}) = J(\varphi)\boldsymbol{w}_{\mathbb{Y},h}$, where $J(\varphi)$ is a matrix with entries involving some first-order derivatives of $\varphi$. Combining the identities (6.6) and (6.13), we obtain

$$\begin{aligned}
&\int_0^T \psi \int_\Omega ((\lambda + 2\mu) \operatorname{div} \boldsymbol{u}_h - p(\rho_h))\rho_h \varphi dxdt \\
&= -(\lambda + 2\mu) \int_0^T \psi \int_\Omega \operatorname{div} \boldsymbol{u}_h \boldsymbol{w}_{\mathbb{Y},h} \cdot \nabla\varphi dxdt - \mu \int_0^T \psi \int_\Omega \operatorname{curl} \boldsymbol{u}_h \cdot J(\varphi)\boldsymbol{w}_{\mathbb{Y},h} dxdt \\
&\quad + \int_0^T \psi \int_\Omega p(\rho_h)\boldsymbol{w}_{\mathbb{Y},h} \cdot \nabla\varphi dxdt + \mathscr{R}_{1,h} + \mathscr{R}_{2,h},
\end{aligned} \tag{6.15}$$

where $\mathscr{R}_{2,h}$ is defined by

$$\mathscr{R}_{2,h} := \int_0^T \psi \int_\Omega ((\lambda + 2\mu) \operatorname{div} \boldsymbol{u}_h - p(\rho_h))(\rho_h\varphi - \Pi_h^{\mathbb{Y}}[\rho_h]\varphi) dxdt.$$

Applying the Hölder inequality and Lemma 6.7 implies

$$\begin{aligned}
|\mathscr{R}_{2,h}| \le & C\| \operatorname{div} \boldsymbol{u}_h\|_{L^2((0,T)\times\Omega)} \|\rho_h - \Pi_h^{\mathbb{Y}}[\rho_h]\|_{L^2((0,T)\times\Omega)} \\
& + C\|p(\rho_h)\|_{L^2((0,T)\times\Omega)} \|\rho_h - \Pi_h^{\mathbb{Y}}[\rho_h]\|_{L^2((0,T)\times\Omega)} \\
\le & Ch^{1-\delta}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + Ch^{1-\delta}\|p(\rho_h)\|_{L^2((0,T)\times\Omega)}.
\end{aligned} \tag{6.16}$$

The identity (6.15) can be rewritten as

$$\begin{aligned}
&\int_0^T \psi \int_\Omega ((\lambda + 2\mu) \operatorname{div} \boldsymbol{u}_h - p(\rho_h))\rho_h \varphi dxdt \\
&= -(\lambda + 2\mu) \int_0^T \psi \int_\Omega \operatorname{div} \boldsymbol{u}_h \boldsymbol{w}_h \cdot \nabla\varphi dxdt - \mu \int_0^T \psi \int_\Omega \operatorname{curl} \boldsymbol{u}_h \cdot J(\varphi)\boldsymbol{w}_h dxdt \\
&\quad + \int_0^T \psi \int_\Omega p(\rho_h)\boldsymbol{w}_h \cdot \nabla\varphi dxdt + \mathscr{R}_{1,h} + \mathscr{R}_{2,h} + \mathscr{R}_{3,h}.
\end{aligned} \tag{6.17}$$

where $\mathscr{R}_{3,h}$ is given by

$$\begin{aligned}
\mathscr{R}_{3,h} := & -(\lambda + 2\mu) \int_0^T \psi \int_\Omega \operatorname{div} \boldsymbol{u}_h (\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h) \cdot \nabla\varphi dxdt \\
& - \mu \int_0^T \psi \int_\Omega \operatorname{curl} \boldsymbol{u}_h \cdot J(\varphi)(\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h) dxdt \\
& + \int_0^T \psi \int_\Omega p(\rho_h)(\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h) \cdot \nabla\varphi dxdt.
\end{aligned}$$

Using the Hölder inequality and the estimate (6.10), we conclude that

$$\begin{aligned}
|\mathscr{R}_{3,h}| \le & C_{\psi,\varphi}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \|\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
& + C_{\psi,\varphi}\|p(\rho_h)\|_{L^2(0,T;L^2(\Omega))} \|\boldsymbol{w}_{\mathbb{Y},h} - \boldsymbol{w}_h\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
\le & C_{\psi,\varphi}h^{1-\delta}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + C_{\psi,\varphi}h^{1-\delta}\|p(\rho_h)\|_{L^2(0,T;L^2(\Omega))}.
\end{aligned} \tag{6.18}$$

Passing to the limit with $h, \tau \to 0$ in (6.17), using (6.1) and (6.11), we find

$$\begin{aligned}&\lim_{h,\tau\to 0}\int_0^T \psi \int_\Omega ((\lambda+2\mu)\operatorname{div}\boldsymbol{u}_h - p(\rho_h))\rho_h\varphi dxdt \\ &\quad = -(\lambda+2\mu)\int_0^T \psi \int_\Omega \operatorname{div}\boldsymbol{u}\boldsymbol{w}\cdot\nabla\varphi dxdt - \mu\int_0^T \psi \int_\Omega \operatorname{curl}\boldsymbol{u}\cdot J(\varphi)\boldsymbol{w}dxdt \\ &\qquad + \int_0^T \psi \int_\Omega p\boldsymbol{w}\cdot\nabla\varphi dxdt + \lim_{h,\tau\to 0}\mathscr{R}_{1,h} + \lim_{h,\tau\to 0}\mathscr{R}_{2,h} + \lim_{h,\tau\to 0}\mathscr{R}_{3,h}. \end{aligned} \tag{6.19}$$

By applying Lemma 3.6, the estimates (6.14), (6.16), (6.18) and $\delta < 1$, we get that

$$\lim_{h,\tau\to 0}\mathscr{R}_{1,h} + \lim_{h,\tau\to 0}\mathscr{R}_{2,h} + \lim_{h,\tau\to 0}\mathscr{R}_{3,h} = 0. \tag{6.20}$$

Taking $\boldsymbol{v} = \psi\varphi\boldsymbol{w}$ in (6.4) and using the identity (6.12), imply

$$\begin{aligned}&\int_0^T \psi \int_\Omega ((\lambda+2\mu)\operatorname{div}\boldsymbol{u} - p)\rho\varphi dxdt = -(\lambda+2\mu)\int_0^T \psi \int_\Omega \operatorname{div}\boldsymbol{u}\boldsymbol{w}\cdot\nabla\varphi dxdt \\ &\quad - \mu\int_0^T \psi \int_\Omega \operatorname{curl}\boldsymbol{u}\cdot J(\varphi)\boldsymbol{w}dxdt + \int_0^T \psi \int_\Omega p\boldsymbol{w}\cdot\nabla\varphi dxdt. \end{aligned} \tag{6.21}$$

Combining the identities (6.19)–(6.21), we have the required discrete effective viscous flux identity (6.7). The proof is thus complete. □

**Lemma 6.8** (Strong convergence of $\rho_h$) *Suppose that the condition of Lemma 5.2 is satisfied, then, passing to a subsequence if necessary*

$$\rho_h \to \rho \ \text{ in } \ L^1((0,T)\times\Omega). \tag{6.22}$$

***Proof*** Firstly, we can show the sequences $p(\rho_h)\rho_h$, $\log(\rho_h)\rho_h$ and $\rho_h \operatorname{div}\boldsymbol{u}_h$ have the following convergent properties:

$$p(\rho_h)\rho_h \rightharpoonup \overline{p(\rho)\rho}, \quad \log(\rho_h)\rho_h \rightharpoonup \overline{\log(\rho)\rho}, \quad \rho_h \operatorname{div}\boldsymbol{u}_h \rightharpoonup \overline{\rho \operatorname{div}\boldsymbol{u}},$$

in a suitable $L^q((0,T)\times\Omega)$ space with $q > 1$, where the overbar is used to denote the weak limit of a nonlinear function. According to the notation introduced above, we write $p = \overline{p(\rho)}$, then it can be easily checked

$$\begin{aligned}&\lim_{h\to 0}\int_0^T \psi \int_\Omega \phi\left[(\lambda+2\mu)\operatorname{div}_h\boldsymbol{u}_h - p(\rho_h)\right]\rho_h dxdt \\ &\quad = \int_0^T \psi \int_\Omega \phi[(\lambda+2\mu)\overline{\rho\operatorname{div}\boldsymbol{u}} - \overline{p(\rho)\rho}]dxdt, \end{aligned} \tag{6.23}$$

for any $\psi \in C_0^\infty(0,T)$ and $\phi \in C_0^\infty(\Omega)$. By applying the discrete effective viscous flux identity (6.7) and the identity (6.23), we conclude that

$$\int_0^T \psi \int_\Omega \phi(\overline{\rho\operatorname{div}\boldsymbol{u}} - \rho\operatorname{div}\boldsymbol{u})dxdt = \int_0^T \psi \int_\Omega \phi\frac{\overline{p(\rho)\rho} - \overline{p(\rho)}\rho}{\lambda+2\mu}dxdt. \tag{6.24}$$

Take the following functions sequence $\psi_m \in C_0^\infty((0,T))$ and $\phi_n \in C_0^\infty(\Omega)$ such that

$$\begin{aligned}&\psi_m \geq 0;\ \psi_m \to 1;\ \psi_m = 1,\ \frac{1}{m} \leq t \leq T - \frac{1}{m}, \\ &\phi_n \geq 0;\ \phi_n \to 1;\ \phi_n = 1,\ \operatorname{dist}(x,\partial\Omega) \geq \frac{1}{n}.\end{aligned}$$

Let $(\psi, \phi) = (\psi_m, \phi_n)$ in (6.24) and $m, n \to +\infty$, by applying Lebesgue's dominated convergence theorem, we obtain

$$\int_0^T \int_\Omega (\overline{\rho \operatorname{div} \boldsymbol{u}} - \rho \operatorname{div} \boldsymbol{u}) dxdt = \int_0^T \int_\Omega \frac{\overline{p(\rho)\rho} - \overline{p(\rho)}\rho}{\lambda + 2\mu} dxdt. \tag{6.25}$$

For the identity (6.25), by employing Lemma A.1, we get that

$$\int_0^T \int_\Omega (\overline{\rho \operatorname{div} \boldsymbol{u}} - \rho \operatorname{div} \boldsymbol{u}) dxdt \geq 0. \tag{6.26}$$

According to Lemmas 2.1 and 6.3, we obtain $(\rho, \boldsymbol{u})$ is a renormalized solution of the continuity equation (2.1). Therefore, taking $\Phi(\rho) = \rho \log(\rho)$ in Definition 2.2 and integrating over $[0, t_F] \times \Omega$ for the results, we can show

$$\int_\Omega \rho \log(\rho)(t_F, \cdot) dx + \int_0^{t_F} \int_\Omega \rho \operatorname{div} \boldsymbol{u} dxdt = \int_\Omega \rho \log(\rho)(0, \cdot) dx, \tag{6.27}$$

for any $t_F \in [0, T]$.

Taking $(\mathcal{B}(\rho), \varphi_h) = (\rho \log(\rho), 1)$ in the discrete renormalized continuity scheme (3.9) and passing to the limit with $h, \tau \to 0$, we have

$$\int_\Omega \overline{\rho \log(\rho)}(t_F, \cdot) dx + \int_0^{t_F} \int_\Omega \overline{\rho \operatorname{div} \boldsymbol{u}} dxdt \leq \int_\Omega \rho \log(\rho)(0, \cdot) dx, \tag{6.28}$$

for any $t_F \in [0, T]$. Subtracting the identity (6.27) from the inequality (6.28), we can show

$$\int_\Omega (\overline{\rho \log(\rho)} - \rho \log(\rho))(t_F, \cdot) dx \leq \int_0^{t_F} \int_\Omega (\rho \operatorname{div} \boldsymbol{u} - \overline{\rho \operatorname{div} \boldsymbol{u}}) dxdt \tag{6.29}$$

for any $t_F \in [0, T]$. Inserting (6.26) into (6.29), we obtain

$$\int_\Omega (\overline{\rho \log(\rho)} - \rho \log(\rho))(t_F, \cdot) dx \leq 0, \tag{6.30}$$

On the other hand, according to Lemma A.2, we have

$$\overline{\rho \log(\rho)} \geq \rho \log(\rho), \ \text{a.e. in } (0, T) \times \Omega. \tag{6.31}$$

Combining the inequalities (6.30) and (6.31) implies

$$\overline{\rho \log(\rho)} = \rho \log(\rho), \ \text{a.e. in } (0, T) \times \Omega.$$

By applying Lemma A.3, we have the required result (6.22). The proof is thus complete. □

**Theorem 6.3** *Suppose that the condition of Lemma 5.2 is satisfied. For any $q_1 \in [1, 2\gamma)$ and $q_2 \in [1, 2)$, then, passing to a subsequence if necessary*

$$\rho_h \to \rho \ \text{in } L^{q_1}((0, T) \times \Omega), \quad p(\rho_h) \to p(\rho) \ \text{in } L^{q_2}((0, T) \times \Omega).$$

***Proof*** By applying (6.22) and Lemma 3.6, we have

$$\rho_h \to \rho \ \text{in } L^{q_1}((0, T) \times \Omega), \quad q_1 \in [1, 2\gamma). \tag{6.32}$$

Noticing $x^\gamma$ and $x^{\frac{1}{\gamma}}$ are increasing functions for $x \in \mathbb{R}_+$ and $(x - y)^\vartheta \leq x^\vartheta - y^\vartheta$ for $x \geq y \geq 0$ and $\vartheta > 0$, we obtain

$$F_h := (p(\rho_h) - p(\rho))(\rho_h - \rho) \geq a|\rho_h^\gamma - \rho^\gamma|^{\frac{1}{\gamma}+1}, \ \text{in } (0, T) \times \Omega. \tag{6.33}$$

By employing the Hölder inequality, (6.32) and Lemma 3.6, we can show

$$\int_0^T\int_\Omega F_h dxdt \leq \|p(\rho_h)\|_{L^2((0,T)\times\Omega)}\|\rho_h-\rho\|_{L^2((0,T)\times\Omega)} + C\|\rho\|^\gamma_{L^{2\gamma}((0,T)\times\Omega)}\|\rho_h-\rho\|_{L^2((0,T)\times\Omega)}. \tag{6.34}$$

Inserting (6.33) into (6.34), using (6.32), we can show

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega |\rho_h^\gamma-\rho^\gamma|^{\frac{1}{\gamma}+1}dxdt \leq \frac{1}{a}\lim_{h,\tau\to 0}\int_0^T\int_\Omega F_h dxdt = 0,$$

which implies that

$$p(\rho_h)\to p(\rho), \text{ in } L^1((0,T)\times\Omega). \tag{6.35}$$

By applying (6.35) and Lemma 3.6, we conclude that

$$p(\rho_h)\to p(\rho) \text{ in } L^{q_2}((0,T)\times\Omega),$$

where $q_2\in[1,2)$. The proof is thus complete. □

Combining Lemmas 6.3 and 6.4, and Theorem 6.3, we can obtain the main result of the first part of this paper:

**Theorem 6.4** *Let $\Omega\subset\mathbb{R}^d$, $d=2,3$ be a bounded domain and assume that the viscosity coefficients $\mu$ and $\lambda$ satisfy $\mu>0$ and $d\lambda+2\mu>0$. Suppose that the pressure $p=p(\rho)$ satisfies the assumption (1.2) with $\gamma>1$. Furthermore, the initial values $\rho_0$ satisfies Hypothesis 2.1. The family $(\rho_h,\boldsymbol{u}_h)$ defined in (3.16)–(3.17) satisfies $\rho_h>0$ for any $h,\tau>0$ with $\tau\approx h$ and $0<\epsilon<\epsilon_0$. Then we have the following convergent properties:*

$$\begin{aligned}
&\boldsymbol{u}_h\rightharpoonup\boldsymbol{u} \text{ in } L^2(0,T;\boldsymbol{H}^1(\Omega)), && \rho_h\rightharpoonup^*\rho \text{ in } L^\infty(0,T;L^\gamma(\Omega)),\\
&\rho_h\rightharpoonup\rho \text{ in } L^{2\gamma}((0,T)\times\Omega), && p(\rho_h)\rightharpoonup p(\rho) \text{ in } L^2((0,T)\times\Omega),\\
&\rho_h\to\rho \text{ in } L^{q_1}((0,T)\times\Omega), && p(\rho_h)\to p(\rho) \text{ in } L^{q_2}((0,T)\times\Omega),
\end{aligned}$$

*for any $1\leq q_1<2\gamma$ and $1\leq q_2<2$, where $(\rho,\boldsymbol{u})$ is a weak solution of the semi-stationary compressible Stokes equations (1.1)–(1.4) in the sense of Definition 2.1.*

***Remark 6.1*** (i) Theorem 6.4 provides an alternative proof of existence of weak solutions via a mixed DG-FEM based on Bernardi–Raugel finite element for the problem (1.1) under the hypothesis $\gamma>1$. (ii) In the case $\gamma>\frac{4}{3}$, the CFL condition $\tau\approx h$ is not required for Theorem 6.4. It is worth noting that the values of adiabatic exponent $\gamma$ in the convergence result without the CFL condition includes the real fluid range of $\gamma\in[\frac{4}{3},\frac{5}{3}]$, such as the monoatomic gas ($\gamma\sim\frac{5}{3}$) and the diatomic gas ($\gamma\sim\frac{7}{5}$). (iii) Theorem 6.4 is also true with the external force $\boldsymbol{f}\neq\boldsymbol{0}\in\boldsymbol{L}^2((0,T)\times\Omega)$ in the momentum equation.

## 7 Error Estimate

An unconditional error estimate for the semi-stationary compressible Stokes equations is established in the section. Note that the existence of weak solution to this model under the assumption of $\gamma>1$ is proved by Theorem 6.4. Now we report the weak-strong uniqueness for this model. To this end, we introduce the following functional $\mathbb{E} : [0,\infty)\times(0,\infty)\to\mathbb{R}$, which is given by

$$\mathbb{E}(\rho\,|\,\underline{\rho}) := \mathcal{H}(\rho) - \mathcal{H}'(\underline{\rho})(\rho-\underline{\rho}) - \mathcal{H}(\underline{\rho}).$$

Noticing that the function $\mathcal{H}$ is strictly convex in $(0,\infty)$, we obtain

$$\mathbb{E}(\rho\,|\,\underline{\rho}) \geq 0, \quad \text{and} \quad \mathbb{E}(\rho\,|\,\underline{\rho}) = 0 \Leftrightarrow \rho = \underline{\rho}. \tag{7.1}$$

Furthermore, the functional $\mathbb{E}(\rho\,|\,\underline{\rho})$ satisfies the following estimates (see, e.g., [13, 17] for more details)

$$\mathbb{E}(\rho\,|\,\underline{\rho}) \geq C(\underline{\rho}) \begin{cases} (\rho-\underline{\rho})^2, & \frac{1}{2}\underline{\rho} < \rho < 2\underline{\rho}, \\ (1+\rho^{\gamma}), & \text{otherwise}, \end{cases} , \tag{7.2}$$

where $C(\underline{\rho})$ is uniformly bounded if $\underline{\rho}$ lies in some compact subset of $(0,\infty)$. Finally, we introduce the relative energy functional of the problem (1.1), which is defined by

$$\mathcal{E}(\rho\,|\,\underline{\rho}) := \int_{\Omega} \mathbb{E}(\rho\,|\,\underline{\rho})dx.$$

**Theorem 7.1** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and assume that the viscosity coefficient $\mu, \lambda$ satisfies $\mu > 0$ and $3\lambda + 2\mu > 0$. Suppose that the pressure $p = p(\rho)$ satisfies the assumption (1.2) with $\gamma > 1$. Let $(\rho, \boldsymbol{u})$ be a weak solution to the problem (1.1) emanating from the initial data $(\rho_0, \boldsymbol{u}_0)$ with the finite energy $E_0 := \int_{\Omega} \mathcal{H}(\rho_0)dx$ and finite mass $M_0 := \int_{\Omega} \rho_0 dx$. Let $(\underline{\rho}, \underline{\boldsymbol{u}})$ be a strong solution of the same problem belonging to the class*

$$\begin{cases} \nabla\underline{\rho} \in L^2(0,T;L^q(\Omega)), \quad 0 < \underline{\rho}_{\min} \leq \underline{\rho} \leq \underline{\rho}_{\max}, \\ \nabla^2\underline{\boldsymbol{u}} \in \boldsymbol{L}^2(0,T;\boldsymbol{L}^q(\Omega)), \quad \underline{\boldsymbol{u}} \in L^2(0,T;\boldsymbol{H}_0^1(\Omega)), \quad q > \max\{3, \frac{6\gamma}{5\gamma-6}\}, \end{cases}$$

*emanating from the same initial data. Then*

$$\rho = \underline{\rho}, \quad \boldsymbol{u} = \underline{\boldsymbol{u}}, \text{ in } (0,T)\times\Omega.$$

The proof of Theorem 7.1 can been found in [13, Theorem 4.1].

Next, we deduce the discrete version of the relative energy inequality from the scheme (3.7)–(3.8), which will play a key role in the subsequent error estimate. To this end, we first introduce the convenient notations

$$\begin{cases} \underline{\rho}_h^n := \Pi_h^{\mathbb{Q}}[\underline{\rho}^n], \quad \underline{\rho}^n := \underline{\rho}(t_n,\cdot), \quad \forall\, 1 \leq n \leq N, \\ \underline{\boldsymbol{u}}_h^n := \Pi_h^{\mathbb{V}}[\underline{\boldsymbol{u}}^n], \quad \underline{\boldsymbol{u}}^n := \underline{\boldsymbol{u}}(t_n,\cdot), \quad \forall\, 1 \leq n \leq N, \end{cases}$$

where $(\underline{\rho}, \underline{\boldsymbol{u}})$ is a strong solution of the problem (1.1) belonging to the class of $C^2$ functions such that $\underline{\boldsymbol{u}}|_{(0,T)\times\partial\Omega} = \boldsymbol{0}$ and $0 < \underline{\rho}_{\min} \leq \underline{\rho} \leq \underline{\rho}_{\max}$. Furthermore, we define the piecewise constant temporal interpolations of $(\underline{\rho}_h^n, \underline{\boldsymbol{u}}_h^n, \underline{\rho}^n, \underline{\boldsymbol{u}}^n)$, $1 \leq n \leq N$, i.e., for any $t \in [t_{n-1}, t_n]$

$$\begin{cases} \underline{\rho}_h(t,\cdot) := \underline{\rho}_h^n, \quad \underline{\rho}_h^{\star}(t,\cdot) := \underline{\rho}_h^{n-1}, \quad \underline{\boldsymbol{u}}_h(t,\cdot) := \underline{\boldsymbol{u}}_h^n, \\ \underline{\rho}_{\tau}(t,\cdot) := \underline{\rho}^n, \quad \underline{\rho}_{\tau}^{\star}(t,\cdot) := \underline{\rho}^{n-1}, \quad \underline{\boldsymbol{u}}_{\tau}(t,\cdot) := \underline{\boldsymbol{u}}^n. \end{cases} \tag{7.3}$$

**Theorem 7.2** *Suppose that Hypothesis 2.1 and $\gamma > 1$ are satisfied. The family $(\rho_h, \boldsymbol{u}_h)$ and $(\underline{\rho}_\tau, \underline{\boldsymbol{u}}_\tau)$ are defined in (3.16)–(3.17) and (7.3), respectively. Then there exists a constant $C > 0$ independent of $h$ and $\tau$ such that*

$$\|\mathcal{E}(\rho_h \,|\, \underline{\rho}_\tau)\|_{L^\infty(0,T)} \le C. \tag{7.4}$$

***Proof*** Using the identity $\mathcal{H}'(\rho)\rho - \mathcal{H}(\rho) = p(\rho)$ and the Hölder inequality, we can show

$$\begin{aligned}\mathcal{E}(\rho_h^n \,|\, \underline{\rho}^n) \le& C\|\rho_h\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} + C\|\underline{\rho}\|^\gamma_{L^\infty(0,T;L^\gamma(\Omega))} \\ &+ C\|\underline{\rho}\|^{\gamma-1}_{L^\infty(0,T;L^{2(\gamma-1)}(\Omega))}\|\rho_h\|_{L^\infty(0,T;L^\gamma(\Omega))}, \quad \forall\, 1 \le n \le N.\end{aligned}$$

By employing Lemma 3.6, we have the estimate (7.4). This proof is thus complete. □

Now we establish the discrete version of the relative energy inequality.

**Theorem 7.3** *Let the families $(\rho_h, \boldsymbol{u}_h)$ and $(\underline{\rho}_\tau, \underline{\boldsymbol{u}}_\tau)$ be defined as in (3.16)–(3.17) and (7.3), respectively. Then the discrete relative energy inequality holds, for any $1 \le m \le N$,*

$$\begin{aligned}&\mathcal{E}(\rho_h^m \,|\, \underline{\rho}^m) + \mu\int_0^{t_m}\int_\Omega |\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 dxdt \\ &+ (\lambda+\mu)\int_0^{t_m}\int_\Omega |\operatorname{div}(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 dxdt \le \mathcal{E}(\rho_h^0 \,|\, \underline{\rho}^0) + \sum_{i=1}^6 \mathcal{R}_i,\end{aligned} \tag{7.5}$$

*where the remainder terms $\mathcal{R}_i$ ($1 \le i \le 5$) are defined by*

$$\begin{aligned}\mathcal{R}_1 &:= \int_0^{t_m}\int_\Omega [\mu\nabla\underline{\boldsymbol{u}}_\tau : \nabla(\underline{\boldsymbol{u}}_\tau - \boldsymbol{u}_h) + (\lambda+\mu)\operatorname{div}\underline{\boldsymbol{u}}_\tau \operatorname{div}(\underline{\boldsymbol{u}}_\tau - \boldsymbol{u}_h)]dxdt, \\ \mathcal{R}_2 &:= \int_0^{t_m}\int_\Omega [\mu\nabla\boldsymbol{u}_h : \nabla(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{u}}_\tau) + (\lambda+\mu)\operatorname{div}\underline{\boldsymbol{u}}_\tau \operatorname{div}(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{u}}_\tau)]dxdt, \\ \mathcal{R}_3 &:= -\int_0^{t_m}\int_\Omega p(\rho_h)\operatorname{div}\underline{\boldsymbol{u}}_\tau dxdt, \quad \mathcal{R}_4 := \int_0^{t_m}\int_\Omega (\underline{\rho}_\tau - \rho_h)D_t\mathcal{H}'(\underline{\rho}_\tau)dxdt, \\ \mathcal{R}_5 &:= -\int_0^{t_m}\int_\Omega \rho_h\boldsymbol{u}_h\cdot\nabla\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt, \quad \mathcal{R}_6 := -\int_0^{t_m}\int_\Omega \mathscr{R}_h\cdot\nabla\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt.\end{aligned}$$

***Proof*** First, taking $\boldsymbol{v}_h = \underline{\boldsymbol{u}}_h^n$ in the discrete momentum equation (3.8), and summing this result with respect to $n$ from $n = 1$ to $n = m$, we conclude that

$$\begin{aligned}&\int_0^{t_m}\int_\Omega [\mu\nabla\boldsymbol{u}_h : \nabla\underline{\boldsymbol{u}}_\tau + (\lambda+\mu)\operatorname{div}\boldsymbol{u}_h\operatorname{div}\underline{\boldsymbol{u}}_\tau - p(\rho_h)\operatorname{div}\underline{\boldsymbol{u}}_\tau]dxdt \\ &+ \int_0^{t_m}\int_\Omega [\mu\nabla\boldsymbol{u}_h : \nabla(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{u}}_\tau) + (\lambda+\mu)\operatorname{div}\boldsymbol{u}_h\operatorname{div}(\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{u}}_\tau)]dxdt = 0.\end{aligned} \tag{7.6}$$

Next, using the same argument as Lemma 4.1 by taking $\varphi = \mathcal{H}'(\underline{\rho}_\tau^\star)$ in Lemma 4.1, we obtain

$$\begin{aligned}-\int_0^{t_m}\int_\Omega D_t\rho_h\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt =& -\int_0^{t_m}\int_\Omega \rho_h\boldsymbol{u}_h\cdot\nabla\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt \\ &-\int_0^{t_m}\int_\Omega \mathscr{R}_h\cdot\nabla\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt.\end{aligned} \tag{7.7}$$

Note that the numerical diffusion terms $\mathcal{D}_{i,h}^n$ $(1 \le i \le 3)$ in the discrete energy identity (3.10) are all positive, we have

$$\int_0^{t_m}\int_\Omega D_t\mathcal{H}(\rho_h)dxdt + \int_0^{t_m}\int_\Omega [\mu|\nabla \boldsymbol{u}_h|^2 + (\lambda+\mu)|\operatorname{div}\boldsymbol{u}_h|^2]dxdt \le 0. \tag{7.8}$$

By applying the identity

$$\rho_h\mathcal{H}'(\underline{\rho}_\tau) - \rho_h^\star\mathcal{H}'(\underline{\rho}_\tau^\star) = \rho_h(\mathcal{H}'(\underline{\rho}_\tau) - \mathcal{H}'(\underline{\rho}_\tau^\star)) + (\rho_h - \rho_h^\star)\mathcal{H}'(\underline{\rho}_\tau^\star),$$

we rewrite

$$\int_0^{t_m}\int_\Omega D_t\rho_h\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt = \int_0^{t_m}\int_\Omega D_t(\rho_h\mathcal{H}'(\underline{\rho}_\tau))dxdt - \int_0^{t_m}\int_\Omega \rho_h D_t\mathcal{H}'(\underline{\rho}_\tau)dxdt,$$

which implies that

$$\begin{aligned}\int_0^{t_m}\int_\Omega D_t\mathcal{H}(\rho_h)dxdt - \int_0^{t_m}\int_\Omega D_t\rho_h\mathcal{H}'(\underline{\rho}_\tau^\star)dxdt = \int_0^{t_m}\int_\Omega D_t\mathbb{E}(\rho_h\,|\,\underline{\rho}_\tau)dxdt\\ + \int_0^{t_m}\int_\Omega \rho_h D_t\mathcal{H}'(\underline{\rho}_\tau)dxdt - \int_0^{t_m}\int_\Omega D_t(\underline{\rho}_\tau\mathcal{H}'(\underline{\rho}_\tau) - \mathcal{H}(\underline{\rho}_\tau))dxdt.\end{aligned} \tag{7.9}$$

According to the convexity of the function $\mathcal{H}$, we obtain

$$\mathcal{H}(\underline{\rho}_\tau) - \mathcal{H}'(\underline{\rho}_\tau^\star)(\underline{\rho}_\tau - \underline{\rho}_\tau^\star) - \mathcal{H}(\underline{\rho}_\tau^\star) \ge 0. \tag{7.10}$$

By using the inequality (7.10), we have

$$\begin{aligned}\int_0^{t_m}\int_\Omega D_t(\underline{\rho}_\tau\mathcal{H}'(\underline{\rho}_\tau) - \mathcal{H}(\underline{\rho}_\tau))dxdt = \int_0^{t_m}\int_\Omega \underline{\rho}_\tau D_t\mathcal{H}'(\underline{\rho}_\tau)dxdt\\ - \frac{1}{\tau}\int_0^{t_m}\int_\Omega \mathcal{H}(\underline{\rho}_\tau) - \mathcal{H}'(\underline{\rho}_\tau^\star)(\underline{\rho}_\tau - \underline{\rho}_\tau^\star) - \mathcal{H}(\underline{\rho}_\tau^\star)dxdt \le \int_0^{t_m}\int_\Omega \underline{\rho}_\tau D_t\mathcal{H}'(\underline{\rho}_\tau)dxdt.\end{aligned} \tag{7.11}$$

Combining the inequalities (7.6)–(7.9) and (7.11), we obtain the inequality (7.5). This proof is thus complete. □

In the next step, we deduce the approximate version of the relative energy inequality from the estimate (7.5).

**Theorem 7.4** *Suppose that Hypothesis 2.1 is satisfied and the pressure* $p = p(\rho)$ *satisfies the hypothesis (1.2) with* $\gamma > 1$*. Let the internal energy* $\mathcal{H}$ *be given by* $\mathcal{H}(\rho) = \frac{p(\rho)}{\gamma-1}$*. Let the families* $(\rho_h, \boldsymbol{u}_h)$ *and* $(\underline{\rho}_\tau, \underline{\boldsymbol{u}}_\tau)$ *be defined as in (3.16)–(3.17) and (7.3), respectively. Then there exists*

$$\begin{aligned}C :=& C(T, \Omega, M_0, E_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])},\\ &\|(\partial_t\underline{\rho}, \partial_{tt}\underline{\rho}, \nabla\underline{\rho}, \partial_t\nabla\underline{\rho})\|_{L^\infty((0,T)\times\Omega)}, \|\underline{\boldsymbol{u}}\|_{L^\infty(0,T;\boldsymbol{H}^2(\Omega))}) > 0,\end{aligned}$$

*such that for any* $1 \le m \le N$*, we have the approximate relative energy inequality holds,*

$$\begin{aligned}&\mathcal{E}(\rho_h^m\,|\,\underline{\rho}^m) + \int_0^{t_m}\int_\Omega [\mu|\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 + (\lambda+\mu)|\operatorname{div}(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2]dxdt\\ \le& \mathcal{E}(\rho_h^0\,|\,\underline{\rho}^0) + \mathcal{R}_1 + \mathcal{R}_3 + \sum_{i=1}^3 \mathcal{L}_3,\end{aligned} \tag{7.12}$$

*where the remainder terms* $\mathcal{L}_i$ *(*$1 \le i \le 3$*) are defined by*

$$\mathcal{L}_1 := \int_0^{t_m} \int_\Omega (\underline{\rho}_\tau - \rho_h) \frac{p'(\underline{\rho}_\tau)}{\underline{\rho}_\tau} [\partial_t \underline{\rho}]_\tau dxdt, \quad \mathcal{L}_2 := -\int_0^{t_m} \int_\Omega \frac{\rho_h}{\underline{\rho}_\tau} p'(\underline{\rho}_\tau) \boldsymbol{u}_h \cdot \nabla \underline{\rho}_\tau dxdt,$$

$$|\mathcal{L}_3| \le C(h^A + \tau), \quad A := \frac{\min\{\epsilon, 1\}}{2}, \quad [\partial_t \underline{\rho}]_\tau := \partial_t \underline{\rho}(t_n, \cdot), \text{ for } [t_{n-1}, t_n].$$

***Proof*** We start the proof from the discrete version of the relative energy inequality (7.5) derived in the previous Theorem 7.3. The terms $\mathcal{R}_i$ $(i = 2, 4, 5, 6)$ will be transformed to a more convenient form, and the other terms $\mathcal{R}_i$ $(i = 1, 3)$ will remain unchanged.

- The term $\mathcal{R}_2$. By applying the Cauchy–Schwarz inequality and the estimate (3.2), we can show

$$\begin{aligned} |\mathcal{R}_2| \le & C \|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \|\underline{\boldsymbol{u}}_h - \underline{\boldsymbol{u}}_\tau\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \\ \le & C(E_0, \|\underline{\boldsymbol{u}}\|_{L^\infty(0,T;\boldsymbol{H}^2(\Omega))}) h. \end{aligned}$$

- The term $\mathcal{R}_4$. Firstly, by applying the Taylor formula, we have

$$\mathcal{H}'(\underline{\rho}^n) - \mathcal{H}'(\underline{\rho}^{n-1}) = \mathcal{H}''(\underline{\rho}^n)(\underline{\rho}^n - \underline{\rho}^{n-1}) - \frac{1}{2}\mathcal{H}'''(\xi_{\underline{\rho}}^n)(\underline{\rho}^n - \underline{\rho}^{n-1})^2, \quad (7.13)$$

where $\xi_{\underline{\rho}}^n \in \text{co}\{\underline{\rho}^n, \underline{\rho}^{n-1}\}$. Let $\xi_{\underline{\rho}}(t, \cdot) := \xi_{\underline{\rho}}^n$ for $t \in [t_{n-1}, t_n]$. By applying the identity (7.13), the term $\mathcal{R}_4$ can be rewritten as

$$\mathcal{R}_4 = \mathcal{L}_1 + \mathcal{L}_{3,1} + \mathcal{L}_{3,2},$$

where the remainder terms $\mathcal{L}_{3,i}$ are given by

$$\begin{aligned} \mathcal{L}_{3,1} := & \int_0^{t_m} \int_\Omega (\underline{\rho}_\tau - \rho_h) \frac{p'(\underline{\rho}_\tau)}{\underline{\rho}_\tau} (D_t \underline{\rho}_\tau - [\partial_t \underline{\rho}]_\tau) dxdt, \\ \mathcal{L}_{3,2} := & \frac{1}{2\tau} \int_0^{t_m} \int_\Omega (\rho_h - \underline{\rho}_\tau) \mathcal{H}'''(\xi_{\underline{\rho}})(\underline{\rho}_\tau - \underline{\rho}_\tau^\star)^2 dxdt. \end{aligned}$$

Using the Taylor formula and the mass conservation (see, Remark 3.1), we obtain

$$\begin{aligned} |\mathcal{L}_{3,1}| \le & \tau C(\underline{\rho}_{\min}, \underline{\rho}_{\max}) |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])} \|\underline{\rho}_\tau - \rho_h\|_{L^1((0,T)\times\Omega)} \|\partial_{tt} \underline{\rho}\|_{L^\infty((0,T)\times\Omega)} \\ \le & \tau C(M_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|\partial_{tt} \underline{\rho}\|_{L^\infty((0,T)\times\Omega)}). \end{aligned}$$

By a similar argument, we conclude that

$$|\mathcal{L}_{3,2}| \le \tau C(M_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|\partial_t \underline{\rho}\|_{L^\infty((0,T)\times\Omega)}).$$

- The term $\mathcal{R}_5$. We may write

$$\begin{aligned} \mathcal{R}_5 = & -\int_0^{t_m} \int_\Omega \rho_h \boldsymbol{u}_h \cdot (\mathcal{H}''(\underline{\rho}_\tau^\star) \nabla \underline{\rho}_\tau^\star - \mathcal{H}''(\underline{\rho}_\tau) \nabla \underline{\rho}_\tau) dxdt \\ & -\int_0^{t_m} \int_\Omega \rho_h \boldsymbol{u}_h \cdot \mathcal{H}''(\underline{\rho}_\tau) \nabla \underline{\rho}_\tau dxdt = \mathcal{L}_{3,3} + \mathcal{L}_2. \end{aligned}$$

By applying the first-order Taylor formula, we obtain

$$\begin{aligned} & \|p'(\underline{\rho}_\tau^\star) - p'(\underline{\rho}_\tau)\|_{L^\infty((0,T)\times\Omega)} \\ & \le C(|p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|\partial_t \underline{\rho}\|_{L^\infty((0,T)\times\Omega)})\tau. \end{aligned} \quad (7.14)$$

Using the estimate (7.14) and the Taylor formula, we have

$$\begin{aligned} &\|\mathcal{H}''(\underline{\rho}_\tau^\star) - \mathcal{H}''(\underline{\rho}_\tau)\|_{L^\infty((0,T)\times\Omega)} \\ &\le C(\underline{\rho}_{\min}, \underline{\rho}_{\max})\|p'(\underline{\rho}_\tau^\star) - p'(\underline{\rho}_\tau)\|_{L^\infty((0,T)\times\Omega)} \\ &\quad + C(\underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])})\|\underline{\rho}_\tau^\star - \underline{\rho}_\tau\|_{L^\infty((0,T)\times\Omega)} \\ &\le C(\underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])}, \|\partial_t\underline{\rho}\|_{L^\infty((0,T)\times\Omega)})\tau. \end{aligned} \tag{7.15}$$

Therefore, by using the Hölder inequality and the estimate (7.15), we obtain

$$\begin{aligned} |\mathcal{L}_{3,3}| \le& C(\Omega, T)\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \\ &\times \|(\mathcal{H}''(\underline{\rho}_\tau^\star) - \mathcal{H}''(\underline{\rho}_\tau))\nabla\underline{\rho}_\tau^\star\|_{L^\infty((0,T)\times\Omega)} \\ \le& C(\Omega, T)\|\rho_h\|_{L^{2\gamma}((0,T)\times\Omega)}\|\boldsymbol{u}_h\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \\ &\times \|\mathcal{H}''(\underline{\rho}_\tau)\nabla(\underline{\rho}_\tau^\star - \underline{\rho}_\tau)\|_{L^\infty((0,T)\times\Omega)} \\ \le& C(\Omega, T, E_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])}, \|(\partial_t\underline{\rho}, \nabla\underline{\rho}, \partial_t\nabla\underline{\rho})\|_{L^\infty((0,T)\times\Omega)})\tau. \end{aligned}$$

- The term $\mathcal{R}_6$. By applying the estimate (4.3) of Lemma 4.2, we get that

$$\begin{aligned} |\mathcal{R}_6| \le& C(\Omega, T)\|\nabla\mathcal{H}'(\underline{\rho}_\tau^\star)\|_{L^\infty((0,T)\times\Omega)}h^A \\ \le& C(\Omega, T, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])}, \|\nabla\underline{\rho}\|_{L^\infty((0,T)\times\Omega)})h^A. \end{aligned}$$

Combining the above analysis, we have $\sum_{i=1}^6 \mathcal{R}_i = \mathcal{R}_1 + \mathcal{R}_3 + \sum_{i=1}^3 \mathcal{L}_i$. After setting $\mathcal{L}_3 := \sum_{i=1}^3 \mathcal{L}_{3,i} + \mathcal{R}_2 + \mathcal{R}_6$, we deduce the approximate relative energy inequality (7.12) from the estimate (7.5). The proof is thus complete. □

Next we derive a discrete identity for the strong solutions.

**Theorem 7.5** *Suppose that Hypothesis 2.1 is satisfied and the pressure $p = p(\rho)$ satisfies the hypothesis (1.2) with $\gamma > 1$. Let the internal energy $\mathcal{H}$ is given by $\mathcal{H}(\rho) = \frac{p(\rho)}{\gamma-1}$. Let the family $(\underline{\rho}_\tau, \underline{\boldsymbol{u}}_\tau)$ be defined as in (7.3). For any $1 \le m \le N$, then the following identity holds:*

$$\mathcal{R}_1 + \sum_{i=4}^{5} \mathcal{L}_i = 0, \tag{7.16}$$

*where the remainder terms $\mathcal{L}_i$ are defined by*

$$\begin{aligned} \mathcal{L}_4 &:= -\int_0^{t_m}\int_\Omega p'(\underline{\rho}_\tau)\boldsymbol{u}_h \cdot \nabla\underline{\rho}_\tau dxdt, \\ \mathcal{L}_5 &:= -\int_0^{t_m}\int_\Omega p(\underline{\rho}_\tau)\,\mathrm{div}\,\underline{\boldsymbol{u}}_\tau dxdt. \end{aligned}$$

***Proof*** Since $(\underline{\rho}, \underline{\boldsymbol{u}})$ is a strong solution of the problem (1.1), the second equation of (1.1) can be rewritten in the form

$$\mu\Delta\underline{\boldsymbol{u}} + (\lambda + \mu)\nabla\,\mathrm{div}\,\underline{\boldsymbol{u}} = \nabla p(\underline{\rho}). \tag{7.17}$$

Taking $t = t_n$ in (7.17), multiplying this identity by $\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n$ and integration over $\Omega$. We get, after summation from $n = 1$ to $n = m$,

$$0 = -\int_0^{t_m}\int_\Omega (\mu\Delta\underline{\boldsymbol{u}}_\tau + (\lambda+\mu)\nabla \operatorname{div}\underline{\boldsymbol{u}}_\tau)\cdot(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)dxdt$$
$$+\int_0^{t_m}\int_\Omega \nabla p(\underline{\rho}_\tau)\cdot \boldsymbol{u}_h dxdt - \int_0^{t_m}\int_\Omega \nabla p(\underline{\rho}_\tau)\cdot\underline{\boldsymbol{u}}_\tau dxdt.$$

which implies that $\mathcal{R}_1 + \sum_{i=4}^5 \mathcal{L}_i = 0$. The proof is thus complete. □

Now, we will derive the unconditional error estimate of the problem (1.1) based on the approximate relative energy inequality (7.13) and the discrete identity (7.16).

**Theorem 7.6** *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a bounded domain and assume that the viscosity coefficient $\mu, \lambda$ satisfies $\mu > 0$ and $d\lambda + 2\mu > 0$. Suppose that the pressure $p = p(\rho)$ satisfies the assumption (1.2) with $\gamma > \frac{6}{5}$. The initial values $(\rho_0, \boldsymbol{u}_0)$ satisfies Hypothesis 2.1 with the finite energy $E_0 := \int_\Omega \mathcal{H}(\rho_0)dx$ and finite mass $M_0 := \int_\Omega \rho_0 dx$. Let $(\underline{\rho}, \underline{\boldsymbol{u}})$ be a strong solution of the problem (1.1) belonging to the class*

$$\begin{cases} \underline{\rho} \in C^2([0,T]\times\Omega), & 0 < \underline{\rho}_{\min} \le \underline{\rho} \le \underline{\rho}_{\max}, \\ \underline{\boldsymbol{u}} \in \boldsymbol{C}^2([0,T]\times\Omega), & \underline{\boldsymbol{u}}|_{(0,T)\times\partial\Omega} = \boldsymbol{0}, \end{cases}$$

*emanating from the initial data $(\underline{\rho}_0, \underline{\boldsymbol{u}}_0)$. Let the families $(\rho_h, \boldsymbol{u}_h)$ and $(\underline{\rho}_\tau, \underline{\boldsymbol{u}}_\tau)$ be defined as in (3.16)–(3.17) and (7.3), respectively. Then there exists*

$$C := C(T, \Omega, M_0, E_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])},$$
$$\|(\partial_t\underline{\rho}, \partial_{tt}\underline{\rho}, \nabla\underline{\rho}, \partial_t\nabla\underline{\rho}, \nabla^2\underline{\boldsymbol{u}})\|_{L^\infty((0,T)\times\Omega)}) > 0,$$

*such that for any $1 \le m \le N$, then we have*

$$\mathcal{E}(\rho_h^m \mid \underline{\rho}^m) + \int_0^{t_m}\int_\Omega [\mu|\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 + (\lambda+\mu)|\operatorname{div}(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2]dxdt$$
$$\le C(\mathcal{E}(\rho_h^0 \mid \underline{\rho}^0) + h^A + \tau), \quad A := \frac{\min\{\epsilon, 1\}}{2}. \tag{7.18}$$

***Proof*** Combining the approximate relative energy inequality (7.12) and the discrete identity (7.16), we can show

$$\mathcal{E}(\rho_h^m \mid \underline{\rho}^m) + \int_0^{t_m}\int_\Omega [\mu|\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 + (\lambda+\mu)|\operatorname{div}(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2]dxdt$$
$$\le \mathcal{E}(\rho_h^0 \mid \underline{\rho}^0) + \sum_{i=6}^{7}\mathcal{L}_i.$$

where the terms $\mathcal{L}_i$ are defined by

$$\mathcal{L}_6 := \int_0^{t_m}\int_\Omega \frac{\underline{\rho}_\tau - \rho_h}{\underline{\rho}_\tau} p'(\underline{\rho}_\tau)\boldsymbol{u}_h\cdot\nabla\underline{\rho}_\tau dxdt + \int_0^{t_m}\int_\Omega (\underline{\rho}_\tau - \rho_h)\frac{p'(\underline{\rho}_\tau)}{\underline{\rho}_\tau}[\partial_t\underline{\rho}]_\tau dxdt,$$
$$+\int_0^{t_m}\int_\Omega (p(\underline{\rho}_\tau) - p(\rho_h))\operatorname{div}\boldsymbol{u}_\tau dxdt, \quad |\mathcal{L}_7| \le C(h^A + \tau).$$

We next bound the term $\mathcal{L}_6$. Since the pair $(\underline{\rho}, \underline{\boldsymbol{u}})$ is a strong solution of the problem (1.1), the first equation of (1.1) can be rewritten in the form

$$[\partial_t \underline{\rho}]_\tau = -\underline{\boldsymbol{u}}_\tau \cdot \nabla \underline{\rho}_\tau - \underline{\rho}_\tau \operatorname{div} \underline{\boldsymbol{u}}_\tau. \tag{7.19}$$

By the identity (7.19), we write

$$\mathcal{L}_6 = \mathcal{L}_{6,1} + \mathcal{L}_{6,2},$$

where

$$\mathcal{L}_{6,1} := -\int_0^{t_m} \int_\Omega (p(\rho_h) - p'(\underline{\rho}_\tau)(\rho_h - \underline{\rho}_\tau) - p(\underline{\rho}_\tau)) \operatorname{div} \boldsymbol{u}_\tau dxdt,$$

$$\mathcal{L}_{6,2} := \int_0^{t_m} \int_\Omega \frac{\underline{\rho}_\tau - \rho_h}{\underline{\rho}_\tau} p'(\underline{\rho}_\tau)(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau) \cdot \nabla \underline{\rho}_\tau dxdt.$$

It is easy to check that

$$|\mathcal{L}_{6,1}| \le C(T, \Omega, \|\operatorname{div} \underline{\boldsymbol{u}}\|_{L^\infty((0,T)\times\Omega)}) \int_0^{t_m} \mathcal{E}(\rho_h \,|\, \underline{\rho}_\tau) dt.$$

Let $\Omega_{h,1} := \{\frac{\underline{\rho}_\tau}{2} < \rho_h < 2\underline{\rho}_\tau\}$ and $\Omega_{h,2} := \Omega \setminus \Omega_{h,1}$. The term $\mathcal{L}_{6,2}$ can be rewritten as

$$\mathcal{L}_{6,2} := \mathcal{L}_{6,2,\Omega_{h,1}} + \mathcal{L}_{6,2,\Omega_{h,2}},$$

where

$$\mathcal{L}_{6,2,\Omega_{h,i}} := \int_0^{t_m} \int_{\Omega_{h,i}} \frac{\underline{\rho}_\tau - \rho_h}{\underline{\rho}_\tau} p'(\underline{\rho}_\tau)(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau) \cdot \nabla \underline{\rho}_\tau dxdt, \quad i = 1, 2.$$

By applying the Poincaré and Young inequalities, the estimate (7.2), $\underline{\rho}_\tau \in (\underline{\rho}_{\min}, \underline{\rho}_{\max})$, we can show

$$\begin{aligned} |\mathcal{L}_{6,2,\Omega_{h,1}}| \le& C(\delta, \Omega, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|\nabla \underline{\rho}\|_{L^\infty((0,T)\times\Omega)}) \int_0^{t_m} \mathcal{E}(\rho_h \,|\, \underline{\rho}_\tau) dt \\ &+ \delta \int_0^{t_m} \int_\Omega |\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 dxdt. \end{aligned}$$

By employing the estimate (7.2) and $\underline{\rho}_\tau \in (\underline{\rho}_{\min}, \underline{\rho}_{\max})$, $\gamma > \frac{6}{5}$, we have

$$\mathbb{E}(\rho_h \,|\, \underline{\rho}_\tau) \ge C(\underline{\rho}_{\min}, \underline{\rho}_{\max}, \gamma)(1 + \rho_h^\gamma) \ge |\rho_h - \underline{\rho}_\tau|^{\frac{6}{5}}, \text{ in } \Omega_{h,2}. \tag{7.20}$$

Using the Poincaré and Young inequalities, the estimates (7.20) and (7.4), $\underline{\rho}_\tau \in (\underline{\rho}_{\min}, \underline{\rho}_{\max})$, we conclude that

$$\begin{aligned} |\mathcal{L}_{6,2,\Omega_{h,2}}| \le& C(\delta, \Omega, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|\nabla \underline{\rho}\|_{L^\infty((0,T)\times\Omega)}) \int_0^{t_m} \mathcal{E}(\rho_h \,|\, \underline{\rho}_\tau) dt \\ &+ \delta \int_0^{t_m} \int_\Omega |\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 dxdt. \end{aligned}$$

Combining the above analysis with $\delta = \frac{\mu}{4}$, we get that

$$\begin{aligned} &\mathcal{E}(\rho_h^m \,|\, \underline{\rho}^m) + \int_0^{t_m} \int_\Omega [\mu |\nabla(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2 + (\lambda + \mu) |\operatorname{div}(\boldsymbol{u}_h - \underline{\boldsymbol{u}}_\tau)|^2] dxdt \\ &\le C(\mathcal{E}(\rho_h^0 \,|\, \underline{\rho}^0) + h^A + \tau) + C \int_0^{t_m} \mathcal{E}(\rho_h \,|\, \underline{\rho}_\tau) dt, \end{aligned}$$

where the constant $C > 0$ is given by

$$C := C(T, \Omega, M_0, E_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|(\partial_t \underline{\rho}, \partial_{tt} \underline{\rho}, \nabla \underline{\rho}, \partial_t \nabla \underline{\rho}, \nabla^2 \underline{\boldsymbol{u}})\|_{L^\infty((0,T)\times\Omega)}).$$

Using the estimate (7.4), we can show

$$\int_0^{t_m} \mathcal{E}(\rho_h \mid \underline{\rho}_\tau) dt = \tau \sum_{n=1}^{m} \mathcal{E}(\rho_h^n \mid \underline{\rho}^n) \le \tau \sum_{n=1}^{m-1} \mathcal{E}(\rho_h^n \mid \underline{\rho}^n) + C\tau,$$

which implies that

$$\mathcal{E}(\rho_h^m \mid \underline{\rho}^m) + \sum_{n=1}^{m} \tau \int_\Omega [\mu |\nabla(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}_\tau^n)|^2 + (\lambda + \mu) |\operatorname{div}(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}_\tau^n)|^2] dx$$
$$\le C(\mathcal{E}(\rho_h^0 \mid \underline{\rho}^0) + h^A + \tau) + C\tau \sum_{n=1}^{m-1} \mathcal{E}(\rho_h^n \mid \underline{\rho}^n). \tag{7.21}$$

By applying the standard discrete version of Gronwall's lemma for the inequality (7.21), the proof is thus complete. □

Finally, we will give an error estimate for the discrete $L^2(L^2)$ norm of $p(\rho_h)$.

**Theorem 7.7** *Suppose that the condition of Theorem 7.6 holds. Let the families $(\rho_h, \boldsymbol{u}_h)$ and $(\underline{\rho}_\tau, \underline{\boldsymbol{u}}_\tau)$ be defined as in (3.16)–(3.17) and (7.3), respectively. Then there exists*

$$C := C(T, \Omega, M_0, E_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|(\partial_t \underline{\rho}, \partial_{tt} \underline{\rho}, \nabla \underline{\rho}, \partial_t \nabla \underline{\rho}, \nabla^2 \underline{\boldsymbol{u}})\|_{L^\infty((0,T)\times\Omega)}) > 0,$$

*such that for any $1 \le m \le N$, we have*

$$\tau \sum_{n=1}^{m} \|p(\rho_h^n) - p(\underline{\rho}^n)\|_{L^2(\Omega)}^2 \le C(\mathcal{E}(\rho_h^0 \mid \underline{\rho}^0) + h^A + \tau), \quad A := \frac{\min\{\epsilon, 1\}}{2}. \tag{7.22}$$

***Proof*** Taking $t = t_n$ in (7.17), multiplying this identity by $\boldsymbol{v}_h \in \mathbb{V}_h$ and integral over $\Omega$, we conclude that

$$\mu \int_\Omega \nabla \underline{\boldsymbol{u}}^n : \nabla \boldsymbol{v}_h dx + (\lambda + \mu) \int_\Omega \operatorname{div} \underline{\boldsymbol{u}}^n \operatorname{div} \boldsymbol{v}_h dx$$
$$- \int_\Omega p(\underline{\rho}^n) \operatorname{div} \boldsymbol{v}_h dx = 0. \tag{7.23}$$

Subtracting (7.23) from (3.8), we can get the error equation

$$\mu \int_\Omega \nabla(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n) : \nabla \boldsymbol{v}_h dx + (\lambda + \mu) \int_\Omega \operatorname{div}(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n) \operatorname{div} \boldsymbol{v}_h dx$$
$$- \int_\Omega (p(\rho_h^n) - p(\underline{\rho}^n)) \operatorname{div} \boldsymbol{v}_h dx = 0, \quad \forall\, \boldsymbol{v}_h \in \mathbb{V}_h. \tag{7.24}$$

Let $r_\rho^n := (p(\rho_h^n) - p(\underline{\rho}_h^n)) - \frac{1}{|\Omega|}\int_\Omega (p(\rho_h^n) - p(\underline{\rho}_h^n))dx$ for $1 \le n \le N$. Taking $\boldsymbol{v}_h = \Pi_h^{\mathbb{V}}\mathbf{B}[r_\rho^n]$ in (7.24), we can show

$$\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)}^2 = |\Omega|^{-1}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^1(\Omega)}^2 + \mu\int_\Omega \nabla(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n) : \nabla\Pi_h^{\mathbb{V}}\mathbf{B}[r_\rho^n]dx$$
$$+ (\lambda + \mu)\int_\Omega \operatorname{div}(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n)\operatorname{div}\Pi_h^{\mathbb{V}}\mathbf{B}[r_\rho^n]dx - \int_\Omega (p(\underline{\rho}_h^n) - p(\underline{\rho}^n))\operatorname{div}\Pi_h^{\mathbb{V}}\mathbf{B}[r_\rho^n]dx.$$

By applying the Cauchy–Schwarz inequality, the estimates (3.3) and (3.12), we obtain

$$\begin{aligned}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)}^2 \le& C(\Omega)\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^1(\Omega)}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)}\\ &+ C(\Omega)\|\nabla(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n)\|_{\boldsymbol{L}^2(\Omega)}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)}\\ &+ C(\Omega)\|p(\underline{\rho}_h^n) - p(\underline{\rho}^n)\|_{L^2(\Omega)}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)},\end{aligned}$$

which implies that

$$\begin{aligned}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)}^2 \le& C(\Omega)\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^1(\Omega)}^2 + C(\Omega)\|\nabla(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n)\|_{\boldsymbol{L}^2(\Omega)}^2\\ &+ C(\Omega)\|p(\underline{\rho}_h^n) - p(\underline{\rho}^n)\|_{L^2(\Omega)}^2.\end{aligned} \tag{7.25}$$

Summing (7.25) from $n = 1$ to $n = m$ and multiplying the resulting inequality by $\tau$, we conclude that

$$\tau\sum_{n=1}^m \|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^2(\Omega)}^2 \le C(\Omega)\sum_{i=8}^{10}\mathcal{L}_i,$$

where the terms $\mathcal{L}_i$ $(8 \le i \le 10)$ are defined by

$$\begin{aligned}\mathcal{L}_8 :=& \tau\sum_{n=1}^m \|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^1(\Omega)}^2, \quad \mathcal{L}_9 := \tau\sum_{n=1}^m \|\nabla(\boldsymbol{u}_h^n - \underline{\boldsymbol{u}}^n)\|_{\boldsymbol{L}^2(\Omega)}^2,\\ \mathcal{L}_{10} :=& \tau\sum_{n=1}^m \|p(\underline{\rho}_h^n) - p(\underline{\rho}^n)\|_{L^2(\Omega)}^2.\end{aligned}$$

Bound on $\mathcal{L}_9$. By applying the estimate (3.1), the mean value theorem and $\underline{\rho}^n, \underline{\rho}_h^n \in [\underline{\rho}_{\min}, \underline{\rho}_{\max}]$, we can show

$$\begin{aligned}\|p(\rho_h^n) - p(\underline{\rho}_h^n)\|_{L^1(\Omega)} \le& \|p(\underline{\rho}^n) - p(\underline{\rho}_h^n)\|_{L^1(\Omega)} + \|p'(\underline{\rho}^n)(\rho_h^n - \underline{\rho}^n)\|_{L^1(\Omega)}\\ &+ \|p(\rho_h^n) - p'(\underline{\rho}^n)(\rho_h^n - \underline{\rho}^n) - p(\underline{\rho}^n)\|_{L^1(\Omega)}\\ \le& C(\Omega, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])}, \|\nabla\underline{\rho}\|_{L^\infty((0,T)\times\Omega)})h\\ &+ C(|p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])})\|\rho_h^n - \underline{\rho}^n\|_{L^1(\Omega)} + C\mathcal{E}(\rho_h^n \,|\, \underline{\rho}^n).\end{aligned}$$

Let $\Omega_{h,1}^n := \{\frac{\underline{\rho}^n}{2} < \rho_h^n < 2\underline{\rho}^n\}$ and $\Omega_{h,2}^n := \Omega \setminus \Omega_{h,1,n}$. By applying the estimate (7.2) and (7.20), we obtain

$$\begin{aligned}\|\rho_h^n - \underline{\rho}^n\|_{L^1(\Omega)} \le& C(\Omega)\|\rho_h^n - \underline{\rho}^n\|_{L^2(\Omega_{h,1}^n)} + C(\Omega)\|\rho_h^n - \underline{\rho}^n\|_{L^{\frac{6}{5}}(\Omega_{h,2}^n)}\\ \le& C(\Omega, \underline{\rho}_{\min}, \underline{\rho}_{\max})\mathcal{E}^{\frac{1}{2}}(\rho_h^n \,|\, \underline{\rho}^n) + C(\Omega, \underline{\rho}_{\min}, \underline{\rho}_{\max})\mathcal{E}^{\frac{5}{6}}(\rho_h^n \,|\, \underline{\rho}^n).\end{aligned}$$

Using the estimate (7.4), we get that

$$\|\rho_h^n - \underline{\rho}^n\|_{L^1(\Omega)} \le C(\Omega, E_0, \underline{\rho}_{\min}, \underline{\rho}_{\max}, |p'|_{C^1([\underline{\rho}_{\min},\underline{\rho}_{\max}])})\mathcal{E}^{\frac{1}{2}}(\rho_h^n \,|\, \underline{\rho}^n).$$

Combining the error estimate of Theorem 7.6, which implies that

$$|\mathcal{L}_8| \leq C(\mathcal{E}(\rho_h^0 \,|\, \underline{\rho}^0) + h^A + \tau).$$

By a similar argument, we can show

$$\begin{aligned}|\mathcal{L}_9| &\leq C(\mathcal{E}(\rho_h^0 \,|\, \underline{\rho}^0) + h^A + \tau),\\ |\mathcal{L}_{10}| &\leq C(\Omega, T, |p'|_{C^1([\underline{\rho}_{\min}, \underline{\rho}_{\max}])}, \|\nabla \underline{\rho}\|_{L^\infty((0,T)\times\Omega)})h^2.\end{aligned}$$

Combining the above analysis, the proof is thus complete. □

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## A Appendix

### A.1 The Proof of Theorem 3.3

Our goal is to show the existence of numerical solutions for the scheme (3.7)–(3.8) by applying Schaeffer's fixed point theorem. For this purpose, we define the mapping

$$\mathcal{L} : \mathbb{V}_h \to \mathbb{V}_h, \quad \mathcal{L}[\boldsymbol{u}] \mapsto \boldsymbol{U},$$

in the following way.

- Given $\boldsymbol{u} \in \mathbb{V}_h$, we will prove the unique solution $\rho \in \mathbb{Q}_h$ of the linear system

$$\begin{aligned}\int_\Omega \frac{\rho - \rho_h^{n-1}}{\tau} \varphi_h dx - \sum_{F \in \mathcal{F}_{h,int}} \int_F \mathrm{Up}[\rho, \boldsymbol{u}] - h^{\epsilon-1} [\![\rho]\!] [\![\varphi_h]\!] dS \\ + h^{\epsilon-1} \sum_{F \in \mathcal{F}_{h,int}} \int_F [\![\rho]\!] [\![\varphi_h]\!] dS = 0,\end{aligned} \tag{A.1}$$

  for any $\varphi_h \in \mathbb{Q}_h$. In order to prove the linear problem (A.1) has a unique solution $\rho(\boldsymbol{u})$, we need prove that the associated homogenous problem

$$\int_\Omega \rho \varphi_h dx - \tau \sum_{F \in \mathcal{F}_{h,int}} \int_F \mathrm{Up}\,[\rho, \boldsymbol{u}]\, [\![\varphi_h]\!] dS + h^{\epsilon-1} \tau \sum_{F \in \mathcal{F}_{h,int}} \int_F [\![\rho]\!] [\![\varphi_h]\!] dS = 0 \tag{A.2}$$

  admits a unique solution $\rho = 0$. By the same proof of [14, Section 4.3], we can show the homogenous problem (A.2) of renormalized equation

$$\begin{aligned}&\int_\Omega \mathcal{B}'(\rho) \rho \varphi_h dx - \tau \sum_{F \in \mathcal{F}_{h,int}} \int_F \mathrm{Up}[\mathcal{B}(\rho), \boldsymbol{u}] [\![\varphi_h]\!] dS \\ &+ h^{\epsilon-1} \tau \sum_{F \in \mathcal{F}_{h,int}} \int_F \mathcal{B}'(\rho_+) [\![\rho]\!] [\![\varphi_h]\!] dS + h^{\epsilon-1} \tau \sum_{F \in \mathcal{F}_{h,int}} \int_F \mathcal{B}''(\overline{\eta}_\rho) [\![\rho]\!]^2 dS \\ &+ \frac{\tau}{2} \sum_{F \in \mathcal{F}_{h,int}} \int_F \varphi_h \mathcal{B}''(\eta_\rho) [\![\rho]\!]^2 |\boldsymbol{u} \cdot \boldsymbol{n}| dS = \tau \int_\Omega \varphi_h (\mathcal{B}(\rho) - \mathcal{B}'(\rho)\rho) \operatorname{div} \boldsymbol{u} dx,\end{aligned} \tag{A.3}$$

  for any $\varphi_h \in \mathbb{Q}_h$, where $\mathcal{B} \in C^2(\mathbb{R}_+)$, $\overline{\eta}_\rho, \eta_\rho \in \mathrm{co}\{\rho, \rho_+\}$ on each face $F \in \mathcal{F}_h$. Any non negative $C^2(\mathbb{R})$ convex approximations function $\mathcal{S}_\epsilon$ such that $\mathcal{S}_\epsilon(\rho) \to \mathcal{S}(\rho)$ and

$\mathcal{S}'_\epsilon(\rho) \to \mathcal{S}'(\rho)$ for all $\rho \neq 0$, where $\mathcal{S}(\rho) = \max\{-\rho, 0\}$. Taking $(\varphi_h, \mathcal{B}) = (1, \mathcal{S}_\epsilon)$ in (A.3), we have

$$\int_\Omega \mathcal{S}_\epsilon(\rho)dx \leq \tau \int_\Omega \varphi_h(\mathcal{S}_\epsilon(\rho) - \mathcal{S}'_\epsilon(\rho)\rho) \operatorname{div} \boldsymbol{u} dx + \int_\Omega (\mathcal{S}_\epsilon(\rho) - \mathcal{S}'_\epsilon(\rho)\rho)dx. \quad (A.4)$$

Combining the inequality (A.4) and $\mathcal{S}(\rho)-\mathcal{S}'(\rho)\rho = 0$ for all $\rho \neq 0$, we obtain $\mathcal{S}(\rho) = 0$ and $\rho \geq 0$. Let $\varphi_h = 1$ in (A.2), we obtain

$$\int_\Omega \rho dx = 0. \quad (A.5)$$

According to $\rho \geq 0$ and (A.5), we have $\rho = 0$, then the problem (A.1) has a unique solution $\rho(\boldsymbol{u})$. By applying the Lemma 3.3, we have $\rho(\boldsymbol{u}) > 0$.

- For given $\rho \in \mathbb{Q}_h$ and $\boldsymbol{u} \in \mathbb{V}_h$, we can show the unique solution $\boldsymbol{U} \in \mathbb{V}_h$ of the algebraic system

$$\int_\Omega [\mu \nabla \boldsymbol{U} : \nabla \boldsymbol{v}_h + (\lambda + \mu) \operatorname{div} \boldsymbol{U} \operatorname{div} \boldsymbol{v}_h]dx = \int_\Omega p(\rho) \operatorname{div} \boldsymbol{v}_h dx, \quad (A.6)$$

for any $\boldsymbol{v}_h \in \mathbb{V}_h$, where $\rho = \rho(\boldsymbol{u})$ is determined by (A.1). Similarly, by applying the Lax-Milgram Lemma for the linear system (A.6), we have a unique solution $\boldsymbol{U} \in \mathbb{V}_h$.

Clearly, any fixed point of the mapping $\mathcal{L}$ is a solution of the scheme (3.7)–(3.8). Next, we need show that the set

$$\{\boldsymbol{z} \in \mathbb{V}_h : \boldsymbol{z} = \Lambda \mathcal{L}(\boldsymbol{z}), \;\; \Lambda \in [0, 1]\}$$

satisfies the conditions of Lemma 3.5. In other words, we need to verify that the set is non empty and bounded. It is obvious show that the set is non empty ($\mathbf{0}$ belongs to the set). Finally, for all $\Lambda \in (0, 1]$, we need to prove the solution $\boldsymbol{u}$ of the equation $\boldsymbol{u} = \Lambda \mathcal{L}[\boldsymbol{u}]$ can be bounded in terms of the local data $(\rho_h^{n-1}, \boldsymbol{u}_h^{n-1})$ uniformly with respect to $\Lambda$. Setting $\rho_h^n = \rho(\boldsymbol{u})$, $\boldsymbol{u}_h^n = \boldsymbol{u}$ in (3.7)–(3.8), where $\boldsymbol{u}$ is a solution of $\boldsymbol{u} = \Lambda \mathcal{L}[\boldsymbol{u}]$, we have

$$\begin{aligned} \int_\Omega d_t \rho_h^n \varphi_h dx - \sum_{F \in \mathcal{F}_{h,int}} \int_F \mathrm{Up}\left[\rho_h^n, \boldsymbol{u}_h^n\right] [\![\varphi_h]\!] dS \\ + h^{\epsilon - 1} \sum_{F \in \mathcal{F}_{h,int}} \int_F [\![\rho_h^n]\!][\![\varphi_h]\!] dS = 0, \\ \Lambda^{-1} \int_\Omega \left[\mu \nabla \boldsymbol{u}_h^n : \nabla \boldsymbol{v}_h + (\lambda + \mu) \operatorname{div} \boldsymbol{u}_h^n \operatorname{div} \boldsymbol{v}_h\right]dx - \int_\Omega p(\rho_h^n) \operatorname{div} \boldsymbol{v}_h dx = 0. \end{aligned}$$

By recalling the steps in the proof of discrete energy estimate (3.10), we can show

$$\int_\Omega \mathcal{H}(\rho_h^n)dx + \frac{1}{\Lambda} \int_\Omega [\mu |\nabla \boldsymbol{u}_h^n|^2 + (\lambda + \mu)|\operatorname{div} \boldsymbol{u}_h^n|^2]dx \leq \int_\Omega \mathcal{H}(\rho_h^{n-1})dx. \quad (A.7)$$

Combining (A.7) and $0 < \Lambda \leq 1$, there exists a constant $C$ independent of $\Lambda$ such that

$$\|\boldsymbol{u}_h^n\|^2_{\mathbb{V}_h} := \mu \|\nabla \boldsymbol{u}_h^n\|^2_{\boldsymbol{L}^2(\Omega)} \leq C.$$

Combining the above conclusions and Lemma 3.5, we can show the schemes (3.7)–(3.8) has at least one solution. By applying the Lemma 3.3, we obtain the density $\rho_h^n > 0$. The proof is thus complete.

### A.2 The Proof of Theorem 6.1

Taking the zero extension of $\boldsymbol{v}_h$ for $\mathbb{R}^d \setminus \Omega$. We show the proof of this Theorem in two steps. Step 1. If $q = 2$, for any $\boldsymbol{x} \in \mathbb{R}^d$, it is easy to check that

$$\boldsymbol{v}_h(\boldsymbol{x}) - \boldsymbol{v}_h(\boldsymbol{x} - \boldsymbol{\xi}) = \int_0^1 \nabla \boldsymbol{v}_h(\boldsymbol{x} - s\boldsymbol{\xi}) \cdot \boldsymbol{\xi} ds. \tag{A.8}$$

For the identity (A.8), by applying Cauchy–Schwarz inequality, we conclude that

$$|\boldsymbol{v}_h(\boldsymbol{x}) - \boldsymbol{v}_h(\boldsymbol{x} - \boldsymbol{\xi})|^2 \leq |\boldsymbol{\xi}|^2 \int_0^1 |\nabla \boldsymbol{v}_h(\boldsymbol{x} - s\boldsymbol{\xi})|^2 ds.$$

Therefor, by employing Fubini theorem and $\nabla \boldsymbol{v}_h$ vanishes outside $\Omega$, we have

$$\int_{\mathbb{R}^d} |\boldsymbol{v}_h(\boldsymbol{x}) - \boldsymbol{v}_h(\boldsymbol{x} - \boldsymbol{\xi})|^2 dx \leq |\boldsymbol{\xi}|^2 \int_{\Omega} |\nabla \boldsymbol{v}_h(\boldsymbol{x})|^2 dx. \tag{A.9}$$

Step 2. For the case of $2 < q \leq 6$, by applying Gagliardo-Nirenberg interpolation inequality and (A.9), we obtain

$$\begin{aligned} \|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|_{\boldsymbol{L}^q(\mathbb{R}^d)} \leq & \|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|^{\theta}_{\boldsymbol{L}^2(\mathbb{R}^d)} \|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|^{1-\theta}_{\boldsymbol{L}^6(\mathbb{R}^d)} \\ \leq & |\boldsymbol{\xi}|^{\theta} \|\nabla \boldsymbol{v}_h\|^{\theta}_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|^{1-\theta}_{\boldsymbol{L}^6(\mathbb{R}^d)}. \end{aligned} \tag{A.10}$$

According to the embedding $\boldsymbol{H}_0^1 \hookrightarrow \boldsymbol{L}^6$ and the Poincaré inequality, we get

$$\|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|_{\boldsymbol{L}^6(\mathbb{R}^d)} \leq C \|\nabla \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}. \tag{A.11}$$

Inserting (A.11) into (A.10), which implies that

$$\|\boldsymbol{v}_h(\cdot) - \boldsymbol{v}_h(\cdot - \boldsymbol{\xi})\|_{\boldsymbol{L}^q(\mathbb{R}^d)} \leq C|\boldsymbol{\xi}|^{\theta} \|\nabla \boldsymbol{v}_h\|_{\boldsymbol{L}^2(\Omega)}. \tag{A.12}$$

Combining the inequalities (A.9) and (A.12), the proof is thus complete.

### A.3 Some Functional Analysis Results

For the convenience of readers, we list some functional analysis results that need to be used in this article. We first recall the following weak convergence and monotonicity properties (see, e.g., [16, Theorem 10.19]):

**Lemma A.1** *Let $I \subset \mathbb{R}$ be an interval, $Q \subset \mathbb{R}^N$ a domain, and $(P, G) \in C(I) \times C(I)$ a couple of non-decreasing functions. Assume that $\rho_n \in L^1(Q; I)$ is a sequence such that*

$$\left\{ \begin{array}{c} P(\rho_n) \rightharpoonup \overline{P(\rho)}, \\ G(\rho_n) \rightharpoonup \overline{G(\rho)}, \\ P(\rho_n)G(\rho_n) \rightharpoonup \overline{P(\rho)G(\rho)}, \end{array} \right\} \quad \text{in } L^1(Q).$$

*(i) Then $\overline{P(\rho)}\ \overline{G(\rho)} \leq \overline{P(\rho)G(\rho)}$. (ii) If, in addition, $G \in C(\mathbb{R})$, $G(\mathbb{R}) = \mathbb{R}$, $G$ is strictly increasing, $P \in C(\mathbb{R})$, $P$ is non-decreasing, and $\overline{P(\rho)}\ \overline{G(\rho)} = \overline{P(\rho)G(\rho)}$, then $\overline{P(\rho)} = P \circ G^{-1}(\overline{G(\rho)})$. (iii) In particular, if $G(z) = z$, then $\overline{P(\rho)} = P(\rho)$.*

Secondly, the convex function have the lower semi-continuous with respect to the weak topology on $L^1(O)$ (see, e.g., [11, Theorem 2.11]).

**Lemma A.2** *Let $O \subset \mathbb{R}^N$ be a measurable set and $\{\boldsymbol{v}_n\}_{n=1}^{\infty}$ a sequence of functions in $L^1(O;\mathbb{R}^M)$ such that*

$$\boldsymbol{v}_n \rightharpoonup \boldsymbol{v}, \text{ in } L^1(O;\mathbb{R}^M).$$

*Let $\Phi : \mathbb{R}^M \to (-\infty, \infty]$ be a lower semi-continuous convex function such that $\Phi(\boldsymbol{v}_n) \in L^1(O)$ for any $n$, and*

$$\Phi(\boldsymbol{v}_n) \rightharpoonup \overline{\Phi(\boldsymbol{v})}, \text{ in } L^1(O).$$

*Then*

$$\Phi(\boldsymbol{v}) \le \overline{\Phi(\boldsymbol{v})} \text{ a. a. on } O.$$

*If, moreover, $\Phi$ is strictly on an open convex set $U \subset \mathbb{R}^M$, and*

$$\Phi(\boldsymbol{v}) = \overline{\Phi(\boldsymbol{v})} \text{ a. a. on } O,$$

*then*

$$\boldsymbol{v}_n(\boldsymbol{y}) \to \boldsymbol{v}(\boldsymbol{y}) \text{ for a. a. } \boldsymbol{y} \in \{\boldsymbol{y} \in O : \boldsymbol{v}(\boldsymbol{y}) \in U\}$$

*extracting subsequence as the case may be.*

Next, we introduce the following sequential compactness (see, e.g., [15, Lemma 3]).

**Lemma A.3** *Let $Q \subset \mathbb{R}^M$, suppose that $\rho_n \rightharpoonup \rho$ in $L^2(Q)$ and $\overline{\rho\log(\rho)} = \rho\log(\rho)$ are satisfied. Then*

$$\rho_n \to \rho \text{ in } L^1(Q).$$

Finally, we recall the following discrete version of the Aubin-Lions compactness Lemma for the Bochner spaces, which is useful in the convergence analysis. (see, e.g., [7, Theorem 1]).

**Lemma A.4** *Let $\mathbb{E}_0$, $\mathbb{E}$ and $\mathbb{E}_1$ be Banach spaces such that the embedding $\mathbb{E}_0 \hookrightarrow \mathbb{E}$ is compact and $\mathbb{E} \hookrightarrow \mathbb{E}_1$ is continuous. Given $T > 0$ and a small number $\tau > 0$, write $(0,T] = \cup_{k=1}^{M}(t_{k-1}, t_k]$ with $t_k = k\tau$ and $M\tau = T$. Let $\{v_\tau\}_{\tau>0}$ be a sequence such that*

- *The mapping $t \mapsto v_\tau(t,\cdot)$ is constant on each interval $(t_{k-1}, t_k]$, $k = 1, 2, \ldots, M$.*
- *Let $D_t v_\tau(t,\cdot) = (v_\tau(t,\cdot) - v_\tau(t-\tau,\cdot))/\tau$ be the discrete time derivative of $v_\tau(t,\cdot)$. The sequence $\{v_\tau\}_{\tau>0}$ satisfies the following estimates:*

$$\|v_\tau\|_{L^{p_0}(0,T;\mathbb{E}_0)} + \|D_t v_\tau\|_{L^{p_1}(\tau,T;\mathbb{E}_1)} \le C,$$

*for any $1 < p_0, p_1 < \infty$, where $C_0$ is a constant which is independent of $\tau$.*

*Then $\{v_\tau\}_{\tau>0}$ is relatively compact in $L^{p_0}(0,T;\mathbb{E})$.*

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