Journal of Scientific Computing (2022) 90:47
https://doi.org/10.1007/s10915-021-01730-6

Unconditionally Optimal Error Analysis of a Linear Euler FEM Scheme for the Navier–Stokes Equations with Mass Diffusion

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Received: 27 April 2021 / Revised: 18 November 2021 / Accepted: 22 November 2021 /
Published online: 16 December 2021

Abstract
In this paper, a linear and decoupled Euler finite element scheme is proposed for solving the 3D incompressible Navier–Stokes equations with mass diffusion numerically by the mini element for the velocity equation and the P_2 conforming element for the density equation. When the time step size τ and the mesh size h both are sufficiently small, the proposed FEM algorithm is unconditionally stable at the full discrete level, which is a key issue in designing the efficient algorithm for the multi-physical field problem. Furthermore, optimal temporal-spatial error estimates are presented for the velocity in $\mathbf{L}^2$-norm and the density in H^1-norm without any constraint of τ and h by using the technique of error splitting.

Keywords Kazhikhov–Smagulov model · Navier–Stokes equations with mass diffusion · Finite element discretization · Unconditional stability · Error estimates

Mathematics Subject Classification 35Q35 · 65M12 · 65M60

1 Introduction

Let $\Omega \subset \mathbf{R}^3$ be a bounded and convex domain with the sufficiently smooth boundary $\partial\Omega$ and $[0, T]$ the time interval with some $T > 0$. We will use notations $Q_T = [0, T] \times \Omega$ and $\Sigma_T = [0, T] \times \partial\Omega$. In this paper, we consider the 3D incompressible Navier–Stokes equations with mass diffusion (or called the Kazhikhov–Smagulov model) in Q_T, which can be deduced from the following 3D compressible Navier–Stokes equations:

$$\rho_t + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{1.1}$$

$$(\rho \mathbf{v})_t + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) - \mu \Delta \mathbf{v} - (\mu + \widetilde{\lambda}) \nabla (\nabla \cdot \mathbf{v}) + \nabla q = \mathbf{f}. \tag{1.2}$$

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In the above system (1.1–1.2), the unknows $\rho : Q_T \to \mathbf{R}$ is the density of the fluid, $\mathbf{v} : Q_T \to \mathbf{R}^3$ is the velocity of the fluid and $q : Q_T \to \mathbf{R}$ is the pressure which depends on the density ρ. $\mathbf{f} : Q_T \to \mathbf{R}^3$ denotes the external force, μ and $\widetilde{\lambda}$ are two constants and present viscosity coefficients which are assumed to satisfy $\mu > 0$ and $3\widetilde{\lambda} + 2\mu > 0$. If the mass diffusion process obeys Fick's law (cf. [12]), the velocity $\mathbf{v}$ of the fluid can be decomposed into a potential part and an incompressible part:

$$\mathbf{v} = \mathbf{u} - \lambda \nabla \ln \rho \quad \text{with} \quad \nabla \cdot \mathbf{u} = 0,$$

where $\lambda > 0$ is the mass diffusion coefficient. Then the compressible Navier–Stokes equations (1.1–1.2) can be rewritten as

$$\rho_t - \lambda \Delta \rho + \nabla \rho \cdot \mathbf{u} = 0, \tag{1.3}$$

$$(\rho \mathbf{u})_t + \nabla \cdot ((\rho \mathbf{u} - \lambda \nabla \rho) \otimes \mathbf{u} - \lambda \mathbf{u} \otimes \nabla \rho) - \mu \Delta \mathbf{u} + \lambda^2 \nabla \cdot (\rho^{-1} \nabla \rho \otimes \nabla \rho) + \nabla P = \mathbf{f}, \tag{1.4}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.5}$$

where $P = q - \lambda \rho_t + \lambda(2\mu + \widetilde{\lambda}) \Delta \ln \rho$. Eliminating the λ^2-term in (1.4) and using the following relations:

$$\begin{aligned} (\rho \mathbf{u})_t + \nabla \cdot ((\rho \mathbf{u} - \lambda \nabla \rho) \otimes \mathbf{u}) &= \rho \mathbf{u}_t + ((\rho \mathbf{u} - \lambda \nabla \rho) \cdot \nabla) \mathbf{u}, \\ -\lambda \nabla \cdot (\mathbf{u} \otimes \nabla \rho) &= -\lambda \nabla (\mathbf{u} \cdot \nabla \rho) + \lambda \nabla \cdot (\rho (\nabla \mathbf{u})^t), \end{aligned}$$

we get the simplified model of (1.3–1.5) in Q_T which is described as

$$\rho_t - \lambda \Delta \rho + \nabla \rho \cdot \mathbf{u} = 0, \tag{1.6}$$

$$\rho \mathbf{u}_t + ((\rho \mathbf{u} - \lambda \nabla \rho) \cdot \nabla) \mathbf{u} - \nabla \cdot \left(\mu \nabla \mathbf{u} - \lambda \rho (\nabla \mathbf{u})^t \right) + \nabla p = \mathbf{f} \tag{1.7}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.8}$$

where $p = P - \lambda \mathbf{u} \cdot \nabla \rho$. The above coupled system (1.6–1.8) are the incompressible Navier–Stokes equations with mass diffusion. It is clear that the system (1.6–1.8) reduce to the incompressible Navier–Stokes equations with variable density if $\lambda = 0$.

We complete (1.6–1.8) by the following boundary conditions

$$\mathbf{u} = 0 \quad \text{and} \quad \partial_{\mathbf{n}} \rho = 0 \quad \text{on } \Sigma_T \tag{1.9}$$

and the initial conditions

$$\rho(0, x) = \rho_0(x) \quad \text{and} \quad \mathbf{u}(0, x) = \mathbf{u}_0(x) \quad \text{in } \Omega, \tag{1.10}$$

where $\mathbf{n}$ denotes the outwards unit normal vector to $\partial \Omega$. Furthermore, we assume that there have two positive constants m and M such that

$$0 < m \le \rho_0(x) \le M \quad \text{in } \Omega, \tag{1.11}$$

which means that there has no vacuum state in Ω.

We recall some known results on the incompressible Navier–Stokes equations with mass diffusion. For the full model (1.3–1.5), Beirão da Veiga in [31] and Secchi in [29] established the local existence of the strong solution in terms of linearization and a fixed point method. Moreover, Secchi in [29] proved the eixstence and uniqueness of a global weak solution to 2D problem by imposing smallness on λ/μ and established the asymptotic behavior towards a weak solution to the incompressible Navier–Stokes problem with variable density when the mass diffusion coefficient $\lambda \to 0$. Guillén-González etc. in [15] proved the global existence of the strong solution for small initial data by means of an iterative method. When the mass diffusion coefficient $\lambda \to 0$ and the viscosity coefficient $\mu \to 0$, Araruna etc. in [4] studied the asymptotic behavior towards a solution to a inhomogeneous, inviscid and incompressible fluid governed by an Euler type system. For the numerical method of (1.3–1.5), Cabrales etc. in [6] proposed a fully discrete decoupled scheme by using a first-order time discretization and a C^0 finite element approximation for all unknowns and proved some stability and convergences results.

For the simplified model (1.6–1.8), Kazhikhov and Smagulov in [21] proved the global existence of the weak solution and the local existence of the strong solution by means of the Galerkin method under the assumptions that the initial density $\rho_0(x)$ satisfies (1.11) and the viscosity and mass diffusion coefficients satisfy $\lambda < 2\mu/(M-m)$. The global existence of the weak solution in the non-cylindrical domain was derived in [26]. Secchi in [28] studied the 3D Cauchy problem and established the local existence and uniqueness of the strong solution. The global existence of the strong solution to the 2D Cauchy problem and the 2D initial-boundary value problem were studied in [8,9], respectively. For the numerical methods, there are not many works concerning numerical analysis of the simplified model (1.6–1.8). By using a first-order time discretization and a C^0 finite element approximation for all unknowns, two decoupled numerical schemes were proposed for solving the 2D problem and the 3D problem in [16] and [17], respectively, where the stabilities of algorithms and the convergences of numerical solutions were investigated. Other numerical schemes can be found in [10] and [11,27], where an hybrid finite volume-finite element scheme and spectral Galerkin schemes were studied, respectively. Furthermore, the stability and convergence of numerical algorithm were investigated in [10].

To our best knowledge, the first error analysis of finite element fully discrete scheme for the simplified model (1.6–1.8) was presented by Guillén-González and Gutiérrez-Santacreu in [18]. To describe error estimates derived in [18], we introduce some notations. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with the time step $\tau = T/N$ and $t_n = n\tau$. If $\{\mathbf{v}^n\}_{n=1}^N$ is a given vector sequence with $\mathbf{v}^n \in X$ for a Banach space X, we introduce the following notations for the discrete-in-time norms:

$$\|\mathbf{v}^n\|_{l^2(X)} = \left(\tau \sum_{n=1}^{N} \|\mathbf{v}^n\|_X^2\right)^{1/2} \quad \text{and} \quad \|\mathbf{v}^n\|_{l^\infty(X)} = \sup_{1\le n\le N} \|\mathbf{v}^n\|_X.$$

Let $(\mathbf{u}_h^n, \rho_h^n)$ be the finite element approximations of $(\mathbf{u}(t_n), \rho(t_n))$ for $1 \le n \le N$. By using the mini-element (cf. [14]) for the approximation of velocity-pressure pair and the P_2 element for the approximation of density, the authors in [18] proved that

$$\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{l^\infty(L^2)} + \|\rho(t_n) - \rho_h^n\|_{l^\infty(H^1)} \le C(\tau + h) \tag{1.12}$$

under the weaker regularity assumptions on the exact solution. Concretely, the authors in [18] avoided using the assumption $\mathbf{u}_{tt} \in L^2(0, T; \mathbf{L}^{6/5}(\Omega))$ which required that the data should satisfy an extra compatibility condition at $t = 0$.

In this paper, a decoupled numerical scheme is proposed by using the mini-element for the velocity-pressure pair and the P_2 element for the density as that in [18]. Inspired by [22], this scheme is slightly different the scheme in [18] by introducing the post-processed velocity in the discretization of the density equation and the stable terms in the discretization of the Navier–Stokes type equation such that the proposed finite element scheme is unconditionally stable. The main result derived in this paper is the following optimal error estimate:

$$\|\mathbf{u}(t_n) - \mathbf{u}_h^n\|_{l^\infty(L^2)} + \|\rho(t_n) - \rho_h^n\|_{l^\infty(H^1)} \le C(\tau + h^2), \tag{1.13}$$

where $h > 0$ is the mesh size and $C > 0$ is some constant independent of h and τ. However, compared to [18], the higher regularities of the exact solution are assumed in this paper. The method of analysis is based on the technique of error splitting for the nonlinear parabolic problems proposed by Li and Sun in [23–25] and further developed in [2,3,7,13,32].

The rest of this paper is organized as follows. In Sect. 2, we state the proposed linear and decoupled Euler finite element scheme, present the stability of numerical scheme in Theorem 2.3 and the main result in Theorem 2.4. The proof of Theorem 2.4 is given in Sect. 3 by using the technique of error splitting. In particular, we firstly derive temporal error estimates and regularities of solutions to the time discrete scheme in Sect. 3.1, and then prove optimal spatial error estimates in Sect. 3.2.

2 Numerical Scheme and Main Result

2.1 Preliminaries

For the mathematical setting, we introduce the following notations. For $k \in \mathbb{N}^+$ and $1 \le p \le +\infty$, we use $W^{k,p}(\Omega)$ to denote the classical Sobolev space. The norm in $W^{k,p}(\Omega)$ is denoted by $\|\cdot\|_{W^{k,p}}$ defined by a classical way (cf. [1]). Denote $W_0^{k,p}(\Omega)$ be the subspace of $W^{k,p}(\Omega)$ where the functions have zero trace on $\partial\Omega$. Especially, $W^{0,p}(\Omega)$ is the Lebesgue space $L^p(\Omega)$ and $W^{k,2}(\Omega)$ is the Hilbert space which is simply denoted by $H^k(\Omega)$. The boldface notations $\mathbf{H}^k(\Omega)$, $\mathbf{W}^{k,p}(\Omega)$ and $\mathbf{L}^p(\Omega)$ are used to denote the vector-value Sobolev spaces corresponding to $H^k(\Omega)^3$, $W^{k,p}(\Omega)^3$ and $L^p(\Omega)^3$, respectively. We use $(\cdot,\cdot)$ to denote the L^2 or $\mathbf{L}^2$ inner product.

Introduce the following function spaces:

$$\begin{aligned}
\mathbf{H} &= \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\},\\
\mathbf{V} &= \mathbf{H}_0^1(\Omega), \quad \mathbf{V}_0 = \{\mathbf{u} \in \mathbf{V},\ \nabla\cdot\mathbf{u} = 0 \text{ in } \Omega\},\\
\mathbf{H}(\mathrm{div},\Omega) &= \{\mathbf{u} \in \mathbf{L}^2(\Omega),\ \nabla\cdot\mathbf{u} \in L^2(\Omega)\},\\
W &= \{r \in H^1(\Omega),\ \int_\Omega r(x)dx = 0\},\\
M &= L_0^2(\Omega) = \{p \in L^2(\Omega),\ \int_\Omega p(x)dx = 0\}
\end{aligned}$$

and

$$\begin{aligned}
H_N^2(\Omega) &= \{\rho \in H^2(\Omega),\ \partial_{\mathbf{n}}\rho = 0 \text{ on } \partial\Omega,\ \int_\Omega \rho(x)dx = \int_\Omega \rho_0(x)dx\},\\
H_{N,0}^2(\Omega) &= \{\rho \in H^2(\Omega),\ \partial_{\mathbf{n}}\rho = 0 \text{ on } \partial\Omega,\ \int_\Omega \rho(x)dx = 0\}.
\end{aligned}$$

It is known that the norms $\|\nabla\rho\|_{H^1}$ and $\|\rho\|_{H^2}$ are equivalent to the seminorm $\|\Delta\rho\|_{L^2}$ for $\rho \in H_N^2(\Omega)$ and $\rho \in H_{N,0}^2(\Omega)$, respectively.

Introduce the trilinear term $a(\rho;\mathbf{u},\mathbf{v})$ by

$$a(\rho;\mathbf{u},\mathbf{v}) = \mu(\nabla\mathbf{u},\nabla\mathbf{v}) - \lambda\int_\Omega \left(\rho - \frac{\widetilde{M}+\widetilde{m}}{2}\right)(\nabla\mathbf{u})^t : \nabla\mathbf{v}dx$$

with

$$\widetilde{M} > M, \quad 0 < \widetilde{m} < m \quad \text{such that} \quad \frac{\lambda(\widetilde{M}-\widetilde{m})}{2} < \mu \tag{2.1}$$

for any $\rho \in L^\infty(\Omega)$ and $\mathbf{u}, \mathbf{v} \in \mathbf{V}$. Under the condition (2.1), we can see that if $\widetilde{m} \le \rho(x) \le \widetilde{M}$, then

$$a(\rho; \mathbf{u}, \mathbf{u}) \ge \mu_1 \|\nabla \mathbf{u}\|_{L^2}^2 \text{ where } \mu_1 = \mu - \frac{\lambda(\widetilde{M} - \widetilde{m})}{2} > 0, \tag{2.2}$$

$$a(\rho; \mathbf{u}, \mathbf{v}) \le \mu_2 \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{v}\|_{L^2}. \tag{2.3}$$

The existence and uniqueness of weak solution to (1.6–1.8) are established by Kazhikhov and Smagulov in [21]. We recall it in the following theorem.

Theorem 2.1 *Let $\boldsymbol{u}_0 \in \boldsymbol{H}$ and $\rho_0 \in W$ satisfying (1.11) and $\boldsymbol{f} \in L^2(0, T; \boldsymbol{L}^2(\Omega))$. Suppose that the constants λ, μ, m and M satisfies*

$$\lambda < \frac{2\mu}{M - m}. \tag{2.4}$$

Then there exists a unique weak solution $(\rho, \boldsymbol{u})$ to (1.6–1.8) such that the solution satisfies

$$\boldsymbol{u} \in L^\infty(0, T; \boldsymbol{H}) \cap L^2(0, T; \boldsymbol{V}_0), \quad \rho \in L^\infty(0, T; W) \cap L^2(0, T; H_N^2(\Omega)), \tag{2.5}$$

$$0 < m \le \rho(t, x) \le M \quad \text{in } Q_T \tag{2.6}$$

and the energy inequalities:

$$\frac{1}{2}\|\sigma(t)\boldsymbol{u}(t)\|_{L^2}^2 + \left(\mu - \frac{\lambda(M - m)}{2}\right) \int_0^t \|\nabla \boldsymbol{u}(\tau)\|_{L^2}^2 d\tau \le \frac{1}{2}\|\sigma_0 \boldsymbol{u}_0\|_{L^2}^2 + \int_0^t (\boldsymbol{f}(\tau), \boldsymbol{u}(\tau)) d\tau,$$

$$\frac{1}{2}\|\rho(t)\|_{L^2}^2 + \lambda \int_0^t \|\nabla \rho(\tau)\|_{L^2}^2 d\tau \le \frac{1}{2}\|\rho_0\|_{L^2}^2$$

for all $0 < t \le T$, where $\sigma(t) = \sqrt{\rho(t)}$ and $\sigma_0 = \sqrt{\rho_0}$.

Throughout this paper, we make the following assumptions on the prescribed data, the regularity of the solution to (1.6–1.10) and the domain Ω.

Assumption (A1): Assume that the prescribed data $\mathbf{f}$, $\mathbf{u}_0$ and ρ_0 satisfy

$$\mathbf{f} \in L^2(0, T; \mathbf{L}^4(\Omega)), \quad \mathbf{u}_0 \in \mathbf{V}_0 \cap \mathbf{H}^2(\Omega) \quad \text{and} \quad \rho_0 \in H_N^2(\Omega) \text{ with (1.11)}.$$

Assumption (A2): Let λ, μ, m, M satisfy (2.4) and $\widetilde{m}, \widetilde{M}$ satisfy (2.1).

Assumption (A3): Assume that the solution $(\rho, \mathbf{u}, p)$ satisfies the following regularities:

$$\rho \in L^\infty(0, T; H^3(\Omega) \cap W), \ \rho_t \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega)),$$
$$\mathbf{u} \in L^\infty(0, T; \mathbf{W}^{2,4}(\Omega) \cap \mathbf{V}_0), \ \mathbf{u}_t \in L^\infty(0, T; \mathbf{H}^1(\Omega)) \cap L^2(0, T; \mathbf{H}^2(\Omega)),$$
$$\rho_{tt} \in L^2(0, T; L^2(\Omega)), \ \mathbf{u}_{tt} \in L^2(0, T; \mathbf{L}^2(\Omega)), \ p \in L^\infty(0, T; H^2(\Omega) \cap M).$$

Assumption (A4): Assume that the boundary $\partial\Omega$ is sufficiently smooth such that the unique solution ϕ of the Neumann problem

$$-\Delta\phi = g \quad \text{in } \Omega, \qquad \partial_{\mathbf{n}}\phi = 0 \quad \text{on } \partial\Omega$$

for prescribed $g \in M \cap H^k(\Omega)$ satisfies

$$\|\phi\|_{H^{2+k}} \le C\|g\|_{H^k}, \quad \text{for } k = 0, 1,$$

and the unique solution $(\mathbf{v}, q)$ of the Stokes problem

$$-\Delta\mathbf{v} + \nabla q = \mathbf{g} \text{ in } \Omega, \quad \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = 0 \text{ on } \partial\Omega$$

for prescribed $\mathbf{g} \in \mathbf{L}^p(\Omega)$ with $1 \le p \le 4$ satisfies

$$\|\mathbf{v}\|_{W^{2,p}} + \|q\|_{W^{1,p}} \le C\|\mathbf{g}\|_{L^p}.$$

Remark 2.1 The verification of the regularity assumption $\mathbf{u}_{tt} \in L^2(0, T; \mathbf{L}^2(\Omega))$ should involve an extra compatibility condition on the data at $t = 0$ which is not generally satisfied (see such condition for Navier–Stokes equations in [19]). We make this assumption merely to simplify the presentation. In [18], such assumption was avoided by using the technique of Euler integrator in the consistency error analysis.

2.2 Time Discrete Scheme

We first describe the time discrete scheme based on the backward Euler method. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with the time step $\tau = T/N$ and $t_n = n\tau$ with $0 \le n \le N$.

Given $\rho^0 = \rho_0$ and $\mathbf{u}^0 = \mathbf{u}_0$, we consider the following first-order Euler time discrete scheme for the simplified system (1.6–1.10).

Euler time discrete scheme:

Step I: For given ρ^n and $\mathbf{u}^n$, we find ρ^{n+1} by

$$D_\tau \rho^{n+1} - \lambda \Delta \rho^{n+1} + \nabla \rho^{n+1} \cdot \mathbf{u}^n = 0 \tag{2.7}$$

with the boundary condition $\partial_{\mathbf{n}} \rho^{n+1} = 0$ on $\partial\Omega$, where

$$D_\tau \rho^{n+1} = \frac{\rho^{n+1} - \rho^n}{\tau}.$$

Step II: For given ρ^n, $\mathbf{u}^n$ and ρ^{n+1} derived from (2.7), we find $(\mathbf{u}^{n+1}, p^{n+1})$ by

$$\begin{aligned} &\rho^n D_\tau \mathbf{u}^{n+1} - \nabla \cdot \left(\mu \nabla \mathbf{u}^{n+1} - \lambda \rho^{n+1} (\nabla \mathbf{u}^{n+1})^t\right) + \rho^{n+1} (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1} \\ &-\lambda (\nabla \rho^{n+1} \cdot \nabla) \mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{f}^{n+1}, \quad \nabla \cdot \mathbf{u}^{n+1} = 0 \end{aligned} \tag{2.8}$$

with the boundary condition $\mathbf{u}^{n+1} = 0$ on $\partial\Omega$.

The weak form of (2.7–2.8) are described as follows. Find the weak solutions $\rho^{n+1} \in W$ and $(\mathbf{u}^{n+1}, p^{n+1}) \in \mathbf{V} \times M$, respectively, by

$$(D_\tau \rho^{n+1}, r) + \lambda(\nabla \rho^{n+1}, \nabla r) + (\nabla \rho^{n+1} \cdot \mathbf{u}^n, r) = 0, \qquad \forall\, r \in W, \tag{2.9}$$

and

$$\begin{aligned} &(\rho^n D_\tau \mathbf{u}^{n+1}, \mathbf{v}) + a(\rho^{n+1}; \mathbf{u}^{n+1}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p^{n+1}) + (\nabla \cdot \mathbf{u}^{n+1}, q) \\ &+(\rho^{n+1} (\mathbf{u}^n \cdot \nabla) \mathbf{u}^{n+1}, \mathbf{v}) - \lambda((\nabla \rho^{n+1} \cdot \nabla) \mathbf{u}^{n+1}, \mathbf{v}) = (\mathbf{f}^{n+1}, \mathbf{v}),\ \forall (\mathbf{v}, q) \in \mathbf{V} \times M. \end{aligned} \tag{2.10}$$

In the above form (2.10), we use

$$\int_\Omega (\nabla \mathbf{u})^t : \nabla \mathbf{v} dx = 0$$

due to $\nabla \cdot \mathbf{u} = 0$ and $\nabla \cdot ((\nabla \mathbf{u})^t) = 0$ and $\mathbf{v} = 0$ on $\partial\Omega$.

From the assumption (**A4**) on the elliptic regularity, the well-posedness of solution to (2.7) was established in [17]. We recall it in the following lemma.

Lemma 2.1 *For each $0 \le n \le N-1$, if*

$$\|\nabla \boldsymbol{u}^n\|_{L^2} \le \kappa_1, \tag{2.11}$$

for some $\kappa_1 > 0$ being independent of τ and n, then for sufficiently small τ, the solution ρ^{n+1} to (2.7) satisfies

$$m \le \rho^{n+1}(x) \le M, \quad \forall\, x \in \Omega, \tag{2.12}$$

$$\|\rho^{n+1}\|_{H^1}^2 + \tau \sum_{i=0}^{n} \|\rho^{i+1}\|_{H^2}^2 \le \kappa_2, \tag{2.13}$$

for some $\kappa_2 > 0$ being independent of τ and n.

Remark 2.2 Although $\mathbf{u}^n$ in (2.7) replaces $\mathbf{u}_h^n$ in [17], the proof of Lemma 2.1 follows immediately from the proof of Lemma 3.4 in [17] by noting the fact that $\tau \sum_{n=1}^{N} \|\nabla \mathbf{u}^n\|_{L^2}^2 \le C$. Please see Appendix A in [17].

Next, we discuss the stability of the time discrete scheme (2.7–2.8). Setting $\phi = 2\tau\rho^{n+1}$ in (2.9) gives

$$\|\rho^{n+1}\|_{L^2}^2 - \|\rho^n\|_{L^2}^2 + \|\rho^{n+1} - \rho^n\|_{L^2}^2 + 2\lambda\tau\|\nabla\rho^{n+1}\|_{L^2}^2 = 0$$

by using

$$2\int_\Omega (\nabla\rho^{n+1}\cdot\mathbf{u}^n)\rho^{n+1}dx = \int_\Omega \nabla|\rho^{n+1}|^2\cdot\mathbf{u}^n dx = -\int_\Omega |\rho^{n+1}|^2\nabla\cdot\mathbf{u}^n = 0.$$

Taking the sum gives

$$\|\rho^{n+1}\|_{L^2}^2 + 2\lambda\tau\sum_{i=0}^{n}\|\nabla\rho^{i+1}\|_{L^2}^2 \le \|\rho_0\|_{L^2}^2$$

for all $0 \le n \le N-1$.

Suppose that

$$m \le \rho^{n+1}(x) \le M, \quad \forall\, 0 \le n \le N-1. \tag{2.14}$$

Setting $(\mathbf{v}, q) = 2\tau(\mathbf{u}^{n+1}, p^{n+1})$ in (2.10) and using (2.2), we have

$$\begin{aligned}
&\|\sigma^n\mathbf{u}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{u}^n\|_{L^2}^2 + \|\sigma^n(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{L^2}^2 + 2\mu_1\tau\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 \\
&+\tau\int_\Omega \rho^{n+1}\mathbf{u}^n\cdot\nabla|\mathbf{u}^{n+1}|^2dx - \lambda\tau\int_\Omega \nabla\rho^{n+1}\cdot\nabla|\mathbf{u}^{n+1}|^2dx \le 2\tau(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}),
\end{aligned}$$

where $\sigma^{n+1} = \sqrt{\rho^{n+1}}$. Setting $\phi = \tau|\mathbf{u}^{n+1}|^2$ in (2.9) leads to

$$\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{u}^{n+1}\|_{L^2}^2 + \lambda\tau\int_\Omega \nabla\rho^{n+1}\cdot\nabla|\mathbf{u}^{n+1}|^2dx + \tau\int_\Omega(\nabla\rho^{n+1}\cdot\mathbf{u}^n)|\mathbf{u}^{n+1}|^2dx = 0.$$

Then we obtain

$$\begin{aligned}
&\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{u}^n\|_{L^2}^2 + \|\sigma^n(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{L^2}^2 + 2\mu_1\tau\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 \\
&= 2\tau(\mathbf{f}^{n+1}, \mathbf{u}^{n+1}) \le \mu_1\tau\|\nabla\mathbf{u}^{n+1}\|_{L^2}^2 + \frac{\tau}{\mu_1}\|\mathbf{f}^{n+1}\|_{L^2}^2.
\end{aligned}$$

Taking the sum gives

$$\|\sigma^{n+1}\mathbf{u}^{n+1}\|_{L^2}^2 + \mu_1\tau\sum_{i=0}^{n}\|\nabla\mathbf{u}^{i+1}\|_{L^2}^2 \le \|\sigma_0\mathbf{u}_0\|_{L^2}^2 + \frac{\tau}{\mu_1}\sum_{i=0}^{N-1}\|\mathbf{f}^{i+1}\|_{L^2}^2$$

for all $0 \le n \le N-1$, where $\sigma_0 = \sqrt{\rho_0}$.

Thus, we get the following stable result of the time discrete scheme (2.7–2.8).

Theorem 2.2 *Under the condition (2.11) and the assumptions on the time step τ in Lemma 2.1, the solutions ρ^{n+1} and $\boldsymbol{u}^{n+1}$ to (2.7) and (2.8) satisfy the following the discrete energy inequalities:*

$$\max_{0\le n\le N-1}\left(\|\rho^{n+1}\|_{L^2}^2 + 2\lambda\tau\sum_{i=0}^{n}\|\nabla\rho^{i+1}\|_{L^2}^2\right) \le \|\rho_0\|_{L^2}^2,$$

$$\max_{0\le n\le N-1}\left(\|\sigma^{n+1}\boldsymbol{u}^{n+1}\|_{L^2}^2 + \mu_1\tau\sum_{i=0}^{n}\|\nabla\boldsymbol{u}^{i+1}\|_{L^2}^2\right) \le \|\sigma_0\boldsymbol{u}_0\|_{L^2}^2 + \frac{\tau}{\mu_1}\sum_{i=0}^{N-1}\|\boldsymbol{f}^{i+1}\|_{L^2}^2,$$

Remark 2.3 From the temporal error analysis in next section, we can see that the condition (2.11) holds for any $0 \le n \le N$. Thus, the above energy inequalities imply that the time discrete scheme (2.7–2.8) is unconditionally stable.

2.3 Finite Element Scheme

We give the finite element fully discretization of (2.7–2.8). Let $\mathcal{T}_h = \{K_j\}_{j=1}^{L}$ be a quasi-uniform tetrahedral partition of Ω with the mesh size $h = \max_{1\le j\le L}\{diam\ K_j\}$. When $\partial\Omega$ has a smooth curve, the element K_j adjacent to the boundary may represent a curved tetrahedron with a curved face. The definitions of finite element spaces on such a partition with curved elements can be dealt with that in [13,25]. We use the mini element $(P_1b - P_1)$ to approximate the velocity field $\mathbf{u}$ and the pressure p, and use the piecewise quadratic Lagrange element (P_2) to approximate the density ρ. The finite element spaces of $\mathbf{V}$, M and W are denoted by $\mathbf{V}_h$, M_h and W_h, respectively. For this choice, the finite element spaces $\mathbf{V}_h$ and M_h satisfy the discrete inf-sup condition. Further, we define the $\mathbf{H}(\text{div}, \Omega)$ conforming Raviart-Thomas finite element spaces of order 1 by

$$\mathbf{RT}_h = \{\mathbf{u}_h \in \mathbf{H}(\text{div}, \Omega),\ \mathbf{u}_h|_K \in P_1(K)^3 + xP_1(K),\ \forall\, K \in \mathcal{T}_h\},$$
$$\mathbf{RT}_{0h} = \{\mathbf{u}_h \in \mathbf{RT}_h,\ \nabla\cdot\mathbf{u}_h = 0 \text{ in } \Omega \text{ and } \mathbf{u}_h\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}.$$

We denote by $\mathbf{P}_{0h}$ the L^2-orthogonal projection operator from $\mathbf{L}^2(\Omega)$ to $\mathbf{RT}_{0h}$ defined by

$$(\mathbf{u} - \mathbf{P}_{0h}\mathbf{u}, \mathbf{v}_h) = 0, \qquad \forall\, \mathbf{v}_h \in \mathbf{RT}_{0h},\ \mathbf{u} \in \mathbf{L}^2(\Omega).$$

Start with $\mathbf{u}_h^0 = I_h\mathbf{u}_0$ and $\rho_h^0 = J_h\rho_0$, where I_h and J_h are the interpolation operators from $\mathbf{V} \to \mathbf{V}_h$ and $W \to W_h$, respectively, and satisfy

$$\|\mathbf{u}_0 - \mathbf{u}_h^0\|_{L^2} + h\|\nabla(\mathbf{u}_0 - \mathbf{u}_h^0)\|_{L^2} \le Ch^2\|\mathbf{u}_0\|_{H^2}, \tag{2.15}$$
$$\|\rho_0 - \rho_h^0\|_{L^2} + h\|\rho_0 - \rho_h^0\|_{H^1} \le Ch^2\|\rho_0\|_{H^2}. \tag{2.16}$$

For $1 \le n \le N$, the finite element fully discrete approximations of (2.7–2.8) are described as follows.

Finite element fully discrete scheme:

Step I: For given $\rho_h^n \in W_h$ and $\mathbf{u}_h^n \in \mathbf{V}_h$, we find $\rho_h^{n+1} \in W_h$ such that

$$(D_\tau \rho_h^{n+1}, r_h) + \lambda(\nabla \rho_h^{n+1}, \nabla r_h) + (\nabla \rho_h^{n+1} \cdot \mathbf{P}_{0h}\mathbf{u}_h^n, r_h) = 0 \tag{2.17}$$

for all $r_h \in W_h$.

Step II: For given $\rho_h^n \in W_h$, $\mathbf{u}_h^n \in \mathbf{V}_h$ and $\rho_h^{n+1} \in W_h$ derived from (2.17), we find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times M_h$ such that

$$\begin{aligned}
&(\rho_h^n D_\tau \mathbf{u}_h^{n+1}, \mathbf{v}_h) + a(\rho_h^{n+1}; \mathbf{u}_h^{n+1}, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, p_h^{n+1}) + (\nabla \cdot \mathbf{u}_h^{n+1}, q_h)\\
&+(\rho_h^{n+1}(\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^{n+1}, \mathbf{v}_h) + \frac{1}{2}(D_\tau \rho_h^{n+1}, \mathbf{u}_h^{n+1} \cdot \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot (\rho_h^{n+1}\mathbf{u}_h^n), \mathbf{u}_h^{n+1} \cdot \mathbf{v}_h)\\
&+\frac{\lambda}{2}(\nabla \rho_h^{n+1}, \nabla(\mathbf{u}_h^{n+1} \cdot \mathbf{v}_h)) - \lambda((\nabla \rho_h^{n+1} \cdot \nabla)\mathbf{u}_h^{n+1}, \mathbf{v}_h) = (\mathbf{f}^{n+1}, \mathbf{v}_h)
\end{aligned} \tag{2.18}$$

for all $(\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h$.

Remark 2.4 In the above algorithm, the post-processed velocity $\mathbf{P}_{0h}\mathbf{u}_h^n$ in (2.17) and the stabilized terms $(D_\tau \rho_h^{n+1}, \mathbf{u}_h^{n+1} \cdot \mathbf{v}_h) + (\nabla \cdot (\rho_h^n \mathbf{u}_h^{n+1}), \mathbf{u}_h^{n+1} \cdot \mathbf{v}_h) + \lambda(\nabla \rho_h^{n+1}, \nabla(\mathbf{u}_h^{n+1} \cdot \mathbf{v}_h))$ in (2.18) are used to preserve the unconditional stability of numerical scheme.

Taking $r_h = 2\tau\rho_h^{n+1}$ in (2.17), we get

$$\|\rho_h^{n+1}\|_{L^2}^2 - \|\rho_h^n\|_{L^2}^2 + \|\rho_h^{n+1} - \rho_h^n\|_{L^2}^2 + 2\lambda\tau\|\nabla\rho_h^{n+1}\|_{L^2}^2 = 0 \tag{2.19}$$

by using

$$2(\nabla \rho_h^{n+1} \cdot \mathbf{P}_{0h}\mathbf{u}_h^n, \rho_h^{n+1}) = \int_\Omega \mathbf{P}_{0h}\mathbf{u}_h^n \cdot \nabla|\rho_h^{n+1}|^2 dx = -\int_\Omega \nabla \cdot (\mathbf{P}_{0h}\mathbf{u}_h^n)|\rho_h^{n+1}|^2 dx = 0.$$

Taking the sum of (2.19) gives

$$\|\rho_h^{n+1}\|_{L^2}^2 + 2\lambda\tau\sum_{i=0}^{n}\|\nabla\rho_h^{i+1}\|_{L^2}^2 \le \|\rho_h^0\|_{L^2}^2$$

for all $0 \le n \le N-1$.

Suppose that the following condition holds:

$$\widetilde{m} < \rho_h^{n+1}(x) < \widetilde{M}, \qquad \forall\, 0 \le n \le N-1. \tag{2.20}$$

Taking $(\mathbf{v}_h, q_h) = 2\tau(\mathbf{u}_h^{n+1}, p_h^{n+1})$ in (2.18) and using (2.2), we have

$$\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}\|_{L^2}^2 - \|\sigma_h^n\mathbf{u}_h^n\|_{L^2}^2 + \|\sigma_h^n(\mathbf{u}_h^{n+1} - \mathbf{u}_h^n)\|_{L^2}^2 + 2\mu_1\tau\|\nabla\mathbf{u}_h^{n+1}\|_{L^2}^2 \le 2\tau(\mathbf{f}^{n+1}, \mathbf{u}_h^{n+1}) \tag{2.21}$$

by using

$$2(\rho_h^{n+1}(\mathbf{u}_h^n \cdot \nabla)\mathbf{u}_h^{n+1}, \mathbf{u}_h^{n+1}) = \int_\Omega \rho_h^{n+1}\mathbf{u}_h^n \cdot \nabla|\mathbf{u}_h^{n+1}|^2 dx = -\int_\Omega \nabla \cdot (\rho_h^{n+1}\mathbf{u}_h^n)|\mathbf{u}_h^{n+1}|^2 dx,$$

where $\sigma_h^{n+1} = \sqrt{\rho_h^{n+1}}$. Taking the sum of (2.21), we can get

$$\|\sigma_h^{n+1}\mathbf{u}_h^{n+1}\|_{L^2}^2 + \mu_1\tau\sum_{i=0}^{n}\|\nabla\mathbf{u}_h^{i+1}\|_{L^2}^2 \le \|\sigma_h^0\mathbf{u}_h^0\|_{L^2}^2 + \frac{\tau}{\mu_1}\sum_{i=0}^{N-1}\|\mathbf{f}^{i+1}\|_{L^2}^2$$

for all $0 \le n \le N-1$, where $\sigma_h^0 = \sqrt{\rho_h^0}$.

Like that for the time discrete scheme (2.7–2.8), we get the following stable result of the fully discrete scheme (2.17–2.18).

Theorem 2.3 *Under the condition (2.20), the solutions $\rho_h^{n+1} \in W_h$ and $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$ to (2.17) and (2.18) satisfy the following the discrete energy inequalities:*

$$\max_{0\le n\le N-1}\left(\|\rho_h^{n+1}\|_{L^2}^2 + 2\lambda\tau\sum_{i=0}^{n}\|\nabla\rho_h^{i+1}\|_{L^2}^2\right) \le \|\rho_h^0\|_{L^2}^2,$$

$$\max_{0\le n\le N-1}\left(\|\sigma_h^{n+1}\boldsymbol{u}_h^{n+1}\|_{L^2}^2 + \mu_1\tau\sum_{i=0}^{n}\|\nabla\boldsymbol{u}_h^{i+1}\|_{L^2}^2\right) \le \|\sigma_h^0\boldsymbol{u}_h^0\|_{L^2}^2 + \frac{\tau}{\mu_1}\sum_{i=0}^{N-1}\|\boldsymbol{f}^{i+1}\|_{L^2}^2.$$

Remark 2.5 From the temporal-spatial error analysis in next section, we can see that the condition (2.20) holds for sufficiently small h and τ. Thus, the above energy inequalities imply that the fully discrete scheme (2.17–2.18) is unconditionally stable. Furthermore, the discrete energy inequalities show the existence and uniqueness of solutions $\rho_h^{n+1} \in W_h$ and $\mathbf{u}_h^{n+1} \in \mathbf{V}_h$ when h and τ are sufficiently small.

2.4 Main Result

We present the optimal error estimate in the following theorem. The proof will be given in Section 3. In the rest of this paper, we denote by C a generic positive constant, which is independent of n, h and τ, and C may be different at different places.

Theorem 2.4 *Under the assumptions (**A1**)-(**A4**), there exist $\tau_0 > 0$ and $h_0 > 0$ such that when $\tau < \tau_0$ and $h < h_0$, the FE solutions ρ_h^{n+1} and $\boldsymbol{u}_h^{n+1}$ to (2.17) and (2.18) satisfy*

$$\max_{0\le n\le N-1}\left(\|\boldsymbol{u}(t_{n+1}) - \boldsymbol{u}_h^{n+1}\|_{L^2} + \|\rho(t_{n+1}) - \rho_h^{n+1}\|_{H^1}\right) \le C(\tau + h^2). \tag{2.22}$$

In the proof of Theorem 2.4, the following inverse inequalities and interpolation inequalities are frequently used (cf. [5]):

$$\|\mathbf{u}_h\|_{L^\infty} \le Ch^{-3/2}\|\mathbf{u}_h\|_{L^2} \quad \text{and} \quad \|\rho_h\|_{L^\infty} \le Ch^{-3/2}\|\rho_h\|_{L^2} \tag{2.23}$$

for any $\mathbf{u}_h \in \mathbf{V}_h$ and $\rho_h \in W_h$, and

$$\|u\|_{L^3} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2} \quad \text{and} \quad \|u\|_{L^4} \le C\|u\|_{L^2}^{1/4}\|u\|_{H^1}^{3/4}, \quad \forall\, u \in H^1(\Omega). \tag{2.24}$$

Finally, we recall the discrete Gronwall's inequality established in [20].

Lemma 2.2 *Let a_k, b_k and γ_k be the nonnegative numbers such that*

$$a_n + \tau\sum_{k=0}^{n} b_k \le \tau\sum_{k=0}^{n}\gamma_k a_k + B, \quad \textit{for } n \ge 1. \tag{2.25}$$

Suppose $\tau\gamma_k < 1$ and set $\sigma_k = (1-\tau\gamma_k)^{-1}$. Then there holds:

$$a_n + \tau\sum_{k=0}^{n} b_k \le \exp\left(\tau\sum_{k=0}^{n}\gamma_k\sigma_k\right)B, \quad \textit{for } n \ge 1 \tag{2.26}$$

Remark 2.6 If the sum on the right-hand side of (2.25) extends only up to $n-1$, then the estimate (2.26) still holds for all $k \ge 1$ with $\sigma_k = 1$.

3 Error Analysis

In this section, we will prove Theorem 2.4 by using the technique of error splitting. We first prove temporal errors in Sect. 3.1 and then prove spatial errors in Sect. 3.2. The finite element error estimates can be derived by combining temporal errors, projection errors and spatial errors.

3.1 Temporal Error Analysis

In this subsection, we will prove the optimal temporal errors. For $0 \le n \le N-1$, we take $t = t_{n+1}$ in (1.6–1.8) to deduce that

$$D_\tau \rho(t_{n+1}) - \lambda \Delta \rho(t_{n+1}) + \nabla \rho(t_{n+1}) \cdot \mathbf{u}(t_n) = R_\rho^{n+1} \tag{3.1}$$

and

$$\begin{aligned} &\rho(t_n) D_\tau \mathbf{u}(t_{n+1}) - \nabla \cdot \left(\mu \nabla \mathbf{u}(t_{n+1}) - \lambda \rho(t_{n+1})(\nabla \mathbf{u}(t_{n+1}))^t\right) + \nabla p(t_{n+1}) \\ &\quad + \rho(t_{n+1})(\mathbf{u}(t_n) \cdot \nabla)\mathbf{u}(t_{n+1}) - \lambda(\nabla \rho(t_{n+1}) \cdot \nabla)\mathbf{u}(t_{n+1}) = \mathbf{f}^{n+1} + R_u^{n+1}, \end{aligned} \tag{3.2}$$

where the truncation functions R_σ^{n+1} and R_u^{n+1} are given by

$$\begin{aligned} R_\rho^{n+1} &= D_\tau \rho(t_{n+1}) - \rho_t(t_{n+1}) - \nabla \rho(t_{n+1}) \cdot \left(\int_{t_n}^{t_{n+1}} \mathbf{u}_t(t) dt \right), \\ R_u^{n+1} &= (\rho(t_n) - \rho(t_{n+1})) D_\tau \mathbf{u}(t_{n+1}) + \rho(t_{n+1}) \left(D_\tau \mathbf{u}(t_{n+1}) - \mathbf{u}_t(t_{n+1})\right) \\ &\quad - \rho(t_{n+1}) \left(\int_{t_n}^{t_{n+1}} \mathbf{u}_t(t) dt \cdot \nabla \right) \mathbf{u}(t_{n+1}). \end{aligned}$$

Under the regularity assumption (**A3**), we have

$$\tau \sum_{n=0}^{N-1} \left(\|R_\rho^{n+1}\|_{L^2}^2 + \|R_u^{n+1}\|_{L^2}^2 \right) \le C\tau^2. \tag{3.3}$$

For $0 \le n \le N$, we introduce temporal error functions by

$$\eta^n = \rho(t_n) - \rho^n, \quad \mathbf{e}^n = \mathbf{u}(t_n) - \mathbf{u}^n, \quad \epsilon^n = p(t_n) - p^n.$$

Then error equations satisfied by $(\eta^{n+1}, \mathbf{e}^{n+1}, \epsilon^{n+1})$ with $0 \le n \le N-1$ are

$$D_\tau \eta^{n+1} - \lambda \Delta \eta^{n+1} + \nabla \rho(t_{n+1}) \cdot \mathbf{e}^n + \nabla \eta^{n+1} \cdot \mathbf{u}^n = R_\rho^{n+1}, \tag{3.4}$$

and

$$\rho^n D_\tau \mathbf{e}^{n+1} - \nabla \cdot \left(\mu \nabla \mathbf{e}^{n+1} - \lambda \rho^{n+1} (\nabla \mathbf{e}^{n+1})^t\right) + \nabla \epsilon^{n+1} + \sum_{i=1}^{7} I_i^{n+1} = R_u^{n+1} \tag{3.5}$$

with $\nabla \cdot \mathbf{e}^{n+1} = 0$ in Ω, where

$$\begin{aligned} I_1^{n+1} &= -\lambda(\nabla \rho^{n+1} \cdot \nabla)\mathbf{e}^{n+1}, \\ I_2^{n+1} &= \rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{e}^{n+1}, \\ I_3^{n+1} &= \rho^{n+1}(\mathbf{e}^n \cdot \nabla)\mathbf{u}(t_{n+1}), \\ I_4^{n+1} &= \eta^n D_\tau \mathbf{u}(t_{n+1}), \end{aligned}$$

$$
\begin{aligned}
I_5^{n+1} &= \lambda \nabla \cdot (\eta^{n+1} (\nabla \mathbf{u}(t_{n+1}))^t), \\
I_6^{n+1} &= -\lambda (\nabla \eta^{n+1} \cdot \nabla) \mathbf{u}(t_{n+1}), \\
I_7^{n+1} &= \eta^{n+1} (\mathbf{u}(t_n) \cdot \nabla) \mathbf{u}(t_{n+1}).
\end{aligned}
$$

Moreover, the weak formulations of (3.4) and (3.5) can be described as: find $\eta^{n+1} \in W$ such that

$$
\left(D_\tau \eta^{n+1}, r\right) + \lambda (\nabla \eta^{n+1}, \nabla r) + \left(\nabla \rho(t_{n+1}) \cdot \mathbf{e}^n, r\right) + \left(\nabla \eta^{n+1} \cdot \mathbf{u}^n, r\right) = (R_\rho^{n+1}, r) \quad (3.6)
$$

for all $r \in W$, and find $(\mathbf{e}^{n+1}, \epsilon^{n+1}) \in \mathbf{V} \times M$ such that

$$
\left(\rho^n D_\tau \mathbf{e}^{n+1}, \mathbf{v}\right) + a(\rho^{n+1}; \mathbf{e}^{n+1}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, \epsilon^{n+1}) + (\nabla \cdot \mathbf{e}^{n+1}, q) + \sum_{i=1}^{7} (I_i^{n+1}, \mathbf{v}) = (R_u^{n+1}, \mathbf{v}) \quad (3.7)
$$

for all $(\mathbf{v}, q) \in \mathbf{V} \times M$.

We estimate η^{n+1} and $\mathbf{e}^{n+1}$ in $l^\infty(L^2)$-norm and $l^2(H^1)$-norm in the following two lemmas.

Lemma 3.1 *Under the regularity assumption (**A3**), there exists some $C > 0$ such that*

$$
\|\eta^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m} \|\eta^{n+1} - \eta^n\|_{L^2}^2 + \lambda \tau \sum_{n=0}^{m} \|\nabla \eta^{n+1}\|_{L^2}^2 \le C \left(\tau^2 + \tau \sum_{n=0}^{m} \|\boldsymbol{e}^n\|_{L^2}^2 \right) \quad (3.8)
$$

for all $0 \le m \le N - 1$.

Proof Taking $r = 2\tau \eta^{n+1}$ in (3.6) and using

$$
\int_\Omega (\nabla \eta^{n+1} \cdot \mathbf{u}^n) \eta^{n+1} dx = -\frac{1}{2} \int_\Omega |\eta^{n+1}|^2 \nabla \cdot \mathbf{u}^n dx + \frac{1}{2} \int_{\partial\Omega} |\eta^{n+1}|^2 \mathbf{u}^n \cdot \mathbf{n} ds = 0,
$$

it is easy to see that

$$
\begin{aligned}
&\|\eta^{n+1}\|_{L^2}^2 - \|\eta^n\|_{L^2}^2 + \|\eta^{n+1} - \eta^n\|_{L^2}^2 + 2\lambda\tau \|\nabla \eta^{n+1}\|_{L^2}^2 \\
&\le \frac{\tau}{2} \|\eta^{n+1}\|_{L^2}^2 + C\tau \left(\|\mathbf{e}^n\|_{L^2}^2 + \|R_\rho^{n+1}\|_{L^2}^2 \right).
\end{aligned}
$$

Summing up the above estimate for n from 0 to m and using (3.3) and the discrete Gronwall's inequality in Lemma 2.2, we complete the proof of (3.8). □

Lemma 3.2 *Under the assumptions (**A2**) and (**A3**), there exists some small $\tau_1 > 0$ such that when $\tau < \tau_1$, there holds*

$$
\|\sigma^{m+1} \boldsymbol{e}^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m} \|\sigma^n (\boldsymbol{e}^{n+1} - \boldsymbol{e}^n)\|_{L^2}^2 + \tau \sum_{n=0}^{m} \|\nabla \boldsymbol{e}^{n+1}\|_{L^2}^2 \le C\tau^2 \quad (3.9)
$$

for all $0 \le m \le N - 1$.

Proof Setting $(\mathbf{v}, q) = 2\tau(\mathbf{e}^{n+1}, \epsilon^{n+1})$ in (3.7), we have

$$
\begin{aligned}
&\|\sigma^n \mathbf{e}^{n+1}\|_{L^2}^2 - \|\sigma^n \mathbf{e}^n\|_{L^2}^2 + \|\sigma^n (\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + 2a(\rho^{n+1}; \mathbf{e}^{n+1}, \mathbf{e}^{n+1}) \\
&-\lambda\tau (\nabla \rho^{n+1}, \nabla |\mathbf{e}^{n+1}|^2) + \tau (\rho^{n+1} \mathbf{u}^n, \nabla |\mathbf{e}^{n+1}|^2) + 2\tau \sum_{i=3}^{7} (I_i^{n+1}, \mathbf{e}^{n+1}) = 2\tau (R_u^{n+1}, \mathbf{e}^{n+1}).
\end{aligned}
$$

Multiplying (2.7) by $\tau|\mathbf{e}^{n+1}|^2$ and integrating over Ω, one has

$$\|\sigma^{n+1}\mathbf{e}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{e}^{n+1}\|_{L^2}^2 + \lambda\tau(\nabla\rho^{n+1}, \nabla|\mathbf{e}^{n+1}|^2) - \tau(\rho^{n+1}\mathbf{u}^n, \nabla|\mathbf{e}^{n+1}|^2) = 0$$

where the integration by parts is used. Taking the sum of the above formulations, we get

$$\begin{aligned} &\|\sigma^{n+1}\mathbf{e}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{e}^n\|_{L^2}^2 + \|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + 2a(\rho^{n+1};\mathbf{e}^{n+1},\mathbf{e}^{n+1}) \\ &+2\tau\sum_{i=3}^{7}(I_i^{n+1},\mathbf{e}^{n+1}) = 2\tau(R_u^{n+1},\mathbf{e}^{n+1}). \end{aligned} \tag{3.10}$$

Now, we suppose that

$$\|\nabla\mathbf{u}^n\|_{L^2} \le 1 + \|\nabla\mathbf{u}_0\|_{L^2} + \|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} := \kappa_1, \quad \forall\, 0 \le n \le N-1. \tag{3.11}$$

According to Lemma 2.1, we have

$$\widetilde{m} < m \le \rho^{n+1}(x) \le M < \widetilde{M}, \qquad \forall\, 0 \le n \le N-1, \tag{3.12}$$

which with (2.2) and (3.10) gives

$$\begin{aligned} &\|\sigma^{n+1}\mathbf{e}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{e}^n\|_{L^2}^2 + \|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + 2\mu_1\tau\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 \\ &\le 2\tau\left|(R_u^{n+1},\mathbf{e}^{n+1})\right| + 2\tau\left|\sum_{i=3}^{7}(I_i^{n+1},\mathbf{e}^{n+1})\right|. \end{aligned} \tag{3.13}$$

To close the mathematical induction (3.11), we need to prove that

$$\|\nabla\mathbf{u}^{n+1}\|_{L^2} \le \kappa_1, \qquad \forall\, 0 \le n \le N-1. \tag{3.14}$$

The right-hand side of (3.13) can be estimated term by term as follows. It is easy to see that

$$2\tau\left|(R_u^{n+1},\mathbf{e}^{n+1})\right| \le \tau\left(\|R_u^{n+1}\|_{L^2}^2 + \|\mathbf{e}^{n+1}\|_{L^2}^2\right).$$

By (**A3**) and (3.12), we have

$$\begin{aligned} 2\tau\left|(I_3^{n+1},\mathbf{e}^{n+1})\right| &\le C\tau\|\rho^{n+1}\|_{L^\infty}\|\mathbf{e}^n\|_{L^2}\|\nabla\mathbf{u}(t_{n+1})\|_{L^3}\|\nabla\mathbf{e}^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{5}\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau\|\mathbf{e}^n\|_{L^2}^2, \\ 2\tau\left|(I_4^{n+1},\mathbf{e}^{n+1})\right| &\le C\tau\|\eta^n\|_{L^2}\|D_\tau\mathbf{u}(t_{n+1})\|_{L^3}\|\nabla\mathbf{e}^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{5}\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau\|\eta^n\|_{L^2}^2, \\ 2\tau\left|(I_5^{n+1},\mathbf{e}^{n+1})\right| &\le C\tau\|\eta^{n+1}\|_{L^6}\|\nabla\mathbf{u}(t_{n+1})\|_{L^3}\|\nabla\mathbf{e}^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{5}\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau\|\eta^{n+1}\|_{L^2}^2 + C\tau\|\nabla\eta^{n+1}\|_{L^2}^2, \\ 2\tau\left|(I_6^{n+1},\mathbf{e}^{n+1})\right| &\le C\tau\|\nabla\eta^{n+1}\|_{L^2}\|\nabla\mathbf{u}(t_{n+1})\|_{L^3}\|\nabla\mathbf{e}^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{5}\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau\|\nabla\eta^{n+1}\|_{L^2}^2, \end{aligned}$$

and

$$\begin{aligned} 2\tau\left|(I_7^{n+1},\mathbf{e}^{n+1})\right| &\le C\tau\|\eta^{n+1}\|_{L^6}\|\mathbf{u}(t_n)\|_{L^\infty}\|\nabla\mathbf{u}(t_{n+1})\|_{L^3}\|\nabla\mathbf{e}^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{5}\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau\|\eta^{n+1}\|_{L^2}^2 + C\tau\|\nabla\eta^{n+1}\|_{L^2}^2, \end{aligned}$$

where the Hölder inequality and the Young inequality are used. Taking into account the above estimates, we get from (3.13) that

$$
\begin{aligned}
&\|\sigma^{n+1}\mathbf{e}^{n+1}\|_{L^2}^2 - \|\sigma^n\mathbf{e}^n\|_{L^2}^2 + \|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + \mu_1\tau\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2\\
&\le C\tau\left(\|R_u^{n+1}\|_{L^2}^2 + \|\mathbf{e}^{n+1}\|_{L^2}^2 + \|\mathbf{e}^n\|_{L^2}^2 + \|\eta^{n+1}\|_{L^2}^2 + \|\eta^n\|_{L^2}^2 + \|\nabla\eta^{n+1}\|_{L^2}^2\right)\\
&\le C\tau\left(\|R_u^{n+1}\|_{L^2}^2 + \|\sigma^{n+1}\mathbf{e}^{n+1}\|_{L^2}^2 + \|\sigma^n\mathbf{e}^n\|_{L^2}^2 + \|\nabla\eta^{n+1}\|_{L^2}^2\right)\\
&\quad + C\tau^3 + C\tau^2\sum_{n=0}^{m}\|\sigma^n\mathbf{e}^n\|_{L^2}^2,
\end{aligned}
\tag{3.15}
$$

where (3.8) in Lemma 3.1 and (3.12) are used. Summing up (3.15) for n from 0 to m, using (3.3), (3.8) and the discrete Gronwall's inequality in Lemma 2.2, we derive (3.9) and complete the mathematical induction (3.14) by taking a sufficiently small τ_1 such that

$$
\begin{aligned}
\|\nabla\mathbf{u}^{n+1}\|_{L^2} &\le \|\nabla\mathbf{e}^{n+1}\|_{L^2} + \|\nabla\mathbf{u}(t_{n+1})\|_{L^2} \le \|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} + (C\tau_1)^{1/2}\\
&\le \|\nabla\mathbf{u}\|_{L^\infty(0,T;L^2)} + 1 \le \kappa_1.
\end{aligned}
$$

□

From the proof of Lemma 3.2, we can see that (3.14) holds for all $0 \le n \le N-1$. It follows from Lemma 2.1 that the solutions ρ^{n+1} to (2.7) and $\mathbf{u}^{n+1}$ to (2.8) satisfy

$$
\widetilde{m} < m \le \rho^{n+1}(x) \le M < \widetilde{M}, \tag{3.16}
$$

$$
\|\nabla\mathbf{u}^{n+1}\|_{L^2} + \|\rho^{n+1}\|_{H^1}^2 + \tau\sum_{i=0}^{n}\|\rho^{i+1}\|_{H^2}^2 \le C \tag{3.17}
$$

for all $0 \le n \le N-1$. By (3.8) and (3.9), we get the following estimate for the density:

$$
\|\eta^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m}\|\eta^{n+1}-\eta^n\|_{L^2}^2 + \lambda\tau\sum_{n=0}^{m}\|\eta^{n+1}\|_{H^1}^2 \le C\tau^2 \tag{3.18}
$$

for $0 \le m \le N-1$. Furthermore, we can estimate η^{n+1} in $l^\infty(H^1)$-norm and $l^2(H^2)$-norm as follows.

Lemma 3.3 *Under the assumptions* (**A2**) *and* (**A3**)*, when* $\tau < \tau_1$*, where* τ_1 *is from Lemma 3.2, there exists some* $C > 0$ *such that*

$$
\|\eta^{m+1}\|_{H^1}^2 + \sum_{n=0}^{m}\|\eta^{n+1}-\eta^n\|_{H^1}^2 + \lambda\tau\sum_{n=0}^{m}\|\eta^{n+1}\|_{H^2}^2 \le C\tau^2 \tag{3.19}
$$

for all $0 \le m \le N-1$.

Proof Multiplying (3.4) by $-2\tau\Delta\eta^{n+1}$ and integrating over Ω, we can prove that

$$
\begin{aligned}
&\|\nabla\eta^{n+1}\|_{L^2}^2 - \|\nabla\eta^n\|_{L^2}^2 + \|\nabla(\eta^{n+1}-\eta^n)\|_{L^2}^2 + 2\lambda\tau\|\Delta\eta^{n+1}\|_{L^2}^2\\
&\le C\tau\left(\|\mathbf{e}^n\|_{L^2} + \|R_\rho^{n+1}\|_{L^2}\right)\|\Delta\eta^{n+1}\|_{L^2} + C\tau\|\nabla\eta^{n+1}\|_{L^2}^{1/2}\|\mathbf{u}^n\|_{L^6}\|\Delta\eta^{n+1}\|_{L^2}^{3/2}\\
&\le \lambda\tau\|\Delta\eta^{n+1}\|_{L^2}^2 + C\tau\left(\|\mathbf{e}^n\|_{L^2} + \|R_\rho^{n+1}\|_{L^2} + \|\nabla\eta^{n+1}\|_{L^2}^2\right)
\end{aligned}
$$

Summing up the above estimate for n from 0 to m and using (3.18), we obtain

$$
\|\nabla\eta^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m}\|\nabla(\eta^{n+1}-\eta^n)\|_{L^2}^2 + \lambda\tau\sum_{n=0}^{m}\|\Delta\eta^{n+1}\|_{L^2}^2 \le C\tau^2
$$

for $0 \le m \le N-1$. By noticing (3.18), again, we complete the proof of (3.19). □

The error estimate (3.19) provides a uniform boundness of ρ^{n+1} in H^2-norm. That is to say that there exists some $C > 0$ such that

$$\|\rho^{n+1}\|_{H^2} \le C, \qquad \forall\, 0 \le n \le N-1. \tag{3.20}$$

Next, we estimate $\mathbf{u}^{n+1}$ in $l^2(\mathbf{H}^2)$-norm under the assumption (**A4**). To do this, we rewrite (2.8) as the Stokes type problem:

$$-\mu\Delta\mathbf{u}^{n+1} + \nabla p^{n+1} = \mathbf{F}^{n+1}, \tag{3.21}$$

where

$$\mathbf{F}^{n+1} = \mathbf{f}^{n+1} - \rho^n D_\tau \mathbf{u}^{n+1} - \lambda\nabla\rho^{n+1}\cdot(\nabla\mathbf{u}^{n+1})^t - \rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1} + \lambda(\nabla\rho^{n+1}\cdot\nabla)\mathbf{u}^{n+1}$$

by using $\nabla\cdot(\nabla\mathbf{u}^{n+1})^t = 0$ due to $\nabla\cdot\mathbf{u}^{n+1} = 0$. From (3.16) and

$$\begin{aligned}
\|\rho^n(\mathbf{u}^{n+1} - \mathbf{u}^n)\|_{L^2}^2 &\le 2\|\rho^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + 2\|\rho^n(\mathbf{u}(t_{n+1}) - \mathbf{u}(t_n))\|_{L^2}^2 \\
&\le 2\|\sigma^n\|_{L^\infty}^2\|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + 2\tau\|\rho^n\|_{L^\infty}^2\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_{L^2}^2 dt \\
&\le C\|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + C\tau\int_{t_n}^{t_{n+1}} \|\mathbf{u}_t(t)\|_{L^2}^2 dt,
\end{aligned}$$

we have

$$\tau\sum_{n=0}^{N-1}\|\rho^n D_\tau\mathbf{u}^{n+1}\|_{L^2}^2 \le C, \tag{3.22}$$

where (3.9) is used. From (3.16), (3.17) and (3.20), one has

$$\begin{aligned}
&\|\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\|_{L^{3/2}} + \|(\nabla\rho^{n+1}\cdot\nabla)\mathbf{u}^{n+1}\|_{L^{3/2}} + \|\nabla\rho^{n+1}\cdot(\nabla\mathbf{u}^{n+1})^t\|_{L^{3/2}} \\
&\le \|\rho^{n+1}\|_{L^\infty}\|\mathbf{u}^n\|_{L^6}\|\nabla\mathbf{u}^{n+1}\|_{L^2} + 2\|\nabla\rho^{n+1}\|_{L^6}\|\nabla\mathbf{u}^{n+1}\|_{L^2} \\
&\le C.
\end{aligned} \tag{3.23}$$

Then (3.22) and (3.23) yield

$$\tau\sum_{n=0}^{N-1}\|\mathbf{F}^{n+1}\|_{L^{3/2}}^2 \le C,$$

which with the assumption (**A4**) gives

$$\tau\sum_{n=0}^{N-1}\left(\|\mathbf{u}^{n+1}\|_{W^{2,3/2}}^2 + \|p^{n+1}\|_{W^{1,3/2}}^2\right) \le C.$$

From the Sobolev imbedding theorem $\mathbf{W}^{2,3/2}(\Omega) \hookrightarrow \mathbf{W}^{1,3}(\Omega)$, we have

$$\tau\sum_{n=0}^{N-1}\|\mathbf{u}^{n+1}\|_{W^{1,3}}^2 \le C. \tag{3.24}$$

By (3.16), (3.17) and (3.20), again, we have

$$\begin{aligned}
&\|\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{u}^{n+1}\|_{L^2} + \|(\nabla\rho^{n+1}\cdot\nabla)\mathbf{u}^{n+1}\|_{L^2} + \|\nabla\rho^{n+1}\cdot(\nabla\mathbf{u}^{n+1})^t\|_{L^2} \\
&\le \|\rho^{n+1}\|_{L^\infty}\|\mathbf{u}^n\|_{L^6}\|\nabla\mathbf{u}^{n+1}\|_{L^3} + 2\|\nabla\rho^{n+1}\|_{L^6}\|\nabla\mathbf{u}^{n+1}\|_{L^3} \\
&\le C\|\nabla\mathbf{u}^{n+1}\|_{L^3}.
\end{aligned} \tag{3.25}$$

Then (3.22) and (3.25) yield

$$\tau \sum_{n=0}^{N-1} \|\mathbf{F}^{n+1}\|_{L^2}^2 \leq C.$$

By the assumption (**A4**), again, we get

$$\tau \sum_{n=0}^{N-1} \left(\|\mathbf{u}^{n+1}\|_{H^2}^2 + \|p^{n+1}\|_{H^1}^2\right) \leq C. \tag{3.26}$$

Thus, the numerical velocity $\mathbf{u}^{n+1}$ is uniformly bound in $l^2(\mathbf{H}^2)$-norm. Based on the regularities (3.20) and (3.26), we can obtain the error estimate of $\mathbf{e}^{n+1}$ in $l^\infty(\mathbf{V})$-norm and $l^2(\mathbf{H}^2)$-norm stated in Lemma 3.4. To make this, we rewrite (3.5) as

$$\rho^n D_\tau \mathbf{e}^{n+1} - \mu\Delta\mathbf{e}^{n+1} + \lambda\nabla\rho^{n+1}\cdot(\nabla\mathbf{e}^{n+1})^t + \nabla\epsilon^{n+1} + \sum_{i=1}^{6} I_i^{n+1} = R_u^{n+1} \tag{3.27}$$

with $\nabla\cdot\mathbf{e}^{n+1} = 0$ in Ω.

Lemma 3.4 *Under the assumptions (**A2**)-(**A4**), there exists some $\tau_2 < \tau_1$ such that when $\tau < \tau_2$, there holds*

$$\|\nabla\boldsymbol{e}^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m} \|\nabla(\boldsymbol{e}^{n+1} - \boldsymbol{e}^n)\|_{L^2}^2 + \tau\sum_{n=0}^{m}\left(\|\boldsymbol{e}^{n+1}\|_{H^2}^2 + \|\epsilon^{n+1}\|_{H^1}^2\right) \leq C\tau^2 \tag{3.28}$$

for all $0 \leq m \leq N-1$.

Proof Testing (3.27) by $2\tau(\mathbf{e}^{n+1} - \mathbf{e}^n)$ leads to

$$\begin{aligned} &2\|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + \mu\tau\left(\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 - \|\nabla\mathbf{e}^n\|_{L^2}^2 + \|\nabla(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2\right) \\ &\quad\leq 2\tau\left|\left(R_u^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| + 2\lambda\tau\left|\left(\nabla\rho^{n+1}\cdot(\nabla\mathbf{e}^{n+1})^t, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| \\ &\qquad+2\tau\sum_{i=1}^{7}\left|\left(I_i^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right|. \end{aligned} \tag{3.29}$$

The right-hand side of (3.29) can be estimated term by term by using the Hölder inequality and the Young inequality. From (3.16), it is easy to show that

$$2\tau\left|\left(R_u^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| \leq \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|R_u^{n+1}\|_{L^2}^2,$$

and

$$\begin{aligned} &2\lambda\tau\left|\left(\nabla\rho^{n+1}\cdot(\nabla\mathbf{e}^{n+1})^t, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| \\ &\leq 2\lambda\tau\left|\left(\nabla\eta^{n+1}\cdot(\nabla\mathbf{e}^{n+1})^t, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| + 2\lambda\tau\left|\left(\nabla\rho(t_{n+1})\cdot(\nabla\mathbf{e}^{n+1})^t, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| \\ &\leq \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau^2\|\nabla\eta^{n+1}\|_{L^3}^2\|\mathbf{e}^{n+1}\|_{H^2}^2. \end{aligned}$$

A similar argument gives

$$2\tau\left|\left(I_1^{n+1}, \mathbf{e}^{n+1} - \mathbf{e}^n\right)\right| \leq \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau^2\|\nabla\eta^{n+1}\|_{L^3}^2\|\mathbf{e}^{n+1}\|_{H^2}^2.$$

Other terms can be bound, respectively, by

$$
\begin{aligned}
2\tau\left|\left(I_2^{n+1},\mathbf{e}^{n+1}-\mathbf{e}^n\right)\right| &\le \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\mathbf{u}^n\|_{H^2}^2\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2\\
2\tau\left|\left(I_3^{n+1},\mathbf{e}^{n+1}-\mathbf{e}^n\right)\right| &\le \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\nabla\mathbf{e}^{n}\|_{L^2}^2,\\
2\tau\left|\left(I_4^{n+1},\mathbf{e}^{n+1}-\mathbf{e}^n\right)\right| &\le \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\eta^n\|_{H^2}^2,\\
2\tau\left|\left(I_5^{n+1},\mathbf{e}^{n+1}-\mathbf{e}^n\right)\right| &\le \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\eta^{n+1}\|_{H^2}^2,\\
2\tau\left|\left(I_6^{n+1},\mathbf{e}^{n+1}-\mathbf{e}^n\right)\right| &\le \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\eta^{n+1}\|_{H^2}^2,\\
2\tau\left|\left(I_7^{n+1},\mathbf{e}^{n+1}-\mathbf{e}^n\right)\right| &\le \frac{1}{9}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\tau^2\|\eta^{n+1}\|_{H^2}^2,
\end{aligned}
$$

where we use the regularity assumption (**A3**) and (3.16). Substituting the above estimates into (3.29) leads to

$$
\begin{aligned}
&\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + \mu\tau\left(\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 - \|\nabla\mathbf{e}^n\|_{L^2}^2 + \|\nabla(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2\right)\\
&\le C\tau^2\left(\|R_u^{n+1}\|_{L^2}^2 + \|\mathbf{u}^n\|_{H^2}^2\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + \|\eta^n\|_{H^2}^2 + \|\eta^{n+1}\|_{H^2}^2\right)\\
&\quad + C\tau^2\left(\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + \|\nabla\mathbf{e}^n\|_{L^2}^2\right) + C\tau^2\|\nabla\eta^{n+1}\|_{L^3}^2\|\mathbf{e}^{n+1}\|_{H^2}^2.
\end{aligned} \tag{3.30}
$$

Summing up (3.30) for n from 0 to m and using (3.9) and (3.19), we obtain

$$
\begin{aligned}
&\mu\tau\|\nabla\mathbf{e}^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + \mu\tau\sum_{n=0}^{m}\|\nabla(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2\\
&\le C\tau^3 + C\tau^2\sum_{n=0}^{m}\|\mathbf{u}^n\|_{H^2}^2\|\nabla\mathbf{e}^{n+1}\|_{L^2}^2 + C\tau^2\sum_{n=0}^{m}\|\nabla\eta^{n+1}\|_{L^3}^2\|\mathbf{e}^{n+1}\|_{H^2}^2.
\end{aligned}
$$

By (3.26) and the discrete Gronwall's inequality in Lemma 2.2, we get

$$
\begin{aligned}
&\tau\|\nabla\mathbf{e}^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + \tau\sum_{n=0}^{m}\|\nabla(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2\\
&\le C\tau^3 + C\tau^3\sum_{n=0}^{m}\|\mathbf{e}^{n+1}\|_{H^2}^2
\end{aligned} \tag{3.31}
$$

by using $\|\nabla\eta^{n+1}\|_{L^3}^2 \le C\|\eta^{n+1}\|_{H^2}^2 \le C\tau$. On the other hand, from (3.27) and the regularity assumption (**A4**) of the solution to the Stokes problem, we have

$$
\begin{aligned}
&\|\mathbf{e}^{n+1}\|_{H^2}^2 + \|\epsilon^{n+1}\|_{H^1}^2\\
&\le C\|\rho^n D_\tau\mathbf{e}^{n+1}\|_{L^2}^2 + C\|R_u^{n+1}\|_{L^2}^2 + C\|\nabla\rho^{n+1}\cdot(\nabla\mathbf{e}^{n+1})^t\|_{L^2}^2 + C\sum_{i=1}^{6}\|I_i^{n+1}\|_{L^2}^2\\
&\le C\tau^{-2}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2 + C\|R_u^{n+1}\|_{L^2}^2 + C\|\nabla\mathbf{e}^{n+1}\|_{L^2}\|\mathbf{e}^{n+1}\|_{H^2}\\
&\quad + C\|\nabla\mathbf{e}^n\|_{L^2}^2 + C\|\eta^{n+1}\|_{H^2}^2 + C\|\eta^n\|_{H^2}^2\\
&\le \frac{1}{2}\|\mathbf{e}^{n+1}\|_{H^2}^2 + C\tau^{-2}\|\sigma^n(\mathbf{e}^{n+1}-\mathbf{e}^n)\|_{L^2}^2\\
&\quad + C\left(\|R_u^{n+1}\|_{L^2}^2 + \|\nabla\mathbf{e}^n\|_{L^2}^2 + \|\eta^{n+1}\|_{H^2}^2 + \|\eta^n\|_{H^2}^2\right).
\end{aligned} \tag{3.32}
$$

Summing up (3.32) for n from 0 to m and using (3.19) and (3.31), we obtain

$$\tau \sum_{n=0}^{m} \left(\|\mathbf{e}^{n+1}\|_{H^2}^2 + \|\epsilon^{n+1}\|_{H^1}^2\right) \le C\tau^2 + C\tau^{-1} \sum_{n=0}^{m} \|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2$$
$$\le C\tau^2 + C\tau^2 \sum_{n=0}^{m} \|\mathbf{e}^{n+1}\|_{H^2}^2. \tag{3.33}$$

Taking a sufficiently small $\tau_2 < \tau_1$ such that $C\tau_2 < 1$, we derive

$$\tau \sum_{n=0}^{m} \left(\|\mathbf{e}^{n+1}\|_{H^2}^2 + \|\epsilon^{n+1}\|_{H^1}^2\right) \le C\tau^2,$$

which with (3.31) leads to

$$\tau \|\nabla \mathbf{e}^{m+1}\|_{L^2}^2 + \sum_{n=0}^{m} \|\sigma^n(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 + \tau \sum_{n=0}^{m} \|\nabla(\mathbf{e}^{n+1} - \mathbf{e}^n)\|_{L^2}^2 \le C\tau^3.$$

Thus, we complete the proof of Lemma 3.4. □

From (3.28), we can see that

$$\|\nabla \mathbf{u}^{n+1}\|_{L^2} \le C\tau + \|\nabla \mathbf{u}\|_{L^\infty(0,T;L^2)}$$
$$\le 1 + \|\nabla \mathbf{u}_0\|_{L^2} + \|\nabla \mathbf{u}\|_{L^\infty(0,T;L^2)} = \kappa_1, \quad \forall\, 0 \le n \le N-1$$

for some small $\tau > 0$. Thus, (3.14) holds and we close the mathematical induction.

The estimate (3.28) provides a uniform boundness of the time discrete solution $(\mathbf{u}^{n+1}, p^{n+1})$ in $l^\infty(\mathbf{H}^2) \times l^\infty(H^1)$-norm, which means that there exists some $C > 0$ such that

$$\|\mathbf{u}^{n+1}\|_{H^2} + \|p^{n+1}\|_{H^1} \le C, \qquad \forall\, 0 \le n \le N-1. \tag{3.34}$$

In addition, the estimates (3.19) and (3.28) imply that

$$\|\nabla(D_\tau \mathbf{u}^{n+1})\|_{L^2} + \|D_\tau \rho^{n+1}\|_{H^1} + \tau \sum_{i=0}^{n} \left(\|D_\tau \mathbf{u}^{n+1}\|_{H^2} + \|D_\tau \rho^{n+1}\|_{H^2}\right) \le C \tag{3.35}$$

for all $0 \le n \le N-1$, if we notice the regularity assumption (**A3**).

Next, we estimate the time discrete solutions $(\rho^{n+1}, \mathbf{u}^{n+1}, p^{n+1})$ in $H^3 \times \mathbf{W}^{2,4} \times W^{1,4}$-norm. We turn back to (2.7) and (3.21). In terms of (3.20) and (3.34), one has

$$\|\nabla \rho^{n+1} \cdot \mathbf{u}^n\|_{L^2} \le C\|\rho^{n+1}\|_{H^2}\|\nabla \mathbf{u}^n\|_{L^2} \le C,$$
$$\|\nabla(\nabla \rho^{n+1} \cdot \mathbf{u}^n)\|_{L^2} \le C\|\rho^{n+1}\|_{H^2}\|\mathbf{u}^n\|_{H^2} \le C$$

and

$$\|\rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{u}^{n+1}\|_{L^4} + \|(\nabla \rho^{n+1} \cdot \nabla)\mathbf{u}^{n+1}\|_{L^4} + \|\nabla \rho^{n+1} \cdot (\nabla \mathbf{u}^{n+1})^t\|_{L^4}$$
$$\le C\|\rho^{n+1}\|_{L^\infty}\|\mathbf{u}^n\|_{H^2}\|\nabla \mathbf{u}^{n+1}\|_{L^4} + C\|\nabla \rho^{n+1}\|_{L^\infty}\|\nabla \mathbf{u}^{n+1}\|_{L^4}$$
$$\le C.$$

From the regularity assumption (**A4**), we obtain

$$\|\rho^{n+1}\|_{H^3} + \|\mathbf{u}^{n+1}\|_{W^{2,4}} + \|p^{n+1}\|_{W^{1,4}} \le C, \quad \forall\, 0 \le n \le N-1. \tag{3.36}$$

3.2 Spatial Error Analysis

In this subsection, we will prove the optimal spatial error estimate for the velocity in $l^\infty(\mathbf{L}^2)$-norm and the density in $l^\infty(H^1)$-norm. The proof is based on the regularities of time discrete solutions derived in Sect. 3.1 and the following new projection operators.

For $1 \le n \le N$, we introduce three new projection operators $(\mathbf{R}_h^n, Q_h^n) : \mathbf{V} \times M \to \mathbf{V}_h \times M_h$ and $\Pi_h^n : W \to W_h$ defined by

$$a(\rho^n; \mathbf{R}_h^n\mathbf{u} - \mathbf{u}, \mathbf{v}_h) - (\nabla \cdot \mathbf{v}_h, Q_h^n p - p) = 0, \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h,$$
$$(\nabla \cdot (\mathbf{R}_h^n\mathbf{u} - \mathbf{u}), q_h) = 0, \quad \forall\, q_h \in M_h,$$

and

$$\lambda(\Pi_h^n\rho - \rho, r_h) + \lambda(\nabla(\Pi_h^n\rho - \rho), \nabla r_h) + (\nabla(\Pi_h^n\rho - \rho) \cdot \mathbf{u}^{n-1}, r_h) = 0, \quad \forall\, r_h \in W_h,$$

where ρ^n and $\mathbf{u}^{n-1}$ are the solutions to (2.7–2.8) and satisfy the point-wise inequality (3.16) and the regularity (3.34), respectively.

Then from the coercive property (2.2), and using a classical argument (cf.[5,14]), the following approximations hold:

$$\|\mathbf{u} - \mathbf{R}_h^n\mathbf{u}\|_{L^2} + h\|\nabla(\mathbf{u} - \mathbf{R}_h^n\mathbf{u})\|_{L^2} + h\|p - Q_h^n p\|_{L^2} \le Ch^2(\|\mathbf{u}\|_{H^2} + \|p\|_{H^1}), \tag{3.37}$$
$$\|\rho - \Pi_h^n\rho\|_{L^2} + h\|\rho - \Pi_h^n\rho\|_{H^1} \le Ch^2\|\rho\|_{H^2}, \tag{3.38}$$
$$\|\rho - \Pi_h^n\rho\|_{L^\infty} + \|\mathbf{u} - \mathbf{R}_h^n\mathbf{u}\|_{L^\infty} \le Ch^{1/2} \tag{3.39}$$

for any $(\rho, \mathbf{u}, p) \in H^2(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{V} \times H^1(\Omega)$. Furthermore, one has

$$\|\rho - \Pi_h^n\rho\|_{L^4} + \|\mathbf{u} - \mathbf{R}_h^n\mathbf{u}\|_{L^4} \le Ch^2(\|\rho\|_{W^{2,4}} + \|\mathbf{u}\|_{W^{2,4}} + \|p\|_{W^{1,4}}), \tag{3.40}$$
$$\|\rho - \Pi_h^n\rho\|_{W^{1,4}} + \|\mathbf{u} - \mathbf{R}_h^n\mathbf{u}\|_{W^{1,4}} \le Ch(\|\rho\|_{W^{2,4}} + \|\mathbf{u}\|_{W^{2,4}} + \|p\|_{W^{1,4}}) \tag{3.41}$$

if $(\rho, \mathbf{u}, p) \in W^{2,4}(\Omega) \cap \mathbf{W}^{2,4}(\Omega) \cap \mathbf{V} \times W^{1,4}(\Omega)$.

We denote by $\mathbf{P}_{1h}$ the standard Raviart-Thomas projection from $\mathbf{H}(\text{div}, \Omega)$ onto $\mathbf{RT}_h$, which satisfies the following properties (cf. [30]):

$$(\nabla \cdot \mathbf{P}_{1h}\mathbf{u}, v_h) = (\nabla \cdot \mathbf{u}, v_h), \qquad \forall\, v_h \in P_1(\mathcal{T}_h),$$
$$\|\mathbf{u} - \mathbf{P}_{1h}\mathbf{u}\|_{L^2} \le Ch^l\|\mathbf{u}\|_{H^l}, \qquad \forall\, \mathbf{u} \in \mathbf{H}^l(\Omega), l = 1, 2,$$

where $P_1(\mathcal{T}_h) \subset H^1(\Omega)$ is the finite element space of functions which are the piecewise linear polynomials on each $K \in \mathcal{T}_h$. For the time discrete solution $\mathbf{u}^n$, since $\nabla \cdot \mathbf{u}^n = 0$ in Ω and $\mathbf{u}^n \cdot \mathbf{n} = 0$ on $\partial\Omega$, then

$$\nabla \cdot \mathbf{P}_{1h}\mathbf{u}^n = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{P}_{1h}\mathbf{u}^n \cdot \mathbf{n} = 0 \text{ on } \partial\Omega,$$

which implies that $\mathbf{P}_{1h}\mathbf{u}^n \in \mathbf{RT}_{0h}$. By noticing the definition of the L^2-projection $\mathbf{P}_{0h}$, there holds that

$$\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}^n\|_{L^2} \le \|\mathbf{u}^n - \mathbf{P}_{1h}\mathbf{u}^n\|_{L^2} \le Ch^2. \tag{3.42}$$

Introduce spatial error functions by

$$\eta_h^0 = J_h\rho_0 - \rho_h^0 = 0, \quad \mathbf{e}_h^0 = I_h\mathbf{u}_0 - \mathbf{u}_h^0 = 0,$$
$$\eta_h^n = \Pi_h^n\rho^n - \rho_h^n, \quad \mathbf{e}_h^n = \mathbf{R}_h^n\mathbf{u}^n - \mathbf{u}_h^n, \quad \epsilon_h^n = Q_h^n p^n - p_h^n, \quad \forall\, 1 \le n \le N,$$

where $(\rho_h^n, \mathbf{u}_h^n, p_h^n)$ and $(\rho^n, \mathbf{u}^n, p^n)$ are numerical solutions to (2.17–2.18) and (2.7–2.8), respectively. Moreover, we denote projection error functions by

$$\theta^0 = J_h\rho_0 - \rho_0, \quad \mathbf{E}^0 = I_h\mathbf{u}_0 - \mathbf{u}_0,$$
$$\theta^n = \Pi_h^n\rho^n - \rho^n, \quad \mathbf{E}^n = \mathbf{R}_h^n\mathbf{u}^n - \mathbf{u}^n, \quad \xi^n = Q_h^n p^n - p^n, \quad \forall\, 1 \le n \le N.$$

From (3.37–3.41) and the regularities (3.20), (3.34) and (3.35), projection error functions satisfy

$$\|\mathbf{E}^n\|_{L^2} + h(\|\nabla\mathbf{E}^n\|_{L^2} + \|\xi^n\|_{L^2}) \le Ch^2, \tag{3.43}$$
$$\|\mathbf{E}^n\|_{L^\infty} + \|\theta^n\|_{L^\infty} \le Ch^{1/2}, \tag{3.44}$$
$$\|D_\tau\mathbf{E}^n\|_{L^2} + \|D_\tau\theta^n\|_{L^2} \le Ch^2(\|D_\tau\mathbf{u}^n\|_{H^2} + \|D_\tau\rho^n\|_{H^2}), \tag{3.45}$$
$$\|\mathbf{E}^n\|_{L^4} + h\|\mathbf{E}^n\|_{W^{1,4}} \le Ch^2, \tag{3.46}$$
$$\|\theta^n\|_{L^2} + h\|\theta^n\|_{H^1} \le Ch^3. \tag{3.47}$$

For $0 \le n \le N-1$, subtracting (2.17–2.18) from (2.9–2.10) with $(r, \mathbf{v}, q) = (r_h, \mathbf{v}_h, q_h)$ and noticing the definitions of projection operators $(\mathbf{R}_h^{n+1}, Q_h^{n+1})$ and Π_h^{n+1}, we get the following error equations satisfied by η_h^{n+1} and $(\mathbf{e}_h^{n+1}, \epsilon_h^{n+1})$, respectively,

$$\begin{aligned}
&(D_\tau\eta_h^{n+1}, r_h) + \lambda(\nabla\eta_h^{n+1}, \nabla r_h)\\
&\quad= (D_\tau\theta^{n+1}, r_h) - \lambda(\theta^{n+1}, r_h) - (\nabla\theta^{n+1}\cdot(\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n), r_h) - (\nabla\rho^{n+1}\cdot(\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n), r_h)\\
&\qquad+(\nabla\eta_h^{n+1}\cdot(\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n), r_h) - (\nabla\eta_h^{n+1}\cdot\mathbf{u}^n, r_h)\\
&\quad:= \sum_{i=1}^{6}(I_{ih}^{n+1}, r_h), \quad \forall\, r_h \in W_h,
\end{aligned} \tag{3.48}$$

and

$$\begin{aligned}
&(\rho_h^n D_\tau\mathbf{e}_h^{n+1}, \mathbf{v}_h) + \frac{1}{2}\left(D_\tau\rho_h^{n+1}, \mathbf{e}_h^{n+1}\cdot\mathbf{v}_h\right) + a(\rho_h^{n+1}; \mathbf{e}_h^{n+1}, \mathbf{v}_h)\\
&\quad-(\nabla\cdot\mathbf{v}_h, \epsilon_h^{n+1}) + (\nabla\cdot\mathbf{e}_h^{n+1}, q_h)\\
&\quad= (\rho_h^n D_\tau\mathbf{E}^{n+1}, \mathbf{v}_h) - \left((\rho^n - \rho_h^n)D_\tau\mathbf{u}^{n+1}, \mathbf{v}_h\right) + \frac{1}{2}\left(D_\tau\theta^{n+1}, \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h\right)\\
&\qquad+\lambda\left((\rho^{n+1} - \rho_h^{n+1}), (\nabla\nabla\mathbf{u}^{n+1})^t : \nabla\mathbf{v}_h\right) - \frac{1}{2}\left(\nabla\cdot(\rho^{n+1}\mathbf{u}^n - \rho_h^{n+1}\mathbf{u}_h^n), \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h\right)\\
&\qquad-\frac{1}{2}\left(\nabla\cdot(\rho_h^{n+1}\mathbf{u}_h^n), \mathbf{e}_h^{n+1}\cdot\mathbf{v}_h\right) - \frac{\lambda}{2}\left(\nabla(\eta_h^{n+1} - \theta^{n+1}), \nabla(\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h)\right)\\
&\qquad+\frac{\lambda}{2}\left(\nabla\rho_h^{n+1}, \nabla(\mathbf{e}_h^{n+1}\cdot\mathbf{v}_h)\right) - \lambda\left((\nabla\rho_h^{n+1}\cdot\nabla)\mathbf{e}_h^{n+1}, \mathbf{v}_h\right)\\
&\qquad-\lambda\left((\nabla(\eta_h^{n+1} - \theta^{n+1})\cdot\nabla)\mathbf{u}^{n+1}, \mathbf{v}_h\right) + \lambda\left((\nabla\rho^{n+1}\cdot\nabla)\mathbf{E}^{n+1}, \mathbf{v}_h\right)\\
&\qquad-\lambda\left((\nabla(\eta_h^{n+1} - \theta^{n+1})\cdot\nabla)\mathbf{E}^{n+1}, \mathbf{v}_h\right) + \left(\rho^{n+1}(\mathbf{u}^n\cdot\nabla)\mathbf{E}^{n+1}, \mathbf{v}_h\right)\\
&\qquad-\left(\rho_h^{n+1}(\mathbf{u}_h^n\cdot\nabla)\mathbf{e}_h^{n+1}, \mathbf{v}_h\right) - \left((\eta_h^{n+1} - \theta^{n+1})(\mathbf{u}^n\cdot\nabla)\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}, \mathbf{v}_h\right)\\
&\qquad-\left(\rho_h^{n+1}((\mathbf{u}^n - \mathbf{u}_h^n)\cdot\nabla)\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}, \mathbf{v}_h\right) - \frac{1}{2}\left(D_\tau\eta_h^{n+1}, \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h\right)\\
&\quad:= \sum_{i=1}^{17}(J_{ih}^{n+1}, \mathbf{v}_h), \quad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{V}_h \times M_h,
\end{aligned} \tag{3.49}$$

where we have noted $\nabla\cdot\mathbf{u}^n = 0$ in Ω and

$$(D_\tau\rho^{n+1}, \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h) + (\nabla\cdot(\rho^{n+1}\mathbf{u}^n), \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h) = -\lambda(\nabla\rho^{n+1}, \nabla(\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{v}_h))$$

by taking $r = \mathbf{R}_h^{n+1}\mathbf{u}^{n+1} \cdot \mathbf{v}_h \in W_h$ in (2.9).

We first estimate η_h^{n+1} and $\nabla\eta_h^{n+1}$ in $l^\infty(L^2)$-norm in the following two lemmas.

Lemma 3.5 *Under the assumptions (**A1–A4**), there exists some $\tau_3 < \tau_2$ such that when $\tau < \tau_3$, there holds*

$$\|\eta_h^{m+1}\|_{L^2}^2 + \tau\sum_{n=0}^{m}\|\eta_h^{n+1}\|_{H^1}^2 \leq Ch^4 + C\tau\sum_{n=0}^{m}\|\boldsymbol{u}^n - \boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^2}^2 + Ch^2\tau\sum_{n=0}^{m}\|\boldsymbol{u}^n - \boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^3}^2 \tag{3.50}$$

for all $0 \leq m \leq N-1$.

Proof Taking $r_h = 2\tau\eta_h^{n+1}$ in (3.48) leads to

$$\|\eta_h^{n+1}\|_{L^2}^2 - \|\eta_h^n\|_{L^2}^2 + \|\eta_h^{n+1} - \eta_h^n\|_{L^2}^2 + 2\lambda\tau\|\nabla\eta_h^{n+1}\|_{L^2}^2 = 2\tau\sum_{i=1}^{4}(I_{ih}^{n+1}, \eta_h^{n+1}) \tag{3.51}$$

by noticing

$$\begin{aligned}(I_{5h}^{n+1}, \eta_h^{n+1}) &= \frac{1}{2}\int_\Omega \nabla|\eta_h^{n+1}|^2 \cdot (\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n)dx \\ &= -\frac{1}{2}\int_\Omega |\eta_h^{n+1}|^2\nabla\cdot(\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n)dx = 0\end{aligned}$$

and

$$(I_{6h}^{n+1}, \eta_h^{n+1}) = \frac{1}{2}\int_\Omega \nabla|\eta_h^{n+1}|^2 \cdot \mathbf{u}^n dx = -\frac{1}{2}\int_\Omega |\eta_h^{n+1}|^2\nabla\cdot\mathbf{u}^n dx = 0.$$

The right-hand side of (3.51) can be bound by using the Hölder inequality and the Young inequality. It follows from (3.43–3.45) that

$$\begin{aligned}2\tau(I_{1h}^{n+1}, \eta_h^{n+1}) &\leq C\tau\|\eta_h^{n+1}\|_{L^2}^2 + C\tau h^4, \\ 2\tau(I_{2h}^{n+1}, \eta_h^{n+1}) &\leq C\tau\|\eta_h^{n+1}\|_{L^2}^2 + C\tau h^4.\end{aligned}$$

For I_3^{n+1} and I_4^{n+1}, we can prove that

$$\begin{aligned}2\tau(I_{3h}^{n+1}, \eta_h^{n+1}) &\leq C\tau\|\nabla\theta^n\|_{L^2}\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^3}\|\eta_h^{n+1}\|_{H^1} \\ &\leq \frac{\lambda\tau}{2}\|\eta_h^{n+1}\|_{H^1}^2 + C\tau h^2\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^3}^2, \\ 2\tau(I_{4h}^{n+1}, \eta_h^{n+1}) &\leq C\tau\|\rho^{n+1}\|_{H^2}\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}\|\eta_h^{n+1}\|_{H^1} \\ &\leq \frac{\lambda\tau}{2}\|\eta_h^{n+1}\|_{H^1}^2 + C\tau\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2.\end{aligned}$$

Substituting the above estimates into (3.51), we obtain

$$\begin{aligned}&\|\eta_h^{n+1}\|_{L^2}^2 - \|\eta_h^n\|_{L^2}^2 + \|\eta_h^{n+1} - \eta_h^n\|_{L^2}^2 + \lambda\tau\|\eta_h^{n+1}\|_{H^1}^2 \\ &\quad \leq C\tau h^4 + C\tau\|\eta_h^{n+1}\|_{L^2}^2 + C\tau\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2 + C\tau h^2\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^3}^2.\end{aligned} \tag{3.52}$$

Taking the sum from 0 to m and using the discrete Gronwall's inequality in Lemma 2.2, we get the desired result (3.50) for some small $\tau < \tau_3 < \tau_2$. □

Lemma 3.6 *Under the assumptions (***A1***)-(***A4***), if*

$$\|\boldsymbol{u}^n - \boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^2}^2 \le Ch^3, \quad \forall\, 0 \le n \le N-1, \tag{3.53}$$

then there exists some $C > 0$ *such that*

$$\tau \sum_{n=0}^{m} \|D_\tau \eta_h^{n+1}\|_{L^2}^2 + \lambda \|\nabla \eta_h^{m+1}\|_{L^2}^2 \le Ch^4 + C\tau \sum_{n=0}^{m} \|\boldsymbol{u}^n - \boldsymbol{P}_{0h}\boldsymbol{u}_h^n\|_{L^2}^2 \tag{3.54}$$

for all $0 \le m \le N-1$.

Proof Taking $r_h = 2D_\tau \eta_h^{n+1}$ in (3.48) leads to

$$2\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + \lambda D_\tau \|\nabla \eta_h^{n+1}\|_{L^2}^2 + \lambda\tau \|\nabla(D_\tau \eta_h^{n+1})\|_{L^2}^2 = 2\sum_{i=1}^{6} (I_{ih}^{n+1}, D_\tau \eta_h^{n+1}). \tag{3.55}$$

We estimate the right-hand side of (3.55) term by term according to the regularities derived in (3.20), (3.34), (3.36) and (3.36). From the Hölder inequality and the Young inequality, we have

$$\begin{aligned}
2(I_{1h}^{n+1}, D_\tau \eta_h^{n+1}) &\le \frac{1}{6}\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + Ch^4 \|D_\tau \rho^{n+1}\|_{H^2}^2,\\
2(I_{2h}^{n+1}, D_\tau \eta_h^{n+1}) &\le \frac{1}{6}\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + Ch^4 \|\rho^{n+1}\|_{H^2}^2,\\
2(I_{3h}^{n+1}, D_\tau \eta_h^{n+1}) &\le C\|\nabla \theta^{n+1}\|_{L^3} \|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2} \|D_\tau \eta_h^{n+1}\|_{L^6}\\
&\le C\|D_\tau \eta_h^{n+1}\|_{L^2} \|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}\\
&\le \frac{1}{6}\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + C\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2,
\end{aligned}$$

where the inverse inequalities (2.23) is used, and

$$\begin{aligned}
2(I_{4h}^{n+1}, D_\tau \eta_h^{n+1}) &\le \|\nabla \rho^{n+1}\|_{L^\infty} \|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2} \|D_\tau \eta_h^{n+1}\|_{L^2}\\
&\le \frac{1}{6}\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + C\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2,
\end{aligned}$$

and

$$\begin{aligned}
2(I_{5h}^{n+1}, D_\tau \eta_h^{n+1}) &\le \|\nabla \eta_h^{n+1}\|_{L^\infty} \|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2} \|D_\tau \eta_h^{n+1}\|_{L^2}\\
&\le \frac{1}{6}\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + Ch^{-3}\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2 \|\nabla \eta_h^{n+1}\|_{L^2}^2,
\end{aligned}$$

where the inverse inequalities (2.23) is used, and

$$2(I_{6h}^{n+1}, D_\tau \eta_h^{n+1}) \le \frac{1}{6}\|D_\tau \eta_h^{n+1}\|_{L^2}^2 + C\|\nabla \eta_h^{n+1}\|_{L^2}^2.$$

Substituting the above estimates into (3.55) and taking the sum from 0 to m, we get

$$\begin{aligned}
&\tau \sum_{n=0}^{m} \|D_\tau \eta_h^{n+1}\|_{L^2}^2 + \lambda \|\nabla \eta_h^{m+1}\|_{L^2}^2\\
&\le Ch^4 + C\tau \sum_{n=0}^{m} \|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2 + C\tau \sum_{n=0}^{m} (1 + h^{-3}\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2)\|\nabla \eta_h^{n+1}\|_{L^2}^2.
\end{aligned}$$

By the condition (3.53) and using the discrete Gronwall inequality in Lemma 2.2, we get the desired result (3.54). □

Next lemma presents the estimate of $\mathbf{e}_h^{n+1}$ in $l^\infty(\mathbf{L}^2)$-norm and $l^2(\mathbf{V})$-norm.

Lemma 3.7 *Under the assumptions (**A1**–**A4**), there exists sufficiently small $h_4 > 0$ and $\tau_4 < \tau_3$ such that when $h < h_4$ and $\tau < \tau_4$, the finite element scheme (2.18) admits a unique solution $(\boldsymbol{u}_h^{n+1}, p_h^{n+1}) \in \boldsymbol{V}_h \times M_h$. Moreover, there holds*

$$\|\boldsymbol{e}_h^{m+1}\|_{L^2}^2 + \tau \sum_{n=0}^{m} \|\nabla \boldsymbol{e}_h^{n+1}\|_{L^2}^2 \le C_0^2 h^4, \qquad \forall\, 0 \le m \le N-1, \tag{3.56}$$

where $C_0 > 0$ is independent of τ, h and m.

Proof We will prove (3.56) by the method of mathematical induction.

• **Initialization** $(m = 0)$

We first prove that (3.56) is valid for $m = 0$. Taking $m = 0$ in (3.50) and (3.54), we can get

$$\tau \|D_\tau \eta_h^1\|_{L^2}^2 + \|\eta_h^1\|_{L^2}^2 \le Ch^4 \tag{3.57}$$

by using

$$\begin{aligned}
\|\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^2}^2 &\le 2\|\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}^0\|_{L^2}^2 + 2\|\mathbf{P}_{0h}\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^2}^2 \\
&\le 2\|\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}^0\|_{L^2}^2 + 2\|\mathbf{u}^0 - \mathbf{u}_h^0\|_{L^2}^2 \\
&\le Ch^4, \\
h^2\|\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^3}^2 &\le 2h^2\|\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}^0\|_{L^3}^2 + 2h^2\|\mathbf{P}_{0h}\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^3}^2 \\
&\le Ch^4 + Ch\|\mathbf{P}_{0h}\mathbf{u}^0 - \mathbf{P}_{0h}\mathbf{u}_h^0\|_{L^2}^2 \\
&\le Ch^4.
\end{aligned}$$

Furthermore, we get from the inverse inequality (2.23) and (3.57) that

$$\|\eta_h^1\|_{L^\infty} \le Ch^{1/2} \quad \text{and} \quad \|\rho^1 - \rho_h^1\|_{L^\infty} \le Ch^{1/2}. \tag{3.58}$$

Then there exists some sufficiently small h_4 such that when $h < h_4$, one has

$$\widetilde{m} < m - Ch^{1/2} < \|\rho_h^1\|_{L^\infty} < M + Ch^{1/2} < \widetilde{M}, \tag{3.59}$$

which with (2.2) implies that the numerical scheme (2.18) with $n = 0$ admits a unique solution $(\mathbf{u}_h^1, p_h^1) \in \mathbf{V}_h \times M_h$. Taking $n = 0$ and $(\mathbf{v}_h, q_h) = 2\tau(\mathbf{e}_h^1, \epsilon_h^1)$ in (3.49) and using (2.2) and $\mathbf{e}_h^0 = 0$, we get

$$\|\sigma_h^0 \mathbf{e}_h^1\|_{L^2}^2 + \|\sigma_h^1 \mathbf{e}_h^1\|_{L^2}^2 + 2\mu_1\tau\|\nabla \mathbf{e}_h^1\|_{L^2}^2 \le 2\tau \sum_{i=1}^{17} (J_{ih}^1, \mathbf{e}_h^1). \tag{3.60}$$

Due to $\eta_h^0 = 0$, then from (3.43–3.45) and (3.35), one has

$$2\tau \sum_{i=1}^{3} (J_{ih}^1, \mathbf{e}_h^1) \le \frac{\mu_1\tau}{11}\|\nabla \mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4. \tag{3.61}$$

By (3.57) and (3.58), we estimate $(J_{4h}^1, \mathbf{e}_h^1)$ by

$$\begin{aligned}
2\tau(J_{4h}^1, \mathbf{e}_h^1) &\le C\tau\|\rho^1 - \rho_h^1\|_{L^\infty}\|\nabla \mathbf{e}_h^1\|_{L^2}^2 + C\tau\|\nabla \mathbf{R}_h^1 \mathbf{u}^1\|_{L^3}\|\rho^1 - \rho_h^1\|_{L^2}\|\nabla \mathbf{e}_h^1\|_{L^2} \\
&\le \frac{\mu_1\tau}{11}\|\nabla \mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4
\end{aligned}$$

for sufficiently small $h < h_4$. Using the integration by parts, we estimate $(J_{5h}^1, \mathbf{e}_h^1)$ by

$$\begin{aligned}2\tau(J_{5h}^1, \mathbf{e}_h^1) &= \tau\left(((\rho^1 - \rho_h^1)\mathbf{u}_0 + \rho_h^1(\mathbf{u}_0 - \mathbf{u}_h^0)) \cdot \nabla(\mathbf{R}_h^1\mathbf{u}^1), \mathbf{e}_h^1\right)\\ &\quad + \tau\left(((\rho^1 - \rho_h^1)\mathbf{u}_0 + \rho_h^1(\mathbf{u}_0 - \mathbf{u}_h^0)) \cdot (\mathbf{R}_h^1\mathbf{u}^1), \nabla\mathbf{e}_h^1\right)\\ &\le C\tau\left(\|\rho^1 - \rho_h^1\|_{L^2} + \|\mathbf{u}_0 - \mathbf{u}_h^0\|_{L^2}\right)\|\nabla\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{\mu_1\tau}{11}\|\nabla\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4,\\ 2\tau(J_{6h}^1, \mathbf{e}_h^1) &= -2\tau(J_{14h}^1, \mathbf{e}_h^1).\end{aligned}$$

Similarly, we can prove

$$\begin{aligned}2\tau(J_{10h}^1, \mathbf{e}_h^1) + 2\tau(J_{12h}^1, \mathbf{e}_h^1) &\le C\tau\|\rho^1 - \rho_h^1\|_{L^2}\|\nabla\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{\mu_1\tau}{11}\|\nabla\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4,\\ 2\tau(J_{8h}^1, \mathbf{e}_h^1) &= -2\tau(J_{9h}^1, \mathbf{e}_h^1).\end{aligned}$$

From the definition of Π_h^1, one has

$$\begin{aligned}2\tau(J_{7h}^1, \mathbf{e}_h^1) &= -\lambda\tau(\nabla\eta_h^1, \nabla(\mathbf{R}_h^1\mathbf{u}^1 \cdot \mathbf{e}_h^1)) - \tau(\nabla\theta^1 \cdot \mathbf{u}_0, \mathbf{R}_h^1\mathbf{u}^1 \cdot \mathbf{e}_h^1) - \lambda\tau(\theta^1, \mathbf{R}_h^1\mathbf{u}^1 \cdot \mathbf{e}_h^1)\\ &\le C\tau(\|\nabla\eta_h^1\|_{L^2} + \|\theta^1\|_{L^2})\|\nabla\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{\mu_1\tau}{11}\|\nabla\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4,\end{aligned}$$

where (3.57) is used. It is easy to see that

$$\begin{aligned}2\tau(J_{11h}^1, \mathbf{e}_h^1) + 2\tau(J_{13h}^1, \mathbf{e}_h^1) &\le C\tau\|\mathbf{E}^1\|_{L^2}\|\nabla\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{\mu_1\tau}{11}\|\nabla\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4,\end{aligned}$$

and

$$\begin{aligned}2\tau(J_{15h}^1, \mathbf{e}_h^1) &\le 2\tau\|\mathbf{u}^0\|_{L^\infty}\|\nabla\mathbf{R}_h^1\mathbf{u}^1\|_{L^3}\|\rho^1 - \rho_h^1\|_{L^2}\|\nabla\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{\mu_1\tau}{11}\|\nabla\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4,\\ 2\tau(J_{16h}^1, \mathbf{e}_h^1) &\le 2\tau\|\rho_h^1\|_{L^\infty}\|\nabla\mathbf{R}_h^1\mathbf{u}^1\|_{L^3}\|\mathbf{E}^0\|_{L^2}\|\nabla\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{\mu_1\tau}{11}\|\nabla\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4.\end{aligned}$$

For J_{17h}, we get from (3.57) that

$$\begin{aligned}2\tau(J_{17h}^1, \mathbf{e}_h^1) &= -\lambda\tau(D_\tau\eta_h^1, \mathbf{R}_h^1\mathbf{u}^1 \cdot \mathbf{e}_h^1)\\ &\le C\tau\|D_\tau\eta_h^1\|_{L^2}\|\mathbf{e}_h^1\|_{L^2}\\ &\le \frac{1}{4}\|\sigma_h^1\mathbf{e}_h^1\|_{L^2}^2 + C\tau h^4.\end{aligned}$$

Taking into account these estimates with (3.60), there exits some $C_1 > 0$ independent of C_0, h and τ such that

$$\|\mathbf{e}_h^1\|_{L^2}^2 + \tau\|\nabla\mathbf{e}_h^1\|_{L^2}^2 \le C_1^2\tau h^4.$$

Thus, (3.56) is valid for $m = 0$ by taking $C_0 \ge C_1$.

• **General step** $(m \ge 1)$

For $0 \le n \le N-1$, we assume that (3.56) is valid for $m = n$, i.e.,

$$\|\mathbf{e}_h^n\|_{L^2}^2 + \tau \sum_{i=1}^{n} \|\nabla \mathbf{e}_h^i\|_{L^2}^2 \le C_0^2 h^4. \tag{3.62}$$

Then

$$\begin{aligned}\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2 &\le 2\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}^n\|_{L^2}^2 + 2\|\mathbf{P}_{0h}(\mathbf{u}^n - \mathbf{u}_h^n)\|_{L^2}^2 \\ &\le Ch^4 + C(\|\mathbf{e}_h^n\|_{L^2}^2 + \|\mathbf{E}^n\|_{L^2}^2) \\ &\le Ch^4 + C\|\mathbf{e}_h^n\|_{L^2}^2 \\ &\le C(1+C_0)^2 h^4. \end{aligned} \tag{3.63}$$

Thus, the condition (3.53) is valid and the estimate (3.54) holds in Lemma 3.6.

By the inverse inequality (2.23), we have

$$\|\mathbf{e}_h^n\|_{L^\infty} \le Ch^{-3/2}\|\mathbf{e}_h^n\|_{L^2} \le CC_0 h^{1/2},$$

which implies that

$$\begin{aligned}\|\mathbf{u}_h^n\|_{L^\infty} &\le \|\mathbf{u}^n\|_{L^\infty} + \|\mathbf{E}^n\|_{L^\infty} + \|\mathbf{e}_h^n\|_{L^\infty} \\ &\le C\|\mathbf{u}^n\|_{H^2} + C(1+C_0)h^{1/2} \\ &\le C \end{aligned} \tag{3.64}$$

for sufficiently small $h < h_4$ such that $(1 + C_0)h_4^{1/2} \le 1$. From (3.50) in Lemma 3.5 and (3.63), one has

$$\|\eta_h^{n+1}\|_{L^2}^2 + \tau \sum_{i=0}^{n} \|\eta_h^{i+1}\|_{H^1}^2 \le C(1+C_0)^2 h^4, \tag{3.65}$$

where we use the inverse inequality (2.23), (3.36) and

$$\begin{aligned} h^2\|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^3}^2 &\le Ch^2\|\mathbf{E}^n\|_{L^3}^2 + Ch\|\mathbf{R}_h^n\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2 \\ &\le Ch^4 + Ch(\|\mathbf{E}^n\|_{L^2}^2 + \|\mathbf{u}^n - \mathbf{P}_{0h}\mathbf{u}_h^n\|_{L^2}^2) \\ &\le Ch^4 \end{aligned} \tag{3.66}$$

for sufficiently small $h < h_4$. Then the finite element solution ρ_h^{n+1} satisfies

$$\begin{aligned}\|\rho^{n+1} - \rho_h^{n+1}\|_{L^\infty} &\le \|\theta^{n+1}\|_{L^\infty} + Ch^{-3/2}\|\eta_h^{n+1}\|_{L^2} \\ &\le C(1+C_0)h^{1/2}, \end{aligned} \tag{3.67}$$

which with (3.16) implies that

$$\widetilde{m} < m - C(1+C_0)h^{1/2} < \rho_h^{n+1} < M + C(1+C_0)h^{1/2} < \widetilde{M}, \quad \forall\, 0 \le n \le N-1 \tag{3.68}$$

for sufficiently small $h < h_4$ such that

$$C(1+C_0)h_4^{1/2} < \max\{m - \widetilde{m},\ \widetilde{M} - M\}.$$

According to (2.2), the fully discrete scheme (2.18) admits a unique solution $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in \mathbf{V}_h \times M_h$.

To close the mathematical induction, we need to prove that (3.56) is valid for $m = n + 1$. Setting $(\mathbf{v}_h, q_h) = 2\tau(\mathbf{e}_h^{n+1}, \epsilon_h^{n+1})$ in (3.49), we get

$$\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2 - \|\sigma_h^n\mathbf{e}_h^n\|_{L^2}^2 + 2\mu_1\tau\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 \le 2\tau\sum_{i=1}^{17}(J_{ih}^{n+1}, \mathbf{e}_h^{n+1}). \tag{3.69}$$

Using the Hölder inequality, the Young inequality, the regularity results (3.34–3.36) derived in Sect. 3 and the projection approximations (3.43–3.47) and the induction assumptions (3.62–3.68), we estimate the right-hand side of (3.69) term by term as follows:

- Estimate of $2\tau(J_{1h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{1h}^{n+1}, \mathbf{e}_h^{n+1}) &\le 2\tau\|\rho_h^n\|_{L^\infty}\|D_\tau\mathbf{E}^{n+1}\|_{L^2}\|\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le C\tau\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau h^4\|D_\tau\mathbf{u}^{n+1}\|_{H^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{2h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{2h}^{n+1}, \mathbf{e}_h^{n+1}) &\le 2\tau\left(\|\eta_h^n\|_{L^2} + \|\theta^n\|_{L^2}\right)\|D_\tau\mathbf{u}^{n+1}\|_{L^3}\|\mathbf{e}_h^{n+1}\|_{L^6} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\eta_h^n\|_{L^2}^2 + C\tau h^4. \end{aligned}$$

- Estimate of $2\tau(J_{3h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{3h}^{n+1}, \mathbf{e}_h^{n+1}) &\le \tau\|D_\tau\theta^{n+1}\|_{L^2}\|\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\|_{L^3}\|\mathbf{e}_h^{n+1}\|_{L^6} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau h^4\|D_\tau\rho^{n+1}\|_{H^2}^2. \end{aligned}$$

- Estimate of $2\tau(J_{4h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{4h}^{n+1}, \mathbf{e}_h^{n+1}) &\le 2\lambda\tau\left(\|\eta_h^{n+1}\|_{L^\infty} + \|\theta^{n+1}\|_{L^\infty}\right)\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 \\ &\quad + 2\lambda\tau\left(\|\eta_h^{n+1}\|_{L^6} + \|\theta^{n+1}\|_{L^6}\right)\|\nabla\mathbf{E}^{n+1}\|_{L^3}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\quad + 2\lambda\tau\left(\|\eta_h^{n+1}\|_{L^2} + \|\theta^{n+1}\|_{L^2}\right)\|\nabla\mathbf{u}^{n+1}\|_{L^\infty}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \left(C(1+C_0)h^{1/2} + \frac{\mu_1}{32}\right)\tau\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau h^4 \\ &\quad + C\tau h^2\|\eta_h^{n+1}\|_{H^1}^2 + C\tau\|\eta_h^{n+1}\|_{L^2}^2 \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\eta_h^{n+1}\|_{L^2}^2 + C\tau h^4 + C\tau h^2\|\eta_h^{n+1}\|_{H^1}^2 \end{aligned}$$

for sufficiently small $h < h_4$ such that $C(1+C_0)h_4^{1/2} < \mu_1/32$.

- Estimate of $2\tau(J_{5h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{5h}^{n+1}, \mathbf{e}_h^{n+1}) &= \tau\left(\rho^{n+1}(\mathbf{u}^n - \mathbf{u}_h^n) + (\rho^{n+1} - \rho_h^{n+1})\mathbf{u}_h^n, \nabla\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\mathbf{e}_h^{n+1}\right) \\ &\quad + \tau\left(\rho^{n+1}(\mathbf{u}^n - \mathbf{u}_h^n) + (\rho^{n+1} - \rho_h^{n+1})\mathbf{u}_h^n, \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\cdot\nabla\mathbf{e}_h^{n+1}\right) \\ &\le C\tau\left(\|\mathbf{e}_h^n\|_{L^2} + \|\mathbf{E}^n\|_{L^2} + \|\eta_h^{n+1}\|_{L^2} + \|\theta^{n+1}\|_{L^2}\right)\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\left(\|\sigma_h^n\mathbf{e}_h^n\|_{L^2}^2 + \|\eta_h^{n+1}\|_{L^2}^2 + h^4\right), \end{aligned}$$

where the integration by parts is used.

- Relation of $2\tau(J_{6h}^{n+1}, \mathbf{e}_h^{n+1})$ and $2\tau(J_{14h}^{n+1}, \mathbf{e}_h^{n+1})$

$$2\tau(J_{6h}^{n+1}, \mathbf{e}_h^{n+1}) = 2\tau(\rho_h^{n+1}\mathbf{u}_h^n, \nabla\mathbf{e}_h^{n+1} \cdot \mathbf{e}_h^{n+1}) = -2\tau(J_{14h}^{n+1}, \mathbf{e}_h^{n+1}),$$

where the integration by parts is used.

- Estimate of $2\tau(J_{7h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{7h}^{n+1}, \mathbf{e}_h^{n+1}) &= -\lambda\tau\left(\nabla\eta_h^{n+1}, \nabla(\mathbf{R}_h^{n+1}\mathbf{u}^{n+1} \cdot \mathbf{e}_h^{n+1})\right) \\ &\quad -\lambda\tau\left(\theta^{n+1}, \mathbf{R}_h^{n+1}\mathbf{u}^{n+1} \cdot \mathbf{e}_h^{n+1}\right) \\ &\quad -\tau\left(\nabla\theta^{n+1} \cdot \mathbf{u}^n, \mathbf{R}_h^{n+1}\mathbf{u}^{n+1} \cdot \mathbf{e}_h^{n+1}\right) \\ &\le C\tau\|\eta_h^{n+1}\|_{H^1}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} + C\tau\|\theta^{n+1}\|_{L^2}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\eta_h^{n+1}\|_{H^1}^2 + C\tau h^4, \end{aligned}$$

where the definition of Π_h^{n+1} is used.

- Relation of $2\tau(J_{8h}^{n+1}, \mathbf{e}_h^{n+1})$ and $2\tau(J_{9h}^{n+1}, \mathbf{e}_h^{n+1})$

$$2\tau(J_{8h}^{n+1}, \mathbf{e}_h^{n+1}) = -2\tau(J_{9h}^{n+1}, \mathbf{e}_h^{n+1})$$

by using the integration by parts.

- Estimate of $2\tau(J_{10h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{10h}^{n+1}, \mathbf{e}_h^{n+1}) &= 2\lambda\tau((\eta_h^{n+1} - \theta^{n+1})\Delta\mathbf{u}^{n+1}, \mathbf{e}_h^{n+1}) \\ &\quad +2\lambda\tau((\eta_h^{n+1} - \theta^{n+1})\nabla\mathbf{u}^{n+1}, \nabla\mathbf{e}_h^{n+1}) \\ &\le C\tau\left(\|\eta_h^{n+1}\|_{L^2} + \|\theta^{n+1}\|_{L^2}\right)\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\eta_h^{n+1}\|_{L^2}^2 + C\tau h^4 \end{aligned}$$

by using the integration by parts.

- Estimate of $2\tau(J_{11h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{11h}^{n+1}, \mathbf{e}_h^{n+1}) &= -2\lambda\tau(\Delta\rho^{n+1}\mathbf{E}^{n+1}, \mathbf{e}_h^{n+1}) - 2\lambda\tau(\nabla\rho^{n+1}, \mathbf{E}^{n+1} \cdot \nabla\mathbf{e}_h^{n+1}) \\ &\le C\tau\|\rho^{n+1}\|_{H^3}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau h^4 \end{aligned}$$

by using the integration by parts.

- Estimate of $2\tau(J_{12h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{12h}^{n+1}, \mathbf{e}_h^{n+1}) &\le C\tau\left(\|\nabla\eta_h^{n+1}\|_{L^2} + \|\nabla\theta^{n+1}\|_{L^2}\right)\|\nabla\mathbf{E}^{n+1}\|_{L^3}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\nabla\eta_h^{n+1}\|_{L^2}^2 + C\tau h^4. \end{aligned}$$

- Estimate of $2\tau(J_{13h}^{n+1}, \mathbf{e}_h^{n+1})$

$$\begin{aligned} 2\tau(J_{13h}^{n+1}, \mathbf{e}_h^{n+1}) &= -2\tau\left(\nabla\rho^{n+1} \cdot \mathbf{u}^n, \mathbf{E}^{n+1} \cdot \mathbf{e}_h^{n+1}\right) - 2\tau\left(\rho^{n+1}(\mathbf{u}^n \cdot \nabla)\mathbf{e}_h^{n+1}, \mathbf{E}^{n+1}\right) \\ &\le C\tau\|\mathbf{u}^n\|_{H^2}\|\rho^{n+1}\|_{H^2}\|\mathbf{E}^{n+1}\|_{L^2}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2} \\ &\le \frac{\mu_1\tau}{16}\|\nabla\mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau h^4 \end{aligned}$$

by using the integration by parts and $\nabla \cdot \mathbf{u}^n = 0$ in Ω.

• Estimate of $2\tau(J_{15h}^{n+1}, \mathbf{e}_h^{n+1})$

$$2\tau(J_{15h}^{n+1}, \mathbf{e}_h^{n+1}) \leq C\tau \left(\|\eta_h^{n+1}\|_{L^2} + \|\theta^{n+1}\|_{L^2}\right) \|\mathbf{u}^n\|_{L^\infty} \|\nabla \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\|_{L^3} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}$$
$$\leq \frac{\mu_1\tau}{16}\|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\eta_h^{n+1}\|_{L^2}^2 + C\tau h^4.$$

• Estimate of $2\tau(J_{16h}^{n+1}, \mathbf{e}_h^{n+1})$

$$2\tau(J_{16h}^{n+1}, \mathbf{e}_h^{n+1}) \leq C\tau \left(\|\mathbf{e}_h^n\|_{L^2} + \|\mathbf{E}^n\|_{L^2}\right) \|\rho_h^{n+1}\|_{L^\infty} \|\nabla \mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\|_{L^3} \|\nabla \mathbf{e}_h^{n+1}\|_{L^2}$$
$$\leq \frac{\mu_1\tau}{16}\|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|\sigma_h^n \mathbf{e}_h^n\|_{L^2}^2 + C\tau h^4.$$

• Estimate of $2\tau(J_{17h}^{n+1}, \mathbf{e}_h^{n+1})$

$$2\tau(J_{17h}^{n+1}, \mathbf{e}_h^{n+1}) \leq C\tau\|D_\tau\eta_h^{n+1}\|_{L^2}\|\mathbf{R}_h^{n+1}\mathbf{u}^{n+1}\|_{L^3}\|\mathbf{e}_h^{n+1}\|_{L^6}$$
$$\leq \frac{\mu_1\tau}{16}\|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2 + C\tau\|D_\tau\eta_h^{n+1}\|_{L^2}^2.$$

Substituting these estimates for J_{1h}^{n+1} to J_{17h}^{n+1} into (3.69), we get

$$\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2 - \|\sigma_h^n \mathbf{e}_h^n\|_{L^2}^2 + \mu_1\tau\|\nabla \mathbf{e}_h^{n+1}\|_{L^2}^2$$
$$\leq C\tau h^4 + C\tau(\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2 + \|\sigma_h^n\mathbf{e}_h^n\|_{L^2}^2) + C\tau h^4(\|D_\tau \mathbf{u}^{n+1}\|_{H^2}^2 + \|D_\tau\rho^{n+1}\|_{H^2}^2)$$
$$+C\tau(\|\eta_h^n\|_{L^2}^2 + \|\eta_h^{n+1}\|_{L^2}^2 + \|\nabla\eta_h^{n+1}\|_{L^2}^2 + \|D_\tau\eta_h^{n+1}\|_{L^2}^2).$$

Taking the sum gives

$$\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2 + \mu_1\tau\sum_{i=0}^n \|\nabla \mathbf{e}_h^{i+1}\|_{L^2}^2$$
$$\leq Ch^4 + C\tau\sum_{i=0}^n(\|\sigma_h^{i+1}\mathbf{e}_h^{i+1}\|_{L^2}^2 + \|D_\tau\eta_h^{i+1}\|_{L^2}^2 + \|\eta_h^{i+1}\|_{L^2}^2 + \|\nabla\eta_h^{i+1}\|_{L^2}^2)$$
$$\leq Ch^4 + C\tau\sum_{i=0}^n\|\sigma_h^{i+1}\mathbf{e}_h^{i+1}\|_{L^2}^2 + Ch^2\tau\sum_{i=0}^n\|\mathbf{u}^i - \mathbf{P}_{0h}\mathbf{u}_h^i\|_{L^3}^2$$
$$+C\tau\sum_{i=0}^n\|\mathbf{u}^i - \mathbf{P}_{0h}\mathbf{u}_h^i\|_{L^2}^2, \tag{3.70}$$

where we used error estimates derived in Lemmas 3.5 and 3.6. From (3.36) (3.63) and (3.66), we have

$$h^2\tau\sum_{i=0}^n\|\mathbf{u}^i - \mathbf{P}_{0h}\mathbf{u}_h^i\|_{L^3}^2 \leq Ch^4$$

and

$$C\tau\sum_{i=0}^n\|\mathbf{u}^i - \mathbf{P}_{0h}\mathbf{u}_h^i\|_{L^2}^2 \leq Ch^4 + C\tau\sum_{i=0}^n\|\mathbf{e}_h^i\|_{L^2}^2$$
$$\leq Ch^4 + C\tau\sum_{i=0}^n\|\sigma_h^i\mathbf{e}_h^i\|_{L^2}^2.$$

Then (3.70) reduces to

$$\begin{aligned}&\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2+\mu_1\tau\sum_{i=0}^{n}\|\nabla\mathbf{e}_h^{i+1}\|_{L^2}^2\\&\le Ch^4+C\tau\sum_{i=0}^{n}\|\sigma_h^{i+1}\mathbf{e}_h^{i+1}\|_{L^2}^2+C\tau\sum_{i=0}^{n}\|\sigma_h^{i}\mathbf{e}_h^{i}\|_{L^2}^2.\end{aligned}$$

Applying the discrete Gronwall's inequality in Lemma 2.2, we derive

$$\|\sigma_h^{n+1}\mathbf{e}_h^{n+1}\|_{L^2}^2+\mu_1\tau\sum_{i=0}^{n}\|\nabla\mathbf{e}_h^{i+1}\|_{L^2}^2\le C\exp(CT)h^4$$

and

$$\|\mathbf{e}_h^{n+1}\|_{L^2}^2+\mu_1\tau\sum_{i=0}^{n}\|\nabla\mathbf{e}_h^{i+1}\|_{L^2}^2\le C\exp(CT)h^4\le C_0^2h^4$$

by using (3.68) and taking $\sqrt{C\exp(CT)}\le C_0$. Thus, we prove that (3.56) is valid for $m=n+1$ and finish the mathematical induction. □

3.3 Proof of Theorem 2.4

By (3.9) in Lemma 3.2 and (3.56) in Lemma 3.7, it is easy to see that

$$\begin{aligned}\|\mathbf{u}(t_{n+1})-\mathbf{u}_h^{n+1}\|_{L^2}&\le\|\mathbf{e}^{n+1}\|_{L^2}+\|\mathbf{E}^{n+1}\|_{L^2}+\|\mathbf{e}_h^{n+1}\|_{L^2}\\&\le C(\tau+h^2),\quad\forall\,0\le n\le N-1,\end{aligned}$$

where the uniform boundness (3.12) of ρ^{n+1} is used. Thus, we get the optimal $\mathbf{L}^2$ error estimate for the velocity. To establish the optimal H^1 error estimate for the density, we have

$$\|\eta_h^{n+1}\|_{H^1}\le Ch^2,$$

where (3.50) in Lemma 3.5 and (3.54) in Lemma 3.6 are used. Then,

$$\begin{aligned}\|\rho(t_{n+1})-\rho_h^{n+1}\|_{H^1}&\le\|\eta^{n+1}\|_{H^1}+\|\theta^{n+1}\|_{H^1}+\|\eta_h^{n+1}\|_{H^1}\\&\le C(\tau+h^2),\quad\forall\,0\le n\le N-1.\end{aligned}$$

Thus, we complete the proof of Theorem 2.4.

Acknowledgements This work was supported by National Natural Science Foundation of China (No. 11771337) and Natural Science Foundation of Zhejiang Province (No. LY18A010021).

Declarations

Conflict of interest Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

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