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Unconditional Stability and Optimal Error Estimates of Euler Implicit/Explicit-SAV Scheme for the Navier–Stokes Equations

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Abstract
The unconditional stability and convergence analysis of the Euler implicit/explicit scheme with finite element discretization are studied for the incompressible time-dependent Navier–Stokes equations based on the scalar auxiliary variable approach. Firstly, a corresponding equivalent system of the Navier–Stokes equations with three variables is formulated, the stable finite element spaces are adopted to approximate these variables and the corresponding theoretical analysis results are provided. Secondly, a fully discrete scheme based on the backward Euler method is developed, the temporal treatment is based on the Euler implicit/explicit scheme, which is implicit for the linear terms and explicit for the nonlinear term. Hence, a constant coefficient algebraic system is formed and it can be solved efficiently. The discrete unconditional energy dissipation and stability of numerical solutions in various norms are established with any restriction on the time step, optimal error estimates are also provided. Finally, some numerical results are provided to illustrate the performances of the considered numerical scheme.

Keywords Time-dependent Navier–Stokes equations · Euler explicit/implicit scheme · Scalar auxiliary variable · Unconditional stability · Optimal error estimates

Mathematics Subject Classification 65M10 · 65N30 · 76Q10

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1 Introduction

Suppose that $\Omega \in \mathbb{R}^2$ is a bounded open set with Lipschitz continuous boundary $\partial\Omega$. Consider the following time-dependent incompressible Navier–Stokes equations in $\Omega \times (0, T]$

$$\begin{cases} u_t - \nu\Delta u + \nabla p + (u \cdot \nabla)u = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1}$$

subject to the homogeneous Dirichlet boundary condition

$$u|_{\partial\Omega} = 0 \tag{2}$$

and the initial condition

$$u|_{t=0} = u_0(x). \tag{3}$$

In these equations $u = u(x, t)$ and $p = p(x, t)$ are the velocity and pressure of the fluid at the space-time location $(x, t) \in \Omega \times (0, T]$, respectively. The parameter $\nu = \frac{1}{Re} > 0$ is the kinematic viscosity, Re is the Reynold number, $f = f(x, t)$ is the body force, T is the final time and $u_0(x)$ is the initial data of the fluid.

As a classical incompressible fluid model, the Navier-Stokes equations (1)–(3) have been widely used in the field of the computational fluid dynamics [5,6,34]. Many important models are formed by coupling the Navier–Stokes equations with other equations, for example, with the Maxwell equation form the MHD equations, with the nonlinear heat equation form the Boussinesq equations and so on. Due to the nonlinear and incompressible properties, to find the exact solutions of the Navier–Stokes equations becomes a difficult work. Therefore, numerous works have devoted to the developments of efficient numerical schemes for the Navier–Stokes equations (1)–(3), such as the nonlinear Galerkin method [1,2], the projection method [27–29], two grid method [12,17] and so on. In these numerical schemes, the treatment of nonlinear term is one of the key points. Generally speaking, the implicit and semi-implicit schemes are unconditional stable, but we need to treat a variable coefficient algebraic discrete problem, for example, [5,16,35] for the Navier–Stokes equations, [10,23] for the MHD equations, [18,25] for nonlinear parabolic problems. The implicit/explicit scheme is an attractive approach to deal with the nonlinear problem, because we just need to treat the constant coefficient algebraic discrete system. Furthermore, the considered problem can be split into a series of linear subproblems, both computational size and storage requirements are reduced. We only mention [7] for the dissipative evolution equations, [13,24,33] for the Navier–Stokes equations, [32] for the Cahn–Hilliard equations and the references therein. However, the stability of numerical solutions in implicit/explicit scheme holds under some restrictions on the time steps [9,11,33]. Namely, the following condition must be satisfied

$$\Delta t \leq C, \tag{4}$$

where Δt is the time step, $C > 0$ is a general constant, independents of Δt and mesh size h.

The scalar auxiliary variable (SAV) method was developed by Shen and his co-authors [20,21,30,31] for the gradient flow. This method can be considered as an extension and improvement of the invariant energy quadratization (IEQ) method given in [36,37,39]. The main advantages of the SAV scheme can be list as follows: (I) unconditional energy dissipation law holds, (II) decoupled equations with constant coefficients need to be solved at each time step, (III) numerical schemes up to second order are accurate. Hence, the SAV method has been used to treat the gradient flow [30,31], phase field model [38] and the references therein, some important and interesting stability and convergence results were established. In recently, the SAV method was extended to solve the incompressible Navier–Stokes equations [22], unconditional energy dissipation of the backward Euler and BDF2 schemes were developed, with a series of numerical examples illustrating the performances of the considered numerical method combining the spectral method. However, the stability and convergence results of numerical solutions in these schemes were not given. Later, Li et al presented the error analysis of the SAV approach for the Navier–Stokes equations based on the finite difference method in [19–21].

The aim of this paper is to establish the rigorous unconditional stability and optimal error estimates of the Euler implicit/explicit-SAV finite element method for the Navier–Stokes equations. We firstly develop the equivalent formulation of the Navier–Stokes equations by introducing the scalar auxiliary variable, the convergence results with finite element discretization are recalled. Then, a fully discrete implicit/explicit SAV finite element scheme is developed, the energy dissipation of numerical scheme, unconditional stability and the optimal error estimates of numerical solutions are provided. Compared with [9,11,13,19,24,33], the main features of this work contain: (I) The unconditional energy dissipation of numerical scheme is presented. (II) Unconditional stability results of numerical solutions in Euler implicit/explicit scheme are developed. (III) Optimal error estimates of numerical approximations are established.

The outline of this paper can be list as follows. Section 2 is devoted to recall some basic results of the Navier–Stokes equations and present the corresponding equivalent form in SAV approach with finite element discretization. Section 3 gives the fully discrete implicit/explicit SAV FEM for the Navier-Stokes equations, unconditional energy dissipation and stability results are established. Section 4 develops the optimal error estimates of numerical solutions. Some numerical results are presented in Sect. 5 to confirm the established theoretical findings, and illustrate the performances of the considered numerical schemes. Finally, a conclusion is given in Sect. 6.

2 Function Setting and the Galerkin Finite Element Method

2.1 Preliminary

Assume that $\Omega \subset \mathbb{R}^2$ satisfies the addition stated in (A1) below. Standard Sobolev spaces and the corresponding norms are used. Denote $H^i(\Omega)$ the function with square integrable distribution derivatives up to order i ($i = 1, 2$) over the domain Ω, $H_0^1(\Omega)$ is the closed subspace of $H^1(\Omega)$ consisting of the functions with zero trace on Ω. We equip the spaces $H^i(\Omega)(i = 1, 2)$ with the norm $\|\cdot\|_i$, $L^i(\Omega)$ with the norm $\|\cdot\|_0$ and inner product $(\cdot, \cdot)$, $H_0^1(\Omega)$ with the scalar product $(\nabla u, \nabla v)$ and norm $\|u\|_1 = (\nabla u, \nabla u)^{1/2}$. Set

$$X = H_0^1(\Omega)^2,\ \ V = \{v \in X; \nabla \cdot v = 0\},\quad M = L_0^2(\Omega) = \{q \in L^2(\Omega); \int_\Omega q dx = 0\},$$
$$Y = L^2(\Omega)^2,\ \ H = \{v \in Y; \mathrm{div} v = 0, v \cdot n|_{\partial\Omega} = 0\},\ \ \mathbb{R} = \{\text{the space of real numbers}\}.$$

We refer readers to [6,14,34] for details on these spaces. We denote the Stokes operator by $A = -P\Delta$, where P is L^2-orthogonal projection of Y onto H and the domain of A by $D(A) = H^2(\Omega)^2 \cap V$. As mentioned above, an additional assumption about the domain Ω is needed (see [1,16,34]).

(A1). Assume that Ω is smooth so that the unique solution $(v, q) \in X \times M$ of the steady Stokes problem

$$-\nu\Delta v + \nabla q = g, \quad \text{div}\, v = 0 \quad x \in \Omega, \quad v|_{\partial\Omega} = 0,$$

for any prescribed $f \in Y$, exists and satisfies

$$\|v\|_2 + \|q\|_1 \le C_1\|g\|_0,$$

where $C_1 > 0$ is a generic constant depending on the data ν and Ω.

We remark that the validity of assumption (A1) is known (see [1,6,14,34]) if $\partial\Omega$ is of C^2 or if Ω is a convex polygon in 2D. Furthermore, it is well known that (see [11,14])

$$\|v\|_{H^2} \le C_1\|Av\|_0, \quad v \in D(A).$$

The following Poincáre inequalities hold

$$\|v\|_0^2 \le \gamma_0\|v\|_1^2, \ \ \forall v \in X, \ \ \|v\|_1^2 \le \gamma_0\|v\|_2^2 \le \|Av\|_0^2, \ \ \forall v \in D(A),$$

where γ_0 is a positive constant depending only on Ω.

Some assumptions about the prescribed data for problem (1) are needed [1,10,12,16].

(A2). The initial data $u_0(x)$ and the body force f satisfy, for the positive constant C_2,

$$u_0 \in H^2(\Omega)^2 \cap V, \ \ f \in L^\infty(0, T; H^1(\Omega)^2) \ \text{ with } \|Au_0\|_0 + \|f\|_1 \le C_2.$$

The continuous bilinear forms $a(\cdot,\cdot)$ on $X \times X$ and $d(\cdot,\cdot)$ on $X \times M$ are defined by

$$a(u, v) = \nu(\nabla u, \nabla v), \ \ d(v, q) = -(v, \nabla q) = (q, \text{div} v) \ \ \forall\, u, v \in X, \ q \in M.$$

Define the trilinear form $b(\cdot,\cdot,\cdot)$ on $X \times X \times X$ with $\nabla \cdot u = 0$ by

$$b(u, v, w) = ((u\cdot\nabla)v, w) + \frac{1}{2}((\nabla\cdot u)v, w) = \frac{1}{2}((u\cdot\nabla)v, w) - \frac{1}{2}((u\cdot\nabla)w, v).$$

The following important estimates of the trilinear form $b(\cdot,\cdot,\cdot)$ can be found in [6,9,11, 13,34] with C_3 is a positive constant depending on Ω

$$b(u, v, v) = 0, \qquad b(u, v, w) = -b(u, w, v), \qquad \forall\, u \in V, \ v, w \in X, \tag{5}$$

$$\begin{aligned}&|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)|\\ &\quad\le \frac{C_3}{2}(\|u\|_0^{1/2}\|u\|_1^{1/2}\|v\|_1 + \|u\|_1\|v\|_0^{1/2}\|v\|_1^{1/2})\|w\|_0^{1/2}\|w\|_1^{1/2}, \ \ \forall\, u, v, w \in V,\end{aligned} \tag{6}$$

$$\begin{aligned}&|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)|\\ &\quad\le \frac{C_3}{2}(\|Av\|_0^{1/2}\|v\|_1^{1/2}\|u\|_0^{1/2}\|u\|_1^{1/2}+\|Av\|_0^{1/2}\|v\|_0^{1/2}\|u\|_1)\|w\|_0, \ \ \forall\, u, v \in V, w \in X,\end{aligned} \tag{7}$$

$$\begin{aligned}&|b(u, v, w)| + |b(v, u, w)| + |b(w, u, v)|\\ &\quad\le \frac{C_3}{3}(\|u\|_0^{1/2}\|Av\|_0^{1/2}\\ &\quad+\|v\|_0^{1/2}\|Au\|_0^{1/2} + \|u\|_1^{1/2}\|v\|_1^{1/2})\|Au\|_0^{1/2}\|Av\|_0^{1/2}\|w\|_{-1}, \ \ \forall\, u, v, w \in V.\end{aligned} \tag{8}$$

With above notations, the variational formulation of problem (1) reads as: for all $(v, q) \in X \times M$, find $(u, p) \in L^\infty(0, T; H) \cap L^2(0, T; X) \times L^2(0, T; M)$ such that

$$\begin{cases} (u_t, v) + a(u, v) - d(v, p) + d(u, q) + b(u, u, v) = (f, v), \\ u(0) = u_0. \end{cases} \tag{9}$$

The following regularity results can be obtained with simple modification to the argument given in [8,14,15] under the compatibility conditions.

Theorem 2.1 *Under the assumptions (A1) and (A2), problem* (9) *admits a unique solution* (u, p) *satisfying the following estimates for all* $t \in [0, T]$

$$\|\nabla u(t)\|_0 + \|u_t\|_0 + \|Au\|_0 + \|p\|_1 + \left(\int_0^t (\|u_{tt}\|_0^2 + \|Au_t\|_0^2 + \|p_t\|_1^2)ds\right)^{1/2} \leq C_4,$$

where C_4 *is a generic positive constant depending on the data* $\nu, \Omega, C_1, C_2, C_3$ *and* T*, which may take different values at its different places.*

2.2 Galerkin Finite Element Method

From now on, let $0 < h < 1$ be a real positive parameter. The finite element subspace (X_h, M_h) of (X, M) is characterized by $\mathcal{T}_h = \mathcal{T}_h(\Omega)$, a partitioning of $\overline{\Omega}$ into triangles K or quadrilaterals K, assumed to be uniformly regular as $h \to 0$. For further details, we can refer to [6,34]. Define the subspace V_h of X_h given by

$$V_h = \{v_h \in X_h : d(v_h, q_h) = 0, \quad \forall q_h \in M_h\}.$$

Set $P_h : Y \to V_h$ denotes the L^2-orthogonal projection, it can be defined by

$$(P_h u, v_h) = (u, v_h), \quad \forall\, u \in Y, \; v_h \in V_h.$$

With above statements, a discrete analogue $A_h = -P_h \Delta_h$ of the Stokes operator $A = -P\Delta$ is defined by $(-\Delta_h u_h, v_h) = (\nabla u_h, \nabla v_h)$ for all $u_h, v_h \in X_h$. The restriction of A_h to V_h is invertible, with the inverse A_h^{-1}. The discrete operator A_h was first introduced in [14] to analyze and obtain the optimal estimates for the transient Navier–Stokes equations.

We set that the finite element spaces X_h and M_h approximating the velocity and pressure are assumed to satisfy the following discrete inf-sup condition: There exists a positive constant $\beta > 0$ independent of h, such that

$$d(v_h, q_h) \geq \beta \|v_h\|_1 \|q_h\|_0, \qquad \forall\, v_h \in X_h, q_h \in M_h. \tag{10}$$

We give an example of the spaces X_h and M_h such that the condition (10) is satisfied. For any nonnegative integer l, we denote by $P_l(K)$ the space of polynomials of degrees less than or equal to l on K.

Example (The MINI element + piecewise constant space)

$$\begin{aligned}
X_h &= \{v_h \in C^0(\Omega)^2 \cap X; v_h|_K \in P_1(K)^2 \oplus \text{span}\{27\lambda_1\lambda_2\lambda_3\}, \; \forall\, K \in \mathcal{T}_h\},\\
M_h &= \{q_h \in C^0(\Omega) \cap M; q_h|_K \in P_1(K), \; \forall\, K \in \mathcal{T}_h\},\\
\mathbb{R}_h &= \{s_h \in C^0(\Omega) \cap R; s_h|_K \in P_0(K), \; \forall\, K \in \mathcal{T}_h\}.
\end{aligned}$$

where λ_1, λ_2 and λ_3 are the barycenter coordinates of the reference element. Other example, such as the Taylor-Hood element for X_h and M_h, we can refer to [6,34].

The Galerkin finite element method for problem (9) is defined as follows: Find $(u_h, p_h) \in L^2(0, T; X_h) \times L^2(0, T; M_h)$, for all $t \in (0, T]$ and $(v_h, q_h) \in X_h \times M_h$, such that

$$\begin{cases} (u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + b(u_h, u_h, v_h) = (f, v_h),\\ u_h(0) = u_{0h} = P_h u_0. \end{cases} \tag{11}$$

Theorem 2.2 *(See [6,8,10]) Under the assumptions (A1)-(A2), problem (11) admits a unique solution (u_h, p_h). Furthermore, for all $t \in (0, T]$ it holds that*

$$\|\nabla u_h\|_0 + \|u_{ht}\|_0 + \|A_h u_h\|_0 + \|\nabla u_{ht}\|_0 + \Big(\int_0^t (\|u_{htt}\|_0^2 + \|A_h u_{ht}\|_0^2) ds\Big)^{1/2} \le C_4,$$
$$\|u - u_h\|_0 + h(\|\nabla(u - u_h)\|_0 + \|p - p_h\|_0) \le C_4 h^2.$$

2.3 The Scalar Auxiliary Variable Approach

This subsection develops the equivalent form of the Navier–Stokes equations based on the scalar auxiliary variable approach. The finite element discretization is considered and the corresponding stability and convergence analysis results are also presented.

Firstly, we introduce the following scaler energy variable

$$E(t) = C_0 + \frac{1}{2}\|u(x,t)\|_0^2, \tag{12}$$

where $C_0 > 0$ is a fixed constant. Denote

$$S(t) = \sqrt{E(t)}. \tag{13}$$

It follows from (13) and $\nabla \cdot u = 0$ that problem (1) can be transformed into

$$\begin{cases} u_t - \nu \Delta u + \nabla p + \frac{S(t)}{\sqrt{E(t)}}(u \cdot \nabla)u = f, \\ \nabla \cdot u = 0, \\ S_t = \frac{1}{2S(t)} \int_\Omega (u_t + \frac{S(t)}{\sqrt{E(t)}}(u \cdot \nabla)u) \cdot u dx. \end{cases} \tag{14}$$

The Galerkin finite element method for (14) reads as: for all $(v_h, q_h, s_h) \in X_h \times M_h \times \mathbb{R}_h$, find $(u_h, p_h, S_h) \in X_h \times M_h \times \mathbb{R}_h$ with $u_h(0) = P_h u_0$ and $S_h(0) = \sqrt{C_0 + \frac{1}{2}\|u_h(0)\|_0^2}$, such that

$$\begin{cases} (u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + \frac{S_h(t)}{\sqrt{E_h(t)}} b(u_h, u_h, v_h) = (f, v_h), \\ (S_{ht}, s_h) = \frac{1}{2S_h(t)}\Big((u_{ht}, u_h) + \frac{S_h(t)}{\sqrt{E_h(t)}} b(u_h, u_h, u_h), s_h\Big). \end{cases} \tag{15}$$

From (15), we can find the numerical solutions u_h, p_h and S_h, then the discrete scaler energy variable E_h is obtained. Furthermore, from (12) it holds

$$E_h(t) = C_0 + \frac{1}{2}\|u_h(t)\|_0^2. \tag{16}$$

From the second equation of (15) and (5), one finds that

$$S_{ht} = \frac{1}{2S_h}(u_{ht}, u_h) \quad \text{with} \quad S_h(0) = \sqrt{C_0 + \frac{1}{2}\|u_h(0)\|_0^2}.$$

This is an ODE, it is easy to obtain that

$$S_h(t) = \sqrt{C_0 + \frac{1}{2}\|u_h(t)\|_0^2}. \tag{17}$$

Combining (16) and (17), problem (15) transforms into (11), by Theorem 2.2, we have

$$|S - S_h| = \left|\sqrt{C_0 + \frac{1}{2}\|u\|_0^2} - \sqrt{C_0 + \frac{1}{2}\|u_h\|_0^2}\right| = \frac{1}{2}\frac{|\|u\|_0^2 - \|u_h\|_0^2|}{\sqrt{C_0 + \frac{1}{2}\|u\|_0^2} + \sqrt{C_0 + \frac{1}{2}\|u_h\|_0^2}}$$

$$\leq \frac{1}{4\sqrt{C_0}}(\|u\|_0 + \|u_h\|_0)\|u - u_h\|_0 \leq C_4 h^2, \tag{18}$$

$$|S_{ht}| = |\frac{1}{2S_h}(u_{ht}, u_h)| = \frac{1}{2\sqrt{C_0 + \frac{1}{2}\|u_h(t)\|_0^2}}|(u_{ht}, u_h)|$$

$$\leq \frac{1}{2\sqrt{C_0}}\|u_{ht}\|_0\|u_h\|_0 \leq C_4, \tag{19}$$

$$\int_0^t |S_{htt}|^2 ds \leq \frac{1}{4C_0}\int_0^t \Big(\|u_{htt}\|_0^2\|u_h\|_0^2 + \|u_{ht}\|_0^4 + \frac{1}{4C_0^2}\|u_{ht}\|_0^4\|u_h\|_0^4\Big)ds \leq C_4. \tag{20}$$

3 Fully Discrete Euler Implicit/Explicit-SAV Method

In this section we consider the time discretization of the Galerkin finite element method with the scalar auxiliary variable. We choose an integer N and define the time step $\Delta t = \frac{T}{N}$ and the discrete times $t_n = n\Delta t, n = 0, 1, .., N$. The Euler implicit/explicit scheme applied to the spatially discrete problem (15) consists of determining functions $(u_h^{n+1}, p_h^{n+1}, S_h^{n+1}) \in X_h \times M_h \times \mathbb{R}_h$ as solutions of the recursive linear equations

$$\begin{cases} (d_t u_h^{n+1}, v_h) + a(u_h^{n+1}, v_h) - d(v_h, p_h^{n+1}) + \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, v_h) = (f^{n+1}, v_h), \\ d(u_h^{n+1}, q_h) = 0, \\ (d_t S_h^{n+1}, s_h) = \frac{1}{2S_h^{n+1}}\Big((d_t u_h^{n+1}, u_h^{n+1}) + \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, u_h^{n+1}), s_h\Big), \end{cases} \tag{21}$$

with $d_t\varphi_h^{n+1} = \frac{\varphi_h^{n+1} - \varphi_h^n}{\Delta t}$, φ_h^n takes u_h^n or S_h^n, $u_h^0 = P_h u_0$, $f^{n+1} = f(t_{n+1})$, $S_h^0 = \sqrt{C_0 + \frac{1}{2}\|u_h^0\|_0^2}$.

Based on the definition (16) of E_h, we have

$$E_h^n = C_0 + \frac{1}{2}\|u_h^n\|_0^2 \geq C_0. \tag{22}$$

Theorem 3.1 *With $f = 0$, scheme (21) is unconditional energy dissipation in the sense that*

$$|S_h^{n+1}|^2 + |S_h^{n+1} - S_h^n|^2 \leq |S_h^n|^2.$$

Proof Choosing $v_h = u_h^{n+1}\Delta t$, $q_h = p_h^{n+1}\Delta t$ and $s_h = 2S_h^{n+1}\Delta t$ in (21), adding them together, using the fact $2(a - b, a) = |a|^2 - |b|^2 + |a - b|^2$, we have

$$|S_h^{n+1}|^2 - |S_h^n|^2 + |S_h^{n+1} - S_h^n|^2$$
$$= \Delta t(d_t u_h^{n+1} + \frac{S_h^n}{\sqrt{E_h^n}}(u_h^n \cdot \nabla)u_h^n, u_h^{n+1}) = -\Delta t\nu\|\nabla u_h^{n+1}\|_0^2 \leq 0.$$

Then we finish the proof. □

Thanks to Theorem 3.1, we know that the total discrete energy of the Navier–Stokes equations in Euler implicit/explicit-SAV scheme (21) is dissipative.

The following classical discrete Gronwall lemma can be found in [26,27].

Lemma 3.2 *Let c and a_k, b_k, c_k, d_k, for integers $k \geq 0$, be non-negative numbers such that*

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq \Delta t \sum_{k=0}^{n-1} d_k a_k + \Delta t \sum_{k=0}^{n-1} c_k + c, \quad \forall\, n \geq 1.$$

Then

$$a_n + \Delta t \sum_{k=0}^{n} b_k \leq exp(\Delta t \sum_{k=0}^{n-1} d_k)(\Delta t \sum_{k=0}^{n-1} c_k + c), \quad \forall\, n \geq 1.$$

Theorem 3.3 *Under the Assumptions (A1)-(A2), $\forall 0 \leq m \leq N$, for problem (21) it holds*

$$|S_h^{m+1}|^2 + \sum_{n=0}^{m} |S_h^{n+1} - S_h^n|^2 + \frac{1}{2}\Delta t \nu \sum_{n=0}^{m} \|\nabla u_h^{n+1}\|_0^2 \leq M_0, \tag{23}$$

$$\|u_h^{m+1}\|_0^2 + \sum_{n=0}^{m} \|u_h^{n+1} - u^n\|_0^2 + \Delta t \nu \sum_{n=0}^{m} \|\nabla u_h^{n+1}\|_0^2 \leq M_1, \tag{24}$$

where $M_0 = |S_h^0|^2 + \frac{\Delta t}{2\nu} \sum_{n=0}^{m} \| f^{n+1}\|_0^2$, $M_1 = \left(\|u_h^0\|_0^2 + \frac{2\gamma_0}{\nu}\Delta t \sum_{n=0}^{m} \| f^{n+1}\|_0^2\right) \exp\left(\frac{4C_3^2 M_0^2}{\nu^2 C_0}\right)$.

Proof Taking $v_h = u_h^{n+1}\Delta t$, $q_h = p_h^{n+1}\Delta t$ and $s_h = 2S_h^{n+1}\Delta t$ in problem (21), adding the resulting equations together, we obtain

$$\begin{aligned} |S_h^{n+1}|^2 - |S_h^n|^2 + |S_h^{n+1} - S_h^n|^2 + \Delta t \nu \|\nabla u_h^{n+1}\|_0^2 &= (f^{n+1}, u_h^{n+1})\Delta t \\ &\leq \frac{\Delta t}{2\nu}\| f^{n+1}\|_0^2 + \frac{\nu}{2}\Delta t \|\nabla u_h^{n+1}\|_0^2. \end{aligned}$$

Eliminating the last term and summing from $n = 0$ to m, we obtain (23).

Choosing $v_h = 2u_h^{n+1}\Delta t$, $q_h = 2p_h^{n+1}\Delta t$ in problem (21), one finds

$$\begin{aligned} &\|u_h^{n+1}\|_0^2 - \|u_h^n\|_0^2 + \|u_h^{n+1} - u_h^n\|_0^2 + 2\Delta t \nu \|\nabla u_h^{n+1}\|_0^2 + 2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, u_h^{n+1}) \\ &= 2\Delta t (f^{n+1}, u_h^{n+1}). \end{aligned} \tag{25}$$

We can treat the trilinear term and right-hand side term as follows

$$\begin{aligned} |2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, u_h^{n+1})| &\leq 2C_3 \Delta t \frac{|S_h^n|}{\sqrt{E_h^n}} \|u_h^n\|_{L^4} \|\nabla u_h^{n+1}\|_{L^2} \|u_h^n\|_{L^4} \\ &\leq 2C_3 \Delta t \frac{|S_h^n|}{\sqrt{E_h^n}} \|u_h^n\|_0 \|\nabla u_h^n\|_0 \|\nabla u_h^{n+1}\|_0 \\ &\leq \frac{\nu}{2}\Delta t \|\nabla u_h^{n+1}\|_0^2 + \frac{2C_3^2 \Delta t}{C_0 \nu} |S_h^n|^2 \|u_h^n\|_0^2 \|\nabla u_h^n\|_0^2, \\ |2\Delta t (f^{n+1}, u_h^{n+1})| \leq 2\Delta t \| f^{n+1}\|_0 \|u_h^{n+1}\|_0 &\leq \frac{\nu}{2}\Delta t \|\nabla u_h^{n+1}\|_0^2 + \frac{2\gamma_0}{\nu}\Delta t \| f^{n+1}\|_0^2. \end{aligned}$$

Combining above inequalities with (25), summing from $n = 0$ to m and using (22), one finds

$$\|u_h^{m+1}\|_0^2 + \sum_{n=0}^{m} \|u_h^{n+1} - u_h^n\|_0^2 + \Delta t \nu \sum_{n=0}^{m} \|\nabla u_h^{n+1}\|_0^2$$
$$\leq \|u_h^0\|_0^2 + \frac{2\gamma_0}{\nu} \Delta t \sum_{n=0}^{m} \|f^{n+1}\|_0^2 + \frac{2C_3^2}{C_0 \nu} \Delta t \sum_{n=0}^{m} |S_h^n|^2 \|\nabla u_h^n\|_0^2 \|u_h^n\|_0^2.$$

With the help of (23) and Lemma 3.2, we complete the proof (24). □

Theorem 3.4 *Under the Assumptions (A1)-(A2), $\forall 0 \leq m \leq N$, for problem (21) it holds*

$$\|\nabla u_h^{m+1}\|_0^2 + \frac{\nu}{2} \Delta t \|A_h u_h^{m+1}\|_0^2 + \sum_{n=0}^{m} \|\nabla(u_h^{n+1} - u_h^n)\|_0^2 + \frac{\nu}{2} \Delta t \sum_{n=0}^{m} \|A_h u_h^{n+1}\|_0^2 \leq M_2,$$

where $M_2 = (\|\nabla u_h^0\|_0^2 + \frac{\nu}{2} \Delta t \|A_h u_h^0\|_0^2 + \frac{2\Delta t}{\nu} \sum_{n=0}^{m} \|f^{n+1}\|_0^2) \exp\left(\frac{4C_3^4 M_0^3 M_1}{\nu^4 C_0^2}\right)$.

Proof It follows from $v_h = -2A_h u_h^{n+1} \Delta t \in V_h$, $q_h = 0$ in the first equation of (21) that

$$\|\nabla u_h^{n+1}\|_0^2 - \|\nabla u_h^n\|_0^2 + \|\nabla(u_h^{n+1} - u_h^n)\|_0^2 + 2\Delta t \nu \|A_h u_h^{n+1}\|_0^2$$
$$= 2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, A_h u_h^{n+1}) - 2\Delta t (f^{n+1}, A_h u_h^{n+1}). \tag{26}$$

For the right-hand side terms, thanks to (22) and the Cauchy inequality, we have

$$\left| 2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, A_h u_h^{n+1}) \right|$$
$$\leq 2C_3 \Delta t \frac{|S_h^n|}{\sqrt{E_h^n}} \|u_h^n\|_0^{1/2} \|\nabla u_h^n\|_0 \|A_h u_h^n\|_0^{1/2} \|A_h u_h^{n+1}\|_0$$
$$\leq \frac{\nu}{2} \Delta t \|A_h u_h^{n+1}\|_0^2 + \frac{2C_3^2}{\nu C_0} \Delta t |S_h^n|^2 \|u_h^n\|_0 \|\nabla u_h^n\|_0^2 \|A_h u_h^n\|_0$$
$$\leq \frac{\nu}{2} \Delta t \|A_h u_h^{n+1}\|_0^2 + \frac{\nu}{2} \Delta t \|A_h u_h^n\|_0^2 + \frac{2C_3^4}{\nu^3 C_0^2} \Delta t |S_h^n|^4 \|u_h^n\|_0^2 \|\nabla u_h^n\|_0^4,$$
$$|2\Delta t (f^{n+1}, A_h u_h^{n+1})|$$
$$\leq 2\Delta t \|f^{n+1}\|_0 \|A_h u_h^{n+1}\|_0 \leq \frac{\nu}{2} \Delta t \|A_h u_h^{n+1}\|_0^2 + \frac{2\Delta t}{\nu} \|f^{n+1}\|_0^2.$$

Combining above inequalities with (26) and summing from $n = 0$ to m, we get

$$\|\nabla u_h^{m+1}\|_0^2 + \frac{\nu}{2} \Delta t \|A_h u_h^{m+1}\|_0^2 + \sum_{n=0}^{m} \|\nabla(u_h^{n+1} - u_h^n)\|_0^2 + \frac{\nu}{2} \Delta t \sum_{n=0}^{m} \|A_h u_h^{n+1}\|_0^2$$
$$\leq \|\nabla u_h^0\|_0^2 + \frac{\nu}{2} \Delta t \|A_h u_h^0\|_0^2 + \frac{2\Delta t}{\nu} \sum_{n=0}^{m} \|f^{n+1}\|_0^2$$
$$+ \Delta t \sum_{n=0}^{m} \frac{2C_3^4}{\nu^3 C_0^2} |S_h^n|^4 \|u_h^n\|_0^2 \|\nabla u_h^n\|_0^2 \|\nabla u_h^n\|_0^2.$$

With the help of (23), (24) and Lemma 3.2, we obtain the desired results. □

Theorem 3.5 *Under the Assumptions (A1)-(A2),* $\forall 0 \le m \le N$, *for problem* (21) *it holds*

$$\|A_h u_h^{m+1}\|_0^2 + \sum_{n=0}^{m} \|A_h(u_h^{n+1} - u_h^n)\|_0^2 + \Delta t \nu \sum_{n=0}^{m} \|A_h^{\frac{3}{2}} u_h^{n+1}\|_0^2 \le M_3,$$

where $M_3 = (\|A_h u_h^0\|_0^2 + \frac{2\Delta t}{\nu} \sum_{n=0}^{m} \|f^{n+1}\|_1^2) \exp\left(\frac{4C_3^2 M_0 M_2}{\nu^2 C_0}\right)$.

Proof Taking $v_h = 2A_h^2 u_h^{n+1}\Delta t \in V_h$, $q_h = 0$ in the first equation of (21), one finds that

$$\begin{aligned} &\|A_h u_h^{n+1}\|_0^2 - \|A_h u_h^n\|_0^2 + \|A_h(u_h^{n+1} - u_h^n)\|_0^2 + 2\Delta t \nu \|A_h^{\frac{3}{2}} u_h^{n+1}\|_0^2 \\ &= 2\Delta t (f^{n+1}, A_h^2 u_h^{n+1}) - 2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, A_h^2 u_h^{n+1}). \end{aligned} \tag{27}$$

For the right-hand side terms, by (8), (22) and the Cauchy inequality, we have

$$\begin{aligned} &\left| 2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, A_h^2 u_h^{n+1}) \right| \\ &\le \frac{4C_3}{3} \Delta t \frac{|S_h^n|}{\sqrt{E_h^n}} \left(\|u_h^n\|_0^{1/2} \|A_h u_h^n\|_0^{3/2} + \|\nabla u_h^n\|_0 \|A_h u_h^n\|_0 \right) \|A_h^2 u_h^{n+1}\|_{-1} \\ &\le \frac{\nu}{2} \Delta t \|A_h^{\frac{3}{2}} u_h^{n+1}\|_0^2 + \frac{C_3^2}{\nu C_0} \Delta t |S_h^n|^2 (\|u_h^n\|_0 \|A_h u_h^n\|_0 + \|\nabla u_h^n\|_0^2) \|A_h u_h^n\|_0^2, \\ &|2\Delta t (f^{n+1}, A_h^2 u_h^{n+1})| \\ &\le 2\Delta t \|f^{n+1}\|_1 \|A_h^2 u_h^{n+1}\|_{-1} \le \frac{\nu}{2} \Delta t \|A_h^{\frac{3}{2}} u_h^{n+1}\|_0^2 + \frac{2\Delta t}{\nu} \|f^{n+1}\|_1^2. \end{aligned}$$

Combining above inequalities with (27) and summing from $n = 0$ to m we obtain

$$\begin{aligned} &\|A_h u_h^{m+1}\|_0^2 + \sum_{n=0}^{m} \|A_h(u_h^{n+1} - u_h^n)\|_0^2 + \Delta t \nu \sum_{n=0}^{m} \|A_h^{\frac{3}{2}} u_h^{n+1}\|_0^2 \\ &\le \|A_h u_h^0\|_0^2 + \frac{2\Delta t}{\nu} \sum_{n=0}^{m} \|f^{n+1}\|_1^2 \\ &\quad + \frac{C_3^2}{\nu C_0} \Delta t \sum_{n=0}^{m} |S_h^n|^2 (\|u_h^n\|_0 \|A_h u_h^n\|_0 + \|\nabla u_h^n\|_0^2) \|A_h u_h^n\|_0^2. \end{aligned}$$

With the help of (23), Theorem 3.4 and Lemma 3.2, we finish the proof. □

4 Error Estimates of the Euler Implicit/Explicit-SAV Scheme

This section is devoted to establish the convergence results of fully discrete implicit/explicit-SAV finite element scheme (21). Firstly, we discrete the Navier–Stokes equations (15) on $n+1$th time level to obtain

$$\begin{cases} (d_t u_h(t_{n+1}), v_h) + a(u_h(t_{n+1}), v_h) - d(p_h(t_{n+1}), v_h) + \frac{S_h(t_{n+1})}{\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), v_h) \\ + d(u_h(t_{n+1}), q_h) = (f^{n+1}, v_h) + (\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t_n - t) u_{htt} dt, v_h), \\ (d_t S_h(t_{n+1}), s_h) = \left(\frac{1}{2S_h(t_{n+1})}(d_t u_h(t_{n+1}) + \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t - t_n) u_{htt} dt, u_h(t_{n+1})), s_h\right) \\ + (\frac{1}{2\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), u_h(t_{n+1})), s_h) + (\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t_n - t) S_{htt} dt, s_h). \end{cases} \tag{28}$$

Denote the errors

$$e_u^n = u_h(t_n) - u_h^n, \quad e_p^n = p_h(t_n) - p_h^n, \quad e_S^n = S_h(t_n) - S_h^n.$$

The following error equations can be obtained by combining (21) with (28)

$$\begin{cases} (d_t e_u^{n+1}, v_h) + a(e_u^{n+1}, v_h) - d(e_p^{n+1}, v_h) + \frac{S_h(t_{n+1})}{\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), v_h) \\ - \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, v_h) + d(e_u^{n+1}, q_h) = (\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} (t_n - t) u_{htt} dt, v_h), \\ (d_t e_S^{n+1}, s_h) = \left(\frac{1}{2S_h(t_{n+1})}(d_t u_h(t_{n+1}), u_h(t_{n+1})) - \frac{1}{2S_h^{n+1}}(d_t u_h^{n+1}, u_h^{n+1}), s_h\right) \\ \qquad + \left(\frac{1}{2\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), u_h(t_{n+1})) - \frac{S_h^n}{2S_h^{n+1}\sqrt{E_h^n}} b(u_h^n, u_h^n, u_h^{n+1}), s_h\right) \\ \qquad + \frac{1}{\Delta t}\left(\int_{t_n}^{t_{n+1}} (t_n - t) S_{htt} dt + \frac{1}{2S_h(t_{n+1})}(\int_{t_n}^{t_{n+1}} (t - t_n) u_{htt} dt, u_h(t_{n+1})), s_h\right), \end{cases} \tag{29}$$

Lemma 4.1 *Under the Assumptions (A1)–(A2) and $e_u^0 = 0$, for all $m \geq 1$, we have*

$$\|e_u^{m+1}\|_0^2 + \sum_{n=0}^{m} \|e_u^{n+1} - e_u^n\|_0^2 + \frac{\nu}{2}\Delta t \|\nabla e_u^{m+1}\|_0^2 + \frac{\nu}{2}\Delta t \sum_{n=0}^{m} \|\nabla e_u^{n+1}\|_0^2$$
$$\leq C_5 \left(\Delta t^2 + \Delta t \sum_{n=0}^{m} |e_S^n|^2\right),$$

where $C_5 > 0$ is a constant depending on the data $\nu, \Omega, C_1, C_2, C_3, C_4$ and T, which may take different values at its different places.

Proof Taking $v_h = 2e_u^{n+1}\Delta t$, $q_h = 2e_p^{n+1}\Delta t$ in problem (29), we have

$$\|e_u^{n+1}\|_0^2 - \|e_u^n\|_0^2 + 2\Delta t \frac{S_h(t_{n+1})}{\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), e_u^{n+1}) - 2\Delta t \frac{S_h^n}{\sqrt{E_h^n}} b(u_h^n, u_h^n, e_u^{n+1})$$
$$+ 2\Delta t \nu \|\nabla e_u^{n+1}\|_0^2 + \|e_u^{n+1} - e_u^n\|_0^2 = 2(\int_{t_n}^{t_{n+1}} (t_n - t) u_{htt} dt, e_u^{n+1}). \tag{30}$$

For the right-hand side term, by the Hölder inequality, one finds

$$\left| 2\left(\int_{t_n}^{t_{n+1}} (t_n - t) u_{htt} dt, e_u^{n+1}\right)\right| \leq \frac{\nu}{4}\Delta t \|\nabla e_u^{n+1}\|_0^2 + \frac{4\gamma_0}{\nu}\Delta t^2 \int_{t_n}^{t_{n+1}} \|u_{htt}\|_0^2 dt.$$

For the trilinear terms, we have

$$
\begin{aligned}
&2\Delta t\Big|\frac{S_h(t_{n+1})}{\sqrt{E_h(t_{n+1})}}b(u_h(t_{n+1}),u_h(t_{n+1}),e_u^{n+1})-\frac{S_h^n}{\sqrt{E_h^n}}b(u_h^n,u_h^n,e_u^{n+1})|\\
&=2\Delta t\Big|b(u_h(t_{n+1}),u_h(t_{n+1}),e_u^{n+1})-b(u_h(t_n),u_h(t_n),e_u^{n+1})+b(u_h(t_n),u_h(t_n),e_u^{n+1})\\
&\quad -b(u_h^n,u_h^n,e_u^{n+1})+\frac{e_S^n}{\sqrt{E_h(t_n)}}b(u_h^n,u_h^n,e_u^{n+1})+S_h^n\Big(\frac{1}{\sqrt{E_h(t_n)}}-\frac{1}{\sqrt{E_h^n}}\Big)b(u_h^n,u_h^n,e_u^{n+1})\Big|.
\end{aligned}
$$

By the Taylor expansion, (6), (7) and (22), we have

$$
\begin{aligned}
&2\Delta t\Big|b(u_h(t_{n+1}),u_h(t_{n+1}),e_u^{n+1})-b(u_h(t_n),u_h(t_n),e_u^{n+1})\Big|\\
&=2\Delta t^2\Big|b(u_{ht}(t_n),u_h(t_{n+1}),e_u^{n+1})+b(u_h(t_n),u_{ht}(t_n),e_u^{n+1})\Big|\\
&\le 2C_3\Delta t^2\|u_{ht}\|_0(\|A_hu_h(t_{n+1})\|_0+\|A_hu_h(t_n)\|_0)\|\nabla e_u^{n+1}\|_0\\
&\le \frac{\nu}{4}\Delta t\|\nabla e_u^{n+1}\|_0^2+\frac{8C_3^2}{\nu}\Delta t^3\|u_{ht}\|_0^2(\|A_hu_h(t_{n+1})\|_0^2+\|A_hu_h(t_n)\|_0^2),\\
&2\Delta t\Big|b(u_h(t_n),u_h(t_n),e_u^{n+1})-b(u_h^n,u_h^n,e_u^{n+1})\Big|=2\Delta t\Big|b(e_u^n,u_h(t_n),e_u^{n+1})+b(u_h^n,e_u^n,e_u^{n+1})\Big|\\
&\le 2C_3\Delta t\|\nabla e_u^{n+1}\|_0\|e_u^n\|_0^{1/2}\|\nabla e_u^n\|_0^{1/2}(\|\nabla u_h^n\|_0+\|\nabla u_h(t_n)\|_0)\\
&\le \frac{\nu}{4}\Delta t\|\nabla e_u^{n+1}\|_0^2+\frac{\nu}{2}\Delta t\|\nabla e_u^n\|_0^2+\frac{8C_3^4}{\nu^3}\Delta t(\|\nabla u_h^n\|_0^2+\|\nabla u_h(t_n)\|_0^2)^2\|e_u^n\|_0^2,\\
&2\Delta t|\frac{e_S^n}{\sqrt{E_h(t_n)}}b(u_h^n,u_h^n,e_u^{n+1})|\le\frac{2C_3}{\sqrt{C_0}}\Delta t|e_S^n|\|\nabla u_h^n\|_0\|\nabla e_u^{n+1}\|_0\|\nabla u_h^n\|_0\\
&\le\frac{\nu}{8}\Delta t\|\nabla e_u^{n+1}\|_0^2+\frac{8C_3^2}{\nu}\Delta t\frac{|e_S^n|^2}{C_0}\|\nabla u_h^n\|_0^4,\\
&2\Delta t|S_h^n(\frac{1}{\sqrt{E_h(t_n)}}-\frac{1}{\sqrt{E_h^n}})b(u_h^n,u_h^n,e_u^{n+1})|\\
&=2\Delta t|S_h^n|\frac{|E_h^n-E_h(t_n)|}{\sqrt{E_h(t_n)}\sqrt{E_h^n}(\sqrt{E_h(t_n)}+\sqrt{E_h^n})}|b(u_h^n,u_h^n,e_u^{n+1})|.
\end{aligned}
$$

Based on the definitions of E_h^n, $E_h(t_n)$ and the following fact that

$$
|E_h^n-E_h(t_n)|=\frac{1}{2}(\|u_h^n\|_0^2-\|u_h(t_n)\|_0^2)\le\|e_u^n\|_0\Big(\sqrt{E_h^n}+\sqrt{E_h(t_n)}\Big),
$$

one finds

$$
\begin{aligned}
2\Delta t|S_h^n(\frac{1}{\sqrt{E_h(t_n)}}-\frac{1}{\sqrt{E_h^n}})b(u_h^n,u_h^n,e_u^{n+1})|&\le\frac{2C_3}{C_0}\Delta t|S_h^n|\|e_u^n\|_0\|\nabla e_u^{n+1}\|_0\|\nabla u_h^n\|_0^2\\
&\le\frac{\nu}{8}\Delta t\|\nabla e_u^{n+1}\|_0^2+\frac{8C_3^2}{C_0^2\nu}\Delta t|S_h^n|^2\|\nabla u^n\|_0^4\|e_u^n\|_0^2.
\end{aligned}
$$

Combining above inequalities with (30), summing from $n=0$ to m, using Lemma 3.2 and Theorems 3.3–3.4, we finish the proof. □

Lemma 4.2 *Under the Assumptions (A1)-(A2) and $e_u^0=0$, for all $m\ge 0$, it holds*

$$
\nu\|\nabla e_u^{m+1}\|_0^2+\Delta t\sum_{n=0}^{m}\|d_te_u^{n+1}\|_0^2\le C_5\Big(\Delta t^2+\Delta t\sum_{n=0}^{m}|e_S^n|^2\Big).
$$

Proof Taking $v_h = 2d_t e_u^{n+1}\Delta t \in V_h, q_h = 0$ in problem (29), one finds

$$\begin{aligned}&\nu(\|\nabla e_u^{n+1}\|_0^2 - \|\nabla e_u^n\|_0^2 + \|\nabla(e_u^{n+1} - e_u^n)\|_0^2) + 2\Delta t\|d_t e_u^{n+1}\|_0^2\\ &\quad -2\Delta t\frac{S_h^n}{\sqrt{E_h^n}}b(u_h^n, u_h^n, d_t e_u^{n+1}) + 2\Delta t\frac{S_h(t_{n+1})}{\sqrt{E_h(t_{n+1})}}b(u_h(t_{n+1}), u_h(t_{n+1}), d_t e_u^{n+1})\\ &\quad = 2\left(\int_{t_n}^{t_{n+1}}(t_n - t)u_{htt}dt, d_t e_u^{n+1}\right). \end{aligned} \tag{31}$$

For the trilinear terms and right-hand side term, we can treat them as follows

$$\begin{aligned}&2\Delta t|\frac{S_h^n}{\sqrt{E_h^n}}b(u_h^n, u_h^n, d_t e_u^{n+1}) - \frac{S_h(t_{n+1})}{\sqrt{E_h(t_{n+1})}}b(u_h(t_{n+1}), u_h(t_{n+1}), d_t e_u^{n+1})|\\ &= 2\Delta t\Big|\frac{S_h^n - S_h(t_n)}{\sqrt{E_h^n}}b(u_h^n, u_h^n, d_t e_u^{n+1}) - b(e_u^n, u_h^n, d_t e_u^{n+1}) - b(u_h(t_n), e_u^n, d_t e_u^{n+1})\\ &\quad +(b(u_h(t_n) - u_h(t_{n+1}), u_h(t_n), d_t e_u^{n+1}) + b(u_h(t_{n+1}), u_h(t_n) - u_h(t_{n+1}), d_t e_u^{n+1})\\ &\quad +S_h(t_n)(\frac{1}{\sqrt{E_h^n}} - \frac{1}{\sqrt{E_h(t_n)}})b(u_h^n, u_h^n, d_t e_u^{n+1})\Big|\\ &\le 2\Delta t\Big|\frac{e_S^n}{\sqrt{E_h^n}}b(u_h^n, u_h^n, d_t e_u^{n+1}) - b(e_u^n, u_h^n, d_t e_u^{n+1}) - b(u_h(t_n), e_u^n, d_t e_u^{n+1})\Big|\\ &\quad +2\Delta t^2\Big|(b(u_{ht}(t_n), u_h(t_n), d_t e_u^{n+1}) + b(u_h(t_{n+1}), u_{ht}(t_n), d_t e_u^{n+1})\Big|\\ &\quad +2\Delta t\frac{|S_h(t_n)|\|e_u^n\|_0}{\sqrt{E_h^n}\sqrt{E_h(t_n)}}|b(u_h^n, u_h^n, d_t e_u^{n+1})|\\ &\le \frac{4}{5}\Delta t\|d_t e_u^{n+1}\|_0^2 + \frac{5C_3^2}{C_0}\|\nabla u_h^n\|_0^2\|A_h u_h^n\|_0^2\Delta t|e_S^n|^2\\ &\quad +\frac{5C_3^2}{C_0^2}|S_h(t_n)|^2\|\nabla u_h^n\|_0^2\|A_h u_h^n\|_0^2\Delta t\|e_u^n\|_0^2\\ &\quad +5C_3^2(\|A_h u_h^n\|_0^2 + \|A_h u_h(t_n)\|_0^2)\Delta t\|\nabla e_u^n\|_0^2 + 5C_3^2\Delta t^3\|\nabla u_{ht}\|_0^2(\|A_h u_h(t_n)\|_0^2\\ &\quad +\|A_h u_h(t_{n+1})\|_0^2),\\ &2\left|\left(\int_{t_n}^{t_{n+1}}(t_n - t)u_{htt}dt, d_t e_u^{n+1}\right)\right| \le \frac{1}{5}\Delta t\|d_t e_u^{n+1}\|_0^2 + 5\Delta t^2\int_{t_n}^{t_{n+1}}\|u_{htt}\|_0^2 dt. \end{aligned}$$

Combining above inequalities with (31), summing from $n = 0$ to m, using Lemmas 3.2, 4.1 and Theorems 3.3–3.5, we complete the proof. □

Lemma 4.3 *Under the Assumptions (A1)-(A2),* $e_S^0 = 0$ *and* $\Delta t \le \frac{C_0}{C_4^2+4}$*, for all* $m \ge 0$*, it holds*

$$|e_S^{m+1}|^2 + \frac{1}{C_4^2 + 4}\sum_{n=0}^{m}|e_S^{n+1} - e_S^n|^2 \le C_5\Delta t^2.$$

Proof Choosing $s_h = 2e_S^{n+1}\Delta t$ in the second equation of problem (29), we have

$$
\begin{aligned}
&|e_S^{n+1}|^2 - |e_S^n|^2 + |e_S^{n+1} - e_S^n|^2 \\
&= \Delta t e_S^{n+1}\Big(\frac{1}{S_h(t_{n+1})}(d_t u_h(t_{n+1}), u_h(t_{n+1})) - \frac{1}{S_h^{n+1}}(d_t u_h^{n+1}, u_h^{n+1})\Big) \\
&\quad + \Delta t e_S^{n+1}\Big(\frac{1}{\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), u_h(t_{n+1})) - \frac{S_h^n}{S_h^{n+1}\sqrt{E_h^n}} b(u_h^n, u_h^n, u_h^{n+1})\Big) \\
&\quad + 2e_S^{n+1}\int_{t_n}^{t_{n+1}}(t_n - t)S_{htt}dt + \frac{e_S^{n+1}}{S_h(t_{n+1})}\Big(\int_{t_n}^{t_{n+1}}(t_n - t)u_{htt}dt, u_h(t_{n+1})\Big).
\end{aligned}
\tag{32}
$$

We are now in the position of treating the right-hand side terms one by one

$$
\begin{aligned}
&\Big|\Delta t e_S^{n+1}\Big(\frac{1}{S_h(t_{n+1})}(d_t u_h(t_{n+1}), u_h(t_{n+1})) - \frac{1}{S_h^{n+1}}(d_t u_h^{n+1}, u_h^{n+1})\Big)\Big| \\
&= \Delta t|e_S^{n+1}|\Big|\Big(\frac{1}{S_h(t_{n+1})} - \frac{1}{S_h^{n+1}}\Big)(d_t u_h(t_{n+1}), u_h(t_{n+1})) \\
&\quad + \frac{1}{S_h^{n+1}}\Big[(d_t u_h(t_{n+1}), e_u^{n+1}) + (d_t e_u^{n+1}, u_h^{n+1})\Big]\Big| \\
&\le \Delta t|e_S^{n+1}|\Big[\frac{|e_S^{n+1}|}{S_h(t_{n+1})S_h^{n+1}}\|u_{ht}\|_0\|u_h(t_{n+1})\|_0 \\
&\quad + \frac{1}{S_h^{n+1}}\Big(\|u_{ht}\|_0\|e_u^{n+1}\|_0 + \|d_t e_u^{n+1}\|_0\|u_h^{n+1}\|_0\Big)\Big] \\
&\le \frac{1}{C_0}\Delta t|e_S^{n+1}|^2\Big(\|u_{ht}\|_0\|u_h(t_{n+1})\|_0 + 1\Big) + \frac{\Delta t}{2}\|u_{ht}\|_0^2\|e_u^{n+1}\|_0^2 + \frac{\Delta t}{2}\|d_t e_u^{n+1}\|_0^2\|u_h^{n+1}\|_0^2, \\
&\Big|\Delta t e_S^{n+1}\Big(\frac{1}{\sqrt{E_h(t_{n+1})}} b(u_h(t_{n+1}), u_h(t_{n+1}), u_h(t_{n+1})) - \frac{S_h^n}{S_h^{n+1}\sqrt{E_h^n}} b(u_h^n, u_h^n, u_h^{n+1})\Big)\Big| \\
&= \frac{\Delta t|e_S^{n+1}|}{\sqrt{E_h^n}}\Big|\frac{S_h^n}{S_h^{n+1}}\Big|\Big|\Delta t b(u_h(t_n), u_h(t_n), u_{ht}(t_n)) + b(e_u^n, u_h(t_n), u_h(t_{n+1})) \\
&\quad + b(u_h^n, e_u^n, u_h(t_{n+1})) + b(u_h^n, u_h^n, e_u^{n+1})\Big| \\
&\le \frac{C_3}{C_0}\Delta t|S_h^n||e_S^{n+1}|\Big(\Delta t\|\nabla u_h(t_n)\|_0\|A_h u_h(t_n)\|_0\|u_{ht}(t_n)\|_0 + \|\nabla u_h^n\|_0^2\|\nabla e_u^{n+1}\|_0 \\
&\quad + \|e_u^n\|_0(\|\nabla u_h(t_n)\|_0 + \|\nabla u_h^n\|_0)\|A_h u_h(t_{n+1})\|_0\Big) \\
&\le \frac{3}{2C_0}\Delta t|e_S^{n+1}|^2 + \frac{C_3^2}{2C_0}\Delta t|S_h^n|^2\Big(\Delta t^2\|\nabla u_h\|_0^2\|A_h u_h\|_0^2\|u_{ht}\|_0^2 + \|\nabla u_h^n\|_0^4\|\nabla e_u^{n+1}\|_0^2 \\
&\quad + \|e_u^n\|_0^2(\|\nabla u_h(t_n)\|_0^2 + \|\nabla u_h^n\|_0^2)\|A_h u_h\|_0^2\Big), \\
&\Big|\frac{e_S^{n+1}}{S_h(t_{n+1})}\Big(\int_{t_n}^{t_{n+1}}(t_n - t)u_{htt}dt, u_h(t_{n+1})\Big)\Big| \\
&\le \frac{\Delta t}{4C_0}|e_S^{n+1}|^2 + \Delta t^2\int_{t_n}^{t_{n+1}}\|u_{htt}\|_0^2 dt\|u_h(t_{n+1})\|_0^2,
\end{aligned}
$$

$$\left|2e_S^{n+1}\int_{t_n}^{t_{n+1}}(t_n-t)S_{htt}dt\right| \le 2|e_S^{n+1}|\Delta t^{\frac{3}{2}}(\int_{t_n}^{t_{n+1}}|S_{htt}|^2dt)^{1/2}$$
$$\le \frac{\Delta t}{4C_0}|e_S^{n+1}|^2 + 4C_0\Delta t^2\int_{t_n}^{t_{n+1}}|S_{htt}|^2dt.$$

Combining above inequalities with (32), summing from $n=0$ to m, using Theorems 2.2, 3.3–3.5, Lemmas 3.2, 4.1–4.2 and the condition $\Delta t \le \frac{C_0}{C_4^2+4}$, we complete the proof. □

Combining Theorem 2.2, (18) with Lemmas 4.1–4.3 and the inf-sup condition, we finally obtain the optimal error estimates of numerical solutions in Euler implicit/explicit-SAV scheme (29) for the Navier–Stokes equations.

Theorem 4.1 *Under the Assumptions of Lemmas 4.2 and 4.3, for all $m \ge 0$ it holds*

$$\|e_u^{m+1}\|_0^2 + |e_S^{m+1}|^2 + h^2(\|\nabla e_u^{m+1}\|_0^2 + \Delta t\sum_{n=0}^{m}\|e_p^{n+1}\|_0^2) \le C_5(\Delta t^2 + h^4). \tag{33}$$

5 Numerical Experiments

In this section, we present some numerical results to illustrate the performances of the fully discrete Euler implicit/explicit-SAV finite element scheme (21) for the Navier–Stokes equations. Due to we treat the nonlinear terms explicitly, so we can split the considered problems into a Stokes equations and a quadratic algebraic equation in one variable. It means that we can solve problem (21) as follows:

$$\begin{aligned}(d_t u_h^{n+1}, v_h) + a(u_h^{n+1}, v_h) - d(v_h, p_h^{n+1}) + d(u_h^{n+1}, q_h)\\ = (f^{n+1}, v_h) - \frac{S_h^n}{\sqrt{E_h^n}}b(u_h^n, u_h^n, v_h),\end{aligned} \tag{34}$$

and

$$(d_t S_h^{n+1}, s_h) = \frac{1}{2S_h^{n+1}}\Big((d_t u_h^{n+1}, u_h^{n+1}) + \frac{S_h^n}{\sqrt{E_h^n}}b(u_h^n, u_h^n, u_h^{n+1}), s_h\Big), \tag{35}$$

with $u_h^0 = P_h u_0$, $S_h^0 = \sqrt{C_0 + \frac{1}{2}\|u_h(0)\|_0^2}$.

Firstly, we solve the Stokes equations (34) with u_h^n, S_h^n and for all $n \ge 0$.

Secondly, taking $v_h = u_h^{n+1}\Delta t$, $q_h = p_h^{n+1}\Delta t$ and $s_h = 2S_h^{n+1}\Delta t$ in (21), one gets

$$2(S_h^{n+1})^2 - 2S_h^{n+1}S_h^n = \Delta t(f^{n+1}, u_h^{n+1}) - \nu\Delta t\|\nabla u_h^{n+1}\|_0^2. \tag{36}$$

Solving the equation (36) with the obtained u_h^{n+1} and the quadratic formula.

Finally, we present some computational results to confirm the established theoretical results and show the performances of the considered numerical scheme (21). The partition of domain Ω uses the triangle mesh with stable MINI element for the velocity and pressure. The mesh is obtained by dividing Ω into squares and then drawing a diagonal in each square. Set $\Omega = [0, 1] \times [0, 1]$, the viscosity parameter $\nu = 1$ the final time $T = 1$ and choose the following analytical solutions for the velocity $u(x, t) = (u_1(x, t), u_2(x, t))$ and pressure

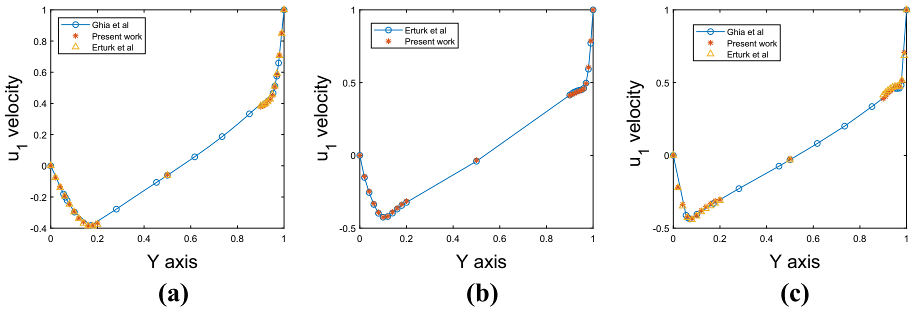

Fig. 1 The computed results of implicit/explicit-SAV scheme of vertical velocity profiles (x=0.5) with different Re. **a** Re = 1000, **b** Re = 2500, **c** Re = 5000

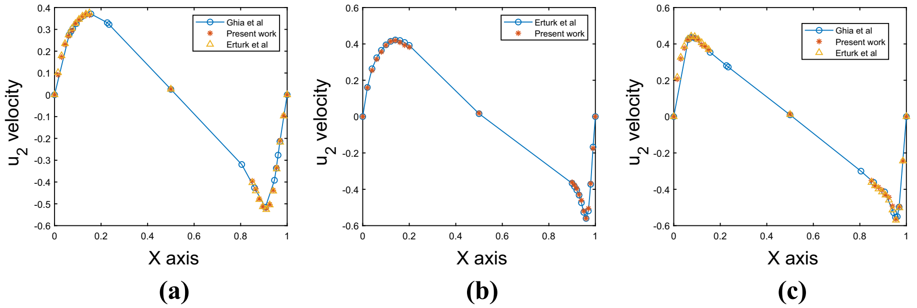

Fig. 2 The computed results of implicit/explicit-SAV scheme of horizontal velocity profiles (y = 0.5) with different Re. **a** Re = 1000, **b** Re = 2500, **c** Re = 5000

$p(x, t)$

$$
\begin{aligned}
u_1(x, t) &= 10x^2(x-1)^2 y(y-1)(2y-1)\exp(-t),\\
u_2(x, t) &= -10x(x-1)(2x-1)y^2(y-1)^2\exp(-t),\\
p(x, t) &= 10(2x-1)(2y-1)\exp(-t).
\end{aligned}
$$

The computational results of Euler implicit/explicit-SAV scheme (21) are presented in Table 1 to verify the established results of Theorem 4.1. From these data, we can see that the convergence orders of velocity in L^2- and H^1-norms are 2 and 1, respectively, which confirm the provided theoretical findings (33) well.

The second example is a classical benchmark model: the lid-driven cavity problem. In this test, we consider the incompressible lid-driven cavity flow problem defined on the unit square. Setting $f = 0$ and the boundary condition $u = 0$ on $[\{0\}\times(0,1)]\cup[(0,1)\times\{0\}]\cup[\{1\}\times(0,1)]$ and $u = (1, 0)^T$ on $(0, 1) \times \{1\}$. The mesh consists of triangular element and the mesh size $h = \frac{1}{60}$, $\Delta t = 0.001$, $C_0 = 10000$, the final time $T = 500$ and the Taylor-Hood element is used to approximate the velocity and pressure. Figures 1 and 2 illustrate the velocity profiles of the lid-driven cavity problem along $x = 0.5$ and $y = 0.5$ in numerical scheme (21). Compared with the results provided by Erturk et al in [3] and Ghia et al in [4], we can see that the results obtained by the fully discrete implicit/explicit-SAV scheme are agree with Ghia's and Erturk's.

Table 1 Numerical results of Euler implicit/explicit SAV scheme (21) with $C_0 = 1.0$, $\Delta t = h^2$

$\frac{1}{h}$	$\frac{\|u-u_h^n\|_0}{\|u\|_0}$	Rate	$\frac{\|\nabla(u-u_h^n)\|_0}{\|\nabla u\|_0}$	Rate	$\frac{\|p-p_h^n\|_0}{\|p\|_0}$	Rate	$\frac{\lvert S-S_h^n\rvert}{\lvert S\rvert}$	Rate
4	0.426715		0.755780		0.0673757		0.0235671	
8	0.127010	1.7483	0.350946	1.1067	0.0211675	1.6704	0.00628828	1.9060
16	0.0322647	1.9769	0.168330	1.0600	0.00655715	1.6907	0.00173324	1.8592
32	0.00801932	2.0084	0.0827566	1.0243	0.00210286	1.6407	0.000446072	1.9581
64	0.00199329	2.0083	0.0411011	1.0097	0.000706242	1.5741	0.000112828	1.9832

6 Conclusion

In this paper, a fully discrete implicit/explicit numerical scheme is considered for the incompressible Navier–Stokes equations. Compared with the published papers [8,9,11,13,33], the main feature of this work is developing the unconditional stability of numerical solutions by introducing the scalar auxiliary variable, which enriches and supplements the theoretical findings of finite element method. Some numerical results are also provided to show the performances of the considered numerical scheme. The constant C_0 in energy variable has an important influence on the computational results, it should be chosen carefully and experimentally, for example, one needs to choose $C_0 \geq 10^4$ in the lid-driven cavity problem with high Reynold numbers. How to design a novel SAV factor independent of the constant C_0 is a meaningful topic, and is the goal of the following works.

Author Contributions Tong Zhang carried out the main theorem and wrote the paper, JinYun Yuan revised and checked the paper, Tong Zhang and JinYun Yuan read and approved the final version.

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Data Availability Statement Raw data were generated at the FreeFEM++ 14.3 64. Derived data supporting the findings of this study are available from the corresponding author upon request.

Declarations

Conflict of interest The authors declare that there is no conflict of competing interests.

Availability of Data and Material All data generated or analyzed during this study are included in this work.

Code Availability Derived data supporting the findings of this study are available from the corresponding author upon request.

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